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On the Application of the Comparison Method to Multivalued Difference Equations

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Presented by M. Putinar

1. Introduction

Mathematical models which are based on difference equations have many applications in different branches of science. Very often we are not interested in exact solutions of an equation, but in their stability. In recent years, numerous definitions of stability having more quantitative nature than most classical stability definitions have been presented by various investigators. There are many methods of examination of the stability of difference equations like the Lyapunov's second method, the so-called algebraic inequalities method, the frequency method. In the last few years, the theory of multivalued difference equations has been developed by several authors. The interest in studying the stability of an equilibrium solution of such equations is due to the fact that it is of importance in various fields like the optimization theory, the problem of minimization algorithms, the mathematical economics and the numerical analysis. In the last discipline multivalued differential equations are derived from a discretization of multivalued differential equations. But the numerical convergence question of the solutions of the multivalued differential equations is closely linked to numerical stability, that is, the stability of the corresponding multivalued difference equations.

The aim of the present paper is to extend part of the results of [1] and [2] to the case of a multivalued difference equation using the Lyapunov's second method. Many stability conditions of multivalued equations are obtained by weakening assumptions about the Lyapunov function. This paper presents a more general procedure, which is based on the comparison method. Certain Lyapunov-type theorems for the singlevalued difference equations [3] and for multivalued difference equations [5] can be obtained as corollaries.

2. Notations, definitions and hypotheses

Let R^k be the Euclidean space with norm $|\cdot|$ and $S(\rho)$ be the open ball in R^k centred at 0 with radius $\rho>0$. Let comp R^k be the family of all compact nonempty subsets of R^k. If n is any integer, we denote by I_n the set $\{n, n+1, \ldots\}$. For some ρ, $0<\rho\leqq\infty$, let $F: I_0\times S(\rho)\to\text{comp } R^k$ be a given multifunction. For fixed $(n_0, x_0)\in I_0\times S(\rho)$, consider on I_{n_0} the multivalued difference system

$$\text{(S)} \qquad x(n+1)\in F(n, x(n)), \quad x(n_0)=x_0.$$

By a solution of (S) on I_{n_0}, we understand a function $x: I_{n_0}\to S(\rho)$, i. e. a sequence, satisfying (S) on I_{n_0}. We denote by $x(.; n_0, x_0)$ any solution of (S) passing through $(n_0, x_0)\in I_0\times S(\rho)$. A point $\bar{x}\in S(\rho)$, such that $\bar{x}\in F(n, \bar{x})$ for any $n\in I_{n_0}$, is called an end-point of (S). Our stability results are based upon the existence of an appropriate scalar function $V: I_0\times S(\rho)\to R$, satisfying $V(n, 0)=0$ for any $n\in I_{n_0}$. In order to measure the growth or decay of such a function along a solution of (S) we define, for any $n\in I_{n_0}$, $x\in S(\rho)$ and $y\in F(n, x)\cap S(\rho)$, the following quantities

$$\text{(2.1)} \qquad \Delta^* V(n, x, y)=V(n+1, y)-V(n, x),$$

$$\Delta^* V(n, x, F)=\sup\{\Delta^* V(n, x, y);\ y\in F(n, x)\}.$$

Let us now consider the scalar difference equation

$$\text{(E)} \qquad \Delta r(n)=\omega(n, r(n)), \quad r(n_0)=r_0,$$

where $\omega: I_0\times A\to R$, $A\subseteq R$, is monotone non-decreasing with respect to r for any fixed $n\in I_0$, and such that (E) possesses a solution on I_{n_0}, for any $(n_0, r_0)\in I_0\times A$ and $\Delta r(n)=r(n+1)-r(n)$. In the sequel, we shall assume that $x=0$ is an end-point of (S) and $\omega(n, 0)=0$ for any $n\in I_0$, i. e., (E) has the zero solution. That solution of (E) passing through $(n_0, r_0)\in I_0\times A$ will be denoted by $r(.; n_0, r_0)$.

The following definitions concern the stability of the end-point 0 of (S).

Definition A. a). *The end-point* 0 *of* (S) *is stable if for any* $\varepsilon>0$ *and* $n_0\in I_0$ *there exists of* $\delta=\delta(n_0, \varepsilon)>0$ *such that* $|x_0|<\delta$ *implies* $|x(n; n_0, x_0)|<\varepsilon$ *for any solution* $x(.; n_0, x_0)$ *through* (n_0, x_0) *and all* $n\in I_{n_0}$;

b). *the end-point* 0 *of* (S) *is uniformly stable if it is stable and, in addition,* $\delta=\delta(\varepsilon)$;

c). *the end-point* 0 *of* (S) *is asymptotically stable if it is stable and if, in addition, there exists a* $\delta_0=\delta_0(n_0)$ *for any* $n_0\in I_0$, *such that* $|x_0|<\delta_0$ *implies* $\lim_{n\to\infty}|x(n; n_0, x_0)|=0$ *for any solution* $x(.; n_0, x_0)$ *of* (S);

d). *the end-point* 0 *of* (S) *is uniformly asymptotically stable if it is uniformly stable, and if, in addition, for any* $\varepsilon>0$ *there exist a* $\delta_0>0$ *and an* $l(\varepsilon)\in I_0$, *such that* $|x_0|<\delta_0$ *implies* $|x(n; n_0, x_0)|<\varepsilon$ *for any solution* $x(.; n_0, x_0)$ *and for all* $n\in I_{n_0+l(\varepsilon)}$.

R e m a r k 2.1. If each time there exists only a solution $x(.; n_0, x_0)$ from those passing through (n_0, x_0), such that one of the above stability definitions holds, we say that it is a matter of the w e a k s t a b i l i t y. Clearly, any type of stability implies the corresponding type of weak stability. This notion of weak stability is pertinent because of non-uniqueness of solutions in the case of multivalued equations.

The results we shall give in the next section are closely linked to certain kinds of stability for the comparison equation (E).

Definition B. a). *The zero solution of* (E) *is* *u p p e r s e m i s t a b l e if for any* $\varepsilon>0$ *and* $n_0 \in I_0$ *there exists a* $\delta=\delta(n_0, \varepsilon)$, *such that* $r_0 \in A$, $r_0<\delta$ *implies* $r(n; n_0, r_0)<\varepsilon$ *for all* $n \in I_{n_0}$;

b). *the zero solution of* (E) *is* *u n i f o r m l y u p p e r s e m i s t a b l e if it is semistable and if, in addition,* $\delta=\delta(\varepsilon)$;

c). *the zero solution of* (E) *is* *a s y m p t o t i c a l l y u p p e r s e m i s t a b l e if it is upper semistable and if, in addition, there exists a* $\delta_0=\delta_0(n_0)$ *for any* $n_0 \in I_0$, *such that* $r_0 \in A$, $r_0<\delta_0$ *implies* $\lim\limits_{n\to\infty} r(n; n_0, r_0)=0$;

d). *the zero solution of* (E) *is* *u n i f o r m l y a s y m p t o t i c a l l y u p p e r s e m i s t a b l e if it is uniformly upper semistable and if, in addition, for any* $\varepsilon>0$ *there exist a* $\delta_0>0$ *and an* $l(\varepsilon) \in I_0$, *such that* $r_0 \in A$, $r_0<\delta_0$ *implies* $r(n; n_0, r_0)<\varepsilon$ *for all* $n \in I_{n_0+l(\varepsilon)}$.

In the next section the function $V(n, x)$ will satisfy by turns some among the following hypotheses:

(H1). for every $n_0 \in I_0$ and any $\eta>0$, there exists a $\delta=\delta(n_0, \eta)>0$, such that for any $x_0 \in S(\rho)$, $|x_0|<\delta$ implies $V(n_0, x_0)<\eta$;

(H1'). for any $\eta>0$ there exists a $\delta=\delta(\eta)>0$, such that for any $n_0 \in I_0$ and $x_0 \in S(\rho)$, $|x_0|<\delta$ implies $V(n_0, x_0)<\eta$;

(H2). for any $n_0 \in I_0$, $n \in I_{n_0}$ and $x \in S(\rho)$ there exists a scalar continuous function $t \to a(t)$ non-decreasing for $t>0$ and $a(0)=0$ (function of class $\mathscr{K}$ such that

$$a(|x|) \leqq V(n, x), \text{ for all } (n, x) \in I_{n_0} \times S(\rho); \tag{2.2}$$

(H3). for all $(n, x) \in I_0 \times S(\rho)$

$$\Delta^* V(n, x, F) \leqq \omega(n, V(n, x)); \tag{2.3.}$$

(H3'). for all $(n, x) \in I_0 \times S(\rho)$ there exists a $y_{n,x} \in F(n, x) \cap S(\rho)$, such that

$$\Delta^* V(n, x, y_{n,x}) \leqq \omega(n, V(n, x)). \tag{2.3}$$

R e m a r k 2.2. Hypotheses (H1) and (H1') represent in fact the upper semicontinuity of $V(n, x)$ in $(n_0, 0)$ simple or, respectively, uniform with respect to n_0. We note that they are weaker than the frequently used hypothesis.

(H4). for any $n_0 \in I_0$, $n \in I_{n_0}$ and $x \in S(\rho)$ there exists a function $t \to b(t)$ of class $\mathcal{K}$, such that

(2.4) $$V(n, x) \leqq b(|x|), \text{ for all } (n, x) \in I_{n_0} \times S(\rho).$$

2.3. The given definitions and the properties of $V(n, x)$ are quite general. No assumption regarding uniqueness of solutions of (S) or the continuity of V are made.

3. Main results

In this section we give some general results concerning the stability of the end-point 0 of (S) connected with the upper semistability of the zero solution of (E).

Theorem 3.1. *Assume that there exists a function $V(n, x)$ satisfying (H2) and (H3). Then the following assertions hold:*
1). *If $V(n, x)$ satisfies* (H1) *and the zero solution of* (E) *is upper semistable, the end-point* 0 *of* (S) *is stable;*
2). *If $V(n, x)$ satisfies* (H1) *and the zero solution of* (E) *is asymptotically upper semistable, the end-point* 0 *of* (S) *is asymptotically stable;*
3). *If $V(n, x)$ satisfies* (H1′) *and the zero solution of* (E) *is uniformly upper semistable, the end-point* 0 *of* (S) *is uniformly stable;*
4). *If $V(n, x)$ satisfies* (H1′) *and the zero solution of* (E) *is uniformly-asymptotically upper semistable, the end-point* 0 *of* (S) *is uniformly-asymptotically stable.*

Proof. Let $x = x(.; n_0, x_0)$ be a solution of (S) through (n_0, x_0), i.e., $y = x(n+1; n_0, x_0) \in F(n, x(n; n_0, x_0)) \cap S(\rho)$. Then, in view of (H3) we get

$$\Delta V(n, x(n; n_0, x_0)) \equiv V(n+1, x(n+1; n_0, x_0)) - V(n, x(n; n_0, x_0))$$

$$= V(n+1, y) - V(n, x) \equiv \Delta^* V(n, x, y) \leqq \Delta^* V(n, x, F) = \Delta^* V(n, x(n; n_0, x_0), F)$$

$$\leqq \omega(n, V(n, x(n; n_0, x_0))).$$

According to Lemma 1 [4], [6] on difference inequalities, it follows:

(3.1) $$V(n, x(n; n_0, x_0)) \leqq r(n; n_0, r_0), \text{ for all } n \in I_{n_0},$$

whenever $V(n_0, x_0) \leqq r_0$.

1). Let $\varepsilon > 0$ and $n_0 \in I_0$ be given. From the upper semistability of the zero solution of (E) it follows that, for every $n_0 \in I_0$ and any $\varepsilon' = a(\varepsilon)$, there exists $\eta = \eta(n_0, \varepsilon) > 0$, such that $r_0 \in A$, $r_0 < \eta$ implies

(3.2.) $$r(n; n_0, r_0) < a(\varepsilon), \text{ for all } n \in I_{n_0}.$$

From (H1), for every $n_0 \in I_0$ and $\eta = \eta(n_0, \varepsilon)$, there exists a $\delta = \delta(n_0, \varepsilon) > 0$ such that for any $x_0 \in S(\rho)$, $|x_0| < \delta$ it follows

$$V(n_0, x_0) < \eta(n_0, \varepsilon). \tag{3.3}$$

Taking into account (3.1) and (H_2) we get

$$a(|x(n;\ n_0, x_0)|) \leqq V(n, x(n;\ n_0, x_0)) \leqq r(n;\ n_0, r_0) < a(\varepsilon),$$

for all $n \in I_{n_0}$ and any $x(.;\ n_0, x_0)$, whenever $V(n_0, x_0) \leqq r_0 < \eta(n_0, \varepsilon)$, and this is possible, according to (3.3), if $|x_0| < \delta$. Therefore, $|x(n;\ n_0, x_0)| < \varepsilon$ for all $n \in I_{n_0}$ and any $x(.;\ n_0, x_0)$ with $|x_0| < \delta(n_0, \varepsilon)$.

2). The stability of the zero solution of (S) is given by part 1). Let now $\varepsilon > 0$ be given. From the asymptotic upper semistability of the zero solution of (E) it follows that, for every $n_0 \in I_0$ and any $\varepsilon' = a(\varepsilon)$, there exist a $\eta_0 > 0$ and an $n_1 \in I_{n_0}$, such that, if $r_0 \in A$, $r_0 < \eta_0$ and $n \in I_{n_1}$, we have $r(n;\ n_0, r_0) < a(\varepsilon)$ for any $r(.;\ n_0, r_0)$. Now, according to (H1), for $n \in I_{n_0}$ and η_0, there exists a $\delta_0 > 0$, such that $x_0 \in S(\rho)$, $|x_0| < \delta_0$, implies $V(n_0, x_0) < \eta_0$. We will show that δ_0 and n_1 are those from the definition of the asymptotic stability. The proof is by contradiction. Assume that there exist x_0^*, $|x_0^*| < \sigma_0$ and $n^* \in I_{ne}$, such that $|x(n^*;\ n_0, x_0^*)| \geqq \varepsilon_0$. If we take $r_0^* = V(n_0, x_0^*)$, then we have $V(n, x(n;\ n_0, x_0^*) \leqq r(n;\ n_0, r_0^*)$ for all $n \in I_{n_1}$ and any solution $x(.;\ n_0, x_0)$. In view of hypothesis (H2) one can now write

$$a(\varepsilon) \leqq V(n, x(n^*;\ n_0, x_0^*) \leqq r(n^*;\ n_0, r_0^*) < a(\varepsilon),$$

a contradiction.

3). Now, from the uniform upper semistability for an arbitrary $\varepsilon > 0$ there exists a $\eta = \eta(\varepsilon) > 0$, such that (3.2) holds and, according to (H1'), $\delta = \delta(\varepsilon)$, such that (3.3) to be fulfilled. In the sequel the proof is similar to the proof of 1) and will, therefore, not be repeated here.

4). The uniform stability of the zero solution of (S) is established by part 3). Now, from the uniform asymptotic upper semistability of the zero solution of (E) it follows that, if $\varepsilon > 0$ is given, there exist a $\eta_0 > 0$ and a natural number $l(\varepsilon) \in I_0$, such that $r_0 \in A$, $r_0 < \eta_0$ implies $r(n;\ n_0, r_0) < \varepsilon$ for any $r(.;\ n_0, r_0)$ and all $n \in I_{n_0 + 1(\varepsilon)}$. The proof continues by a similar reasoning to than from 2).

Remark 3.1. From Theorem 3.1 one can easily obtain the results of J. Schinas and A. Meimaridou [5]. Thus, Theorems 1 and 3 will be obtained if we take $\omega(n, r) \equiv 0$. Then the zero solution of (E) is uniformly upper semistable and hypothesis (H3) becomes $V(n, x, F) \leqq 0$ for all $(n, x) \in I_0 \times S(\rho)$.

Let now consider $\omega(n, r) = -\varphi(n)\, c(r)$, where $c \in \mathscr{K}$ and $\varphi : I_0 \to R^+$ is such that $\sum_{n=0}^{\infty} \varphi(n)=\infty$. We can show that the zero solution of $\Delta r(n)=-\varphi(n)\ c(r(n))$, where $A=[0,\infty)$, is asymptotically stable and, therefore, it is asymptotically upper semistable. Actually, if $r(s)\leqq z\leqq r(s+1)$, then $c(r(s))\leqq c(z)\leqq c(r(s+1))$ and from here,

$$\frac{\Delta r(s)}{c(r(s))}=\int_{r(s)}^{r(s+1)}\frac{dz}{c(r(s))}\geqq\int_{r(s)}^{r(s+1)}\frac{dz}{c(z)}.$$

Taking into account our equation, we obtain

$$\text{(3.4)}\qquad \sum_{s=n_0}^{n-1}\int_{r(s)}^{r(s+1)}\frac{dz}{c(z)}=\int_{r(n_0)}^{r(n)}\frac{dz}{c(z)}\leqq-\sum_{s=n_0}^{n-1}\varphi(s),\quad r(n)=r(n;\ n_0,\ r_0).$$

From here, according to our hypothesis on $\varphi(n)$, it follows that $\lim_{n\to\infty} r(n;n_0,r_0)=0$. Theorem 5 [5] follows, since (H3) becomes $\Delta^* V(n,\ x,\ F)\leqq-\varphi(n,\ x))$, for all $(n,\ x)\in I_0\times S(\rho)$, and Theorem 7 follows if we take $\varphi(n)\equiv 1$, $n\in I_0$, because condition $\Delta^* V(n,x,F)\leqq -c(V(n,x))$ is weaker than $\Delta^* V(n,x,F)\leqq -c(|x|)$ for all $(n,\ x)\in I_0\times S(\rho)$.

In order to obtain sufficient conditions assuring the upper semistability of the zero solution of (E) we shall extend some of those given by L.Yu. Anapolskiĭ [1].

3.2. Assume that for any $(n_0,\ r_0)\in I_0\times A$ there exists a neighbourhood U of $r=0$, such that $U\subset A$. If for every $n_0\in I_0$ there exists $\alpha\in R^+\cap U$, such that for any $r\in(0,\ \alpha)$ and $n\in I_{n_0}$.

$$\text{(3.5)}\qquad \bar\omega(n,\ r)=\lim_{\delta\to 0}\ \sup_{|r'-r|<\delta}\ \omega(n,\ r')<0,$$

then the zero solution of (E) is upper semistable. Indeed, we have $\Delta r(n)=\omega(n,r(n))\leqq\bar\omega(n,\ r(n))<0$, from where $r(n)=r(n;\ n_0,\ r_0)<r_0$ for all $n\in I_{n_0}$. If we choose arbitrary $n_0\in I_0$ and $\varepsilon>0$, then it will suffice to take $\delta=\min(\alpha,\ \varepsilon)$.

Let now consider $\omega(n,\ r)=\lambda(n)\ \varphi(r)$, where $\varphi(r)$ and $\lambda(n)$ are such that for any $a>0$ there exists $b>0$, such that

$$\text{(3.6)}\qquad J(u)=\int_0^u\frac{ds}{\varphi(s)}<b\ \text{implies}\ u<a,$$

and, respectively, for every $n_0\in I_0$ there exists $\alpha\in \mathrm{R}^+\cap U$ such that

$$\text{(3.7)}\qquad \sum_{s=n_0}^{\infty}\lambda(s)\leqq -J(r_0)\ \text{for}\ r_0\in(0,\ \alpha).$$

If we assume that (3.5), (3.6), and (3.7) hold for $\omega(n,r)=\lambda(n)\ \varphi(r)$, then the zero solution of (E) is asymptotically upper semistable. We observe that from (3.4) we have $J(r(n;n_0,r_0))\leqq J(r_0)+\sum^{n-1}\lambda(s)$, and for $n_0\in I_0$ and $\varepsilon>0$ we take $\delta_0=\alpha$ and

$s=n_0$

$\gamma>0$ such that $J(u)<\gamma$ implies $u<\varepsilon$. According to (3.7), there exists $n_1\in I_{n_0}$ such that $n\in I_n$ implies $J(r(n;n_0,r_0))<0$ uniformly with respect to $r_0\in(0,\alpha)$. When (3.5) holds uniformly with respect to $n\in I_0$, the zero solution of (E) is uniformly upper semistable.

If we consider $\omega(n,r)=\lambda(n)\varphi(r)$ and we assume that (3.6) holds and, in addition, (3.5) and (3.7) are satisfied uniformly with respect to $n_0\in I_0$, then it follows the uniform asymptotic upper semistability of the zero solution of (E).

In the following we shall give a result concerning the weak stability of the zero solution of (S) analogous to that from Theorem 3.1.

Theorem 3.2. *Assume that there exists a function* $V(n,x)$ *satisfying* (H2) *and* (H3′) . *Then the following assertions hold:*

1). *If* $V(n,x)$ *satisfies* (H1) *and the zero solution of* (E) *is upper semistable, the end-point* 0 *of* (S) *is weakly stable;*

2). *If* $V(n,x)$ *satisfies* (H1) *and the zero solution of* (E) *is asymptotically upper semistable, the end-point* 0 *of* (S) *is weakly asymptotically stable;*

3). *If* $V(n,x)$ *satisfies* (H1′) *and the zero solution of* (E) *is uniformly upper semistable, the end-point* 0 *of* (S) *is weakly uniformly stable;*

4). *If* $V(n,x)$ *satisfies* (H1′) *and the zero solution of* (E) *is uniformly asymptotically upper semistable, the end-point* 0 *of* (S) *is weakly uniformly asymptotically stable.*

The proof of this theorem is similar to the proof of Theorem 3.1 and will, therefore, be outlined only for the part 1). Let $x(n;n_0,x_0): I_{n_0}\to S(\rho)$ be defined by $x(n_0;n_0,x_0)=x_0$, $x(n+1;n_0,x_0)=y_{n,x(n;n_0x_0)}\in F(n,x(n;n_0,x_0))\cap S(\rho)$, $n\in I_{n_0}$. Clearly, $x(.;n_0,x_0)$ is a solution of (S) passing through (n_0,x_0). Then, as in Theorem 3.1, part 1), we get:

$V(n,x(n;n_0,x_0))\leqq r(n;n_0r_0)$ for all $n\in I_{n_0}$ whenever $V(n_0,x_0)\leqq r_0$, and the proof can be continued in the same way.

Remark 3.3. Again, if we take $\omega(n,r)\equiv 0$, hypothesis (H3′) becomes $\Delta^*V(n,x,y_{n,y})\leqq 0$ for all $(n,x)\in I_0\times S(\rho)$ and $y_{n,x}\in F(n,x)\cap S(\rho)$, and we obtain Theorems 2 and 4 [5]. Taking $\omega(n,r)=-\varphi(n)c(r)$ as in Remark 3.1, we can obtain the other results.

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