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Relationships between Invariant Subspaces and Deflating Subspaces of a Regular Pair (F, G)

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For a regular pair (F, G) the relationship between the deflating subspaces [1] and Φ-(F, G)-invariant subspaces [2] [3] of the domain of (F, G) is established. The characterization of deflating subspaces by invariants allows the extension of perturbation results to the case of $\Phi-(F, G)$-invariant subspaces.

1. Introduction

In the study of perturbation theory of the generalised eigenvalue eigenvector problem defined on a regular pair $(F, G)\in\mathscr{L}^r_{n,n}$ a key notion has emerged: the notion of the deflating subspace introduced by W. Stewart [1]. The aim of this paper is to establish the links between the invariant subspaces of a regular pair of matrices [2], [3] and the notion of deflating subspaces. For a regular pair $(F, G)\in\mathscr{L}^r_{n,n}$ a d-dimensional subspace of R^n, $\mathscr{U}$, is defined as a deflating subspace iff $\dim(F\mathscr{U}+G\mathscr{U})\leqq\dim\mathscr{U}=d$. It will be shown that a d-dimensional subspace $\mathscr{U}$ is a deflating subspace of the regular pair $(F, G)\in\mathscr{L}^r_{n,n}$, iff $\mathscr{U}$ is a $\Phi-(F, G)$ elementary divisor subspace (i. e., the set of strict equivalence invariants of the restriction pencil $(F, G)/\mathscr{U}$ [4] is characterised by elementary divisors and possibly zero row minimal indices). Our analysis also demonstrates that in the definition of deflating subspaces given by Stewart [1] the equality sign should be used instead of "$\leqq$" since there is no subspace of R^n for which strict inequality of the "$<$" holds true.

So, the perturbation results established by Stewart for this case may be transferred to the invariant subspaces of a regular pair $(F, G)\in\mathscr{L}^r_{n,n}$. The results presented here for regular pairs extend to the more general case of entirely right regular pairs.

2. Characterization of invariant subspaces of a regular pair

Definition (2.1) [5]: *The pair (F, G), $F, G\in\mathsf{R}^{n\times n}$ is said to be regular if $p=\operatorname{rank}_{\mathsf{R}(s,\hat{s})}\{sF-\hat{s}G\}=n$, where $sF-\hat{s}G$ is the associated homogeneous matrix pencil.*■

Note that the terms given for the pencils will also be used for the pairs (F, G) which "generate" the pencil $sF-\hat{s}G$. By factorising the invariant polynomials $f_i(s, \hat{s})$ of the Smith form into powers of homogeneous polynomials irreducible over C we obtain the set of elementary divisors (e. d.) of the pencil $sF-\hat{s}G$; these are of the following type: s^p, $\hat{s}^q$, and pairs of complex conjugate e. d. $(s-a\hat{s})^\tau$, $(s-\bar{a}\hat{s})^\tau$, a, $\bar{a}\in$C. For the pair (F, G) we define: e. d. of the type $\hat{s}^q$ are called infinite elementary divisors (i. e. d.), e. d. of the type s^p are called zero elementary divisors (z. e. d.), and e. d. of the type $(s-a\hat{s})^\tau$ are called non zero finite element divisors (nz. f. e. d.). Elementary divisors of the type s^p, $(s-a\hat{s})^\tau$, $a\neq 0$ will be referred to as finite elementary divisors (f. e. d.). In the following $\mathscr{L}^r_{n,n}$ shall denote the set of regular pairs of dimensions $n\times n$.

The classical theory of matrix pencils deals with the study of invariants and canonical forms of the strict equivalence classes [5] $\mathscr{E}_s(L)$ when $L\in\mathscr{L}^r_{n,n}$. The map f : $\mathscr{L}^r_{n,n}\to\{$f. e. d.$\}\times\{$i. e. d$\}$ is a complete invariant for $\mathscr{E}_s(L)$. The existence of a complete set of invariants for pairs $L\in\mathscr{L}^r_{n,n}$ implies the existence of a canonical form known as Weirstrass canonical form [5].

Definition (2.2): *A pair $(F, G)\in\mathscr{L}^r_{m,n}$, $m\geqq n$, will be called entirely right regular* (e. r. r.) *iff $sF-\hat{s}G$ is characterized by* e. d. *and possibly zero row minimal indices* (r. m. i.).■

For a given e. r. r. pair (F, G) the types of invariants of $sF-\hat{s}G$ will be denoted in short by D_0, D_∞, D_α, where D_0 is the set of z. e. d. of the pair (F, G), D_∞ is the set of infinite e. d., and D_α is the set of finite and non zero e. d.

The key tool in our study of invariant subspaces is the notion of the $\mathscr{U}$ – restricted ordered pair. This definition is stated next.

Definition (2.3), [2]: *Let $(F, G)\in$R$^{m\times n}\times$R$^{m\times n}$, $\mathscr{U}\subseteq$R^n be a subspace of R^n and V a basis matrix of $\mathscr{U}$. The pair (FV, GV) will be called a $\mathscr{U}$-restricted ordered pair; the assosiated pencil $sFV-\hat{s}GV$ will be termed $(F, G)-\mathscr{U}$ restriction pencil and shall be denoted by $(F, G)/\mathscr{U}$.*■

Note that for a given $\mathscr{U}$ the pair (FV, GV) or the pencil $sFV-\hat{s}GV$ is not uniquely defined since the definition involves a particular choice of basis matrix for $\mathscr{U}$; it is clear, however, that for a given $\mathscr{U}$ all restriction pencils are strict equivalent [5] and thus they are characterized by the same set of strict equivalence invariants. It is due to this fact that the algebraic structure of $\mathscr{U}$ with respect to the pair (F, G) is independent of the particular choice of basis.

Definition (2,4): *Let $(F, G)\in$R$^{m\times n}\times$R$^{m\times n}$, $\mathscr{U}\subseteq$R^n, be a subspace, $I_\mathscr{U}$ be the set of strict equivalence invariants of $(F, G)/\mathscr{U}$, and let Φ be the root range of $(F, G)/\mathscr{U}$ (i. e., the set of all roots of elementary divisors of the associated pencil $sFV-\hat{s}GV$). The subspace $\mathscr{U}$ will be called $\Phi-(F, G)$ -elementary divisor subspace ($\Phi-(F, G)-$e. d. s.) if $I_\mathscr{U}=\{D_{\alpha_i}\forall\alpha_i\in\Phi\}\cup I^0_r$, where I^0_r denotes the set of zero row minimal indices of the pencil $sFV-\hat{s}GV$.* ■

3. Deflating subspaces and their properties

Definition (3.1) [1]: *Let $(F, G) \in \mathscr{L}^r_{n,n}$ and $\mathscr{U}$ be a d-dimensional subspace of* R^n. *$\mathscr{U}$ is a* d e f l a t i n g s u b s p a c e *for* (F, G) *iff* $\dim(F\mathscr{U}+G\mathscr{U}) \leqq \dim \mathscr{U} = d$.■

Before we examine the exact relationship of deflating subspace and the invariant subspaces defined above, we state the following result.

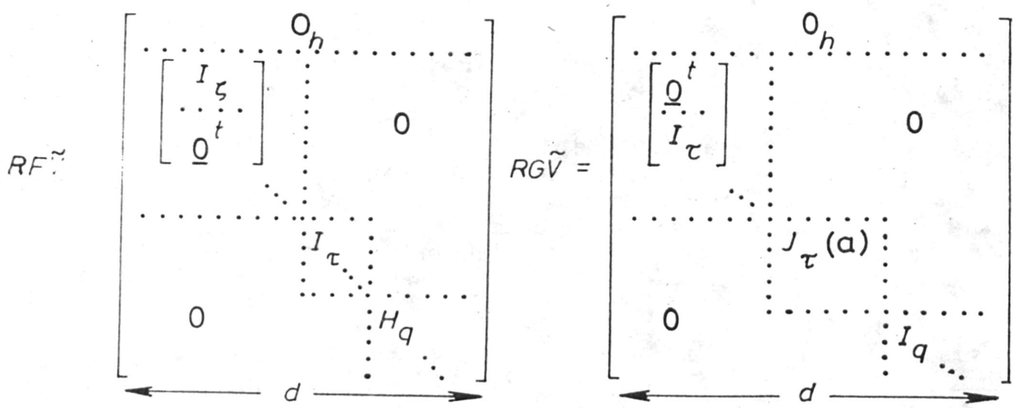
Proposition (3.1): *Let $(F, G) \in \mathscr{L}^r_{n,n}$ and $\mathscr{U} \subseteq \mathrm{R}^n$ be a d-dimensional subspace. The restriction pencil $(F, G)/\mathscr{U}$ has no* c. m. i. *(column minimal indices) in its set $I_{\mathscr{U}}$ of strict equivalence invariants.*

P r o o f. Let $(sF-G)V$, V be a basis matrix of $\mathscr{U}$, and let $N_r\{sFV-GV\} \neq \{0\}$. Then, $\exists \; \underline{v}(s) \in \mathrm{R}^n[s] \underline{v}(s) \neq 0$, such that $(sF-G)\underline{V}\underline{v}(s)=0$. Define $\underline{x}(s)=V.\underline{v}(s)$; since V has full rank and $\underline{v}(s)\neq 0$, then $(sF-G)\,x(s)=0$, and, thus, $sF-G$ is singular, Q. E. D.■

From the above result it is clear that $I_{\mathscr{U}}$ may contain f. e. d., i. e. d., and r. m. i.

Theorem (3.1): *Let $(F, G) \in \mathscr{L}^r_{n,n}$ and let $\mathscr{U} \subseteq \mathrm{R}^n$ be a d-dimensional subspace. $\mathscr{U}$* is a deflating subspace iff $\mathscr{U}$ is a $\Phi-(F, G)-$ e. d. s. (i. e., $I_{\mathscr{U}}$ *is characterized by* e. d. *and possibly zero* r. m. i.).

P r o o f. First note that $\dim(F\mathscr{U}+G\mathscr{U})=\mathrm{rank}[FV \vdots GV]$, where V is any basis matrix of $\mathscr{U}$. Clearly, if $R \in \mathrm{R}^{n\times n}$, $|R|\neq 0$, then $\mathrm{rank}_{\mathrm{R}}[RFV \vdots RGV] = \mathrm{rank}_{\mathrm{R}}[FV \vdots GV]$. We can always choose a special basis $\tilde{V}$ and an R such that $(RF\tilde{V}, RG\tilde{V})$ is in the Kronecker form [5]. By proposition (3.1), the possible set of invariants are e. d. and r. m. i. The typical blocks in $(RF\tilde{V}, RGV)$ are:



where $\left(\begin{bmatrix} I_\zeta \\ \cdots \\ \underline{0}^t \end{bmatrix}, \begin{bmatrix} \underline{0}^t \\ \cdots \\ I_\zeta \end{bmatrix}\right)$, $(I_\tau, J_\tau(\alpha))$, (H_q, I_q) are typical blocks associated with a r. m. i. ζ, a f. e. d. $(s-\alpha)^\tau$, and an i. e. d. $\hat{s}^q$.

Because of the block diagonal structure in (3.1) the independent columns in $[RE\tilde{V} \vdots RG\tilde{V}]$ may be found block by block. Thus, by inspection:

(i) $\left\{ \vdots \begin{bmatrix} I_\zeta \vdots 0 \\ 0^t \vdots I_\zeta \end{bmatrix} \right.$ has $\zeta+1$ independent columns $\forall\ \zeta \in \mathrm{N}$.

(ii) $[I_\tau \vdots J_\tau(\alpha)]$ has τ independent columns $\forall\ \alpha \in \mathrm{C}$, $\tau \in \mathrm{N}$.

(iii) $[H_q \vdots I_q]$ has q independent columns $\forall\ q \in \mathrm{N}$, and the $\dim(F\mathcal{U}+G\mathcal{U}) = \operatorname{rank}[FV \vdots GV] = \operatorname{rank}[RF\tilde{V} \vdots RG\tilde{V}] = \Sigma_{i=1}^{t} \quad (\zeta_i+1) + \Sigma_{i=1}^{\rho} t_i + \Sigma_{i=1}^{\mu} q_i$, where t is the total number of non-zero r. m. i., ρ is the total number of f. e. d., and μ is the total number of infinite el. div.

By setting $g = \Sigma_{i=1}^{t} \zeta_i + \Sigma_{i=1}^{\rho} \tau_i + \Sigma_{i=1}^{\mu} q_i$ we have that

$$\dim(F\mathcal{U}+G\mathcal{U}) = g+t. \tag{3.2}$$

However, it is clear from (3.1) that

$$d = \dim \mathcal{U} = \sum_{i=1}^{t} \zeta_i + \sum_{i=1}^{\rho} \tau_i + \sum_{i=1}^{\mu} q_i = g. \tag{3.3}$$

By (3.2) and (3.3) we have that $\dim(F\mathcal{U}+G\mathcal{U}) = g+t \geqq g = d = \dim\ \mathcal{U}$ and equality holds if and only if there are no non-zero r. m. i. in $I_{\mathcal{U}}$ (zero r. m. i. do not affect the above inequality).

Now if $\mathcal{U}$ is deflating, then for $\dim(F\mathcal{U}+G\mathcal{U}) = d$ all r. m. i. must be zero, and, thus, $\mathcal{U}$ is a $\Phi-(F, G)-$ e. d. s. Conversely, if $\mathcal{U}$ is a $\Phi-(F, G)-$ e. d. s., then all ζ_i are zero and $\dim(F\mathcal{U}+G\mathcal{U}) = \dim \mathcal{U}$, i. e., $\mathcal{U}$ is deflating subspace. ∎

Remark (3.1): Let $(F, G) \in \mathcal{L}^r_{n,n}$ and $\mathcal{U} \subseteq \mathrm{R}^n$ be a d-dimensional subspace. Then $\dim(F\mathcal{U}+G\mathcal{U}) \geqq \dim \mathcal{U} = d$ and equality holds true iff $\mathcal{U}$ is a $\Phi-(F, G)$ $-$ e. d. s.

The above remark demonstrates that in the definition of deflating subspaces given by Stewart the equality sign should be used instead of "$\leqq$" since there is no subspace of R^n for which strict inequality "$<$" holds true. However, the "$<$" inequality sign may hold true if the pair (F, G) is not right regular.

Theorem (3.2): *Let $(F, G) \in \mathrm{R}^{n\times n} \times \mathrm{R}^{n\times n}$ be a matrix pair (not necessarily regular), and let $\mathcal{U}$ be a d-dimensional subspace. Then, $\dim(F\mathcal{U}+G\mathcal{U}) \leqq \dim \mathcal{U}$ iff the total number of non zero row minimal indices is less than total number of column minimal indices of the restriction pair $(F, G)/\mathcal{U}$.*

Proof. By choossing a special basis $\tilde{V}$ *of* $\mathcal{U}$ and $R \in \mathrm{R}^{n\times n}$, $|R| \neq 0$, such that $(RF\tilde{V}, RG\tilde{V})$ is in Kronecker form, we have that $\dim(F\mathcal{U}+G\mathcal{U}) = \operatorname{rank}_{\mathrm{R}}[RF\tilde{V} \vdots RG\tilde{V}]$. The typical blocks in $(RF\tilde{V}, RG\tilde{V})$ are:

$$R F \tilde{V} = \underbrace{\begin{bmatrix} 0_{h,g} & & & & & \\ & [I_\varepsilon \vdots \underline{0}] & & & 0 & \\ & & \begin{bmatrix} I_\zeta \\ \cdots \\ \underline{0}^t \end{bmatrix} & & & \\ & & & I_\tau & & \\ & 0 & & & H_q & \end{bmatrix}}_{d}, \quad R G \tilde{V} = \begin{bmatrix} 0_{h,g} & & & & & \\ & [\underline{0} \vdots I_\varepsilon] & & & 0 & \\ & & \begin{bmatrix} \underline{0}^t \\ \cdots \\ I_\zeta \end{bmatrix} & & & \\ & & & J_\tau(a) & & \\ & 0 & & & I_q & \end{bmatrix}$$

where $([I_\varepsilon \vdots \underline{0}], [\underline{0} \vdots I_\varepsilon])$, $\left(\begin{bmatrix} I_\zeta \\ \cdots \\ \underline{0}^t \end{bmatrix}, \begin{bmatrix} \underline{0}^t \\ \cdots \\ I_\zeta \end{bmatrix}\right)$, $(I_\tau, J_\tau(\alpha))$, (H_q, I_q) are typical blocks associated with a col. m. i. ε, a r. m. i. ζ, a f. e. d. $(s-\alpha)^\tau$, and an inf. e. d. $\hat{s}^q$. Following the same arguments as in the theorem (3.1) we have by inspection

$$(3.5)\quad \dim \mathscr{U} = d = g + \sum_{i=1}^{p} (\varepsilon_i + 1) + \sum_{j=1}^{t} \zeta_j + \sum_{\kappa=1}^{\rho} \tau_\kappa + \sum_{\nu=1}^{\mu} q_\nu = g + p + r,$$

where $r = \Sigma_{i=1}^{p} \varepsilon_i + \Sigma_{j=1}^{t} \zeta_j + \Sigma_{\kappa=1}^{\rho} \tau_\kappa + \Sigma_{\nu=1}^{\mu} q_\nu$, however, it is clear from (3.4) that

$$(3.6)\quad \dim (F\mathscr{U} + G\mathscr{U}) = \sum_{j=1}^{t} (\zeta_j + 1) + \sum_{i=1}^{p} \varepsilon_i + \sum_{\kappa=1}^{\rho} \tau_\kappa + \sum_{\nu=1}^{\mu} q_\nu = t + r.$$

So, $\dim (F\mathscr{U} + G\mathscr{U}) \leqq \dim \mathscr{U}$ iff $t + r \leqq g + p + r \leftrightarrow t \leqq g + p$.

4. Perturbation analysis for Φ-(F, G) – invariant subspaces

Given that deflating subspaces of a regular pair (F, G) may also be expressed for $\Phi-(F, G)$-e. d. subspaces. Thus, following the results of Stewart we may state:

Proposition (4.1): *Let $\mathscr{U}$ be a $\Phi-(F, G)$-e. d. subspace. Then, there are orthogonal matrices K and L such that the first d columns of L span $\mathscr{U}$ and*

$$K^t F L = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}, \quad K^t F L = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix},$$

where G_{11} and F_{11} are $d\times d$ matrices.

Proof. Let $L=[L_1 \vdots L_2]$, where $L_1\in R^{n\times d}$ with orthonormal columns such that colspan $L_1=\mathscr{U}$, and $L_2\in R^{n\times(n-d)}$ such that $L=[L_1 \vdots L_2]$ be orthogonal. Also, let $K_2\in R^{n\times(n-d)}$ have orthonormal columns lying in the orthogonal complement of $F\mathscr{U}+G\mathscr{U}((F\mathscr{U}+G\mathscr{U})^1)$ and $K_1\in R^{n\times d}$ be chosen such that $K=[K_1 \vdots K_2]$ is orthogonal.

Since colspan $L_1=\mathscr{U}\Rightarrow$colspan$(GL_1)\subseteq G\mathscr{U}\subseteq F\mathscr{U}+G\mathscr{U}$, and because K_2 is lying in the orthogonal complement of $F\mathscr{U}+G\mathscr{U}$ we have that $K_2^+(GL_1)=0$. Following the same arguments we have that $K_2^+(FL_1)=0$. So,

$$K^t GL=\begin{bmatrix} K_1^t \\ \hline K_2^t \end{bmatrix} G[L_1 \vdots L_2]=\begin{bmatrix} K_1^t GL_1 & K_1^t GL_2 \\ K_2^t GL_1 & K_2^t GL_2 \end{bmatrix}=\begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} \text{ and}$$

$$K^t FL=\begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \quad \text{thus} \quad K^t(sF-G)L=s\begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}-\begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}. \blacksquare$$

Also if $\lambda\in\sigma(F_{11}, G_{11})$ with corresponding eigenvector $\underline{z}$, then $\lambda\in\sigma(F, G)$ with corresponding eigenvector $L_1\underline{z}\in\mathscr{U}$. Next, we shall define an operator which plays an important role in deriving the error bounds in the perturbation analysis discussed next.

Definition (4.1.), (1): *Let G_1, $F_1\in R^{l\times l}$ and G_2, $F_2\in R^{m\times m}$ are fixed matrices. For any $X=[P \vdots Q]\in R^{m\times 2l}$, P, $Q\in R^{m\times l}$ we define the operator T on $R^{m\times 2l}$ as follows:*

$$(4.1) \qquad T\colon R^{m\times 2l}\to R^{m\times 2l}\colon \forall\, X=[P \vdots Q]\in R^{m\times 2l}$$

$$\to T(X)\colon =[PG_1-G_2Q \vdots PF_1-F_2Q]\in R^{m\times 2l}\blacksquare$$

Lemma (4.1): *The operator T is linear.*$\blacksquare$

Lemma (4.2), 1: *The operator T is non-singular if and only if: $\sigma(F_1, G_1)\cap\sigma(F_2, G_2)=\emptyset$.*

The spectrum above is defined on the associated pencils sF_1-G_1, sF_2-G_2 in the usual manner).

Remark (4.1): T is non singular is equivalent that, $\forall\,[R \vdots S]=Y\in R^{m\times 2l}$ $\exists X=[P \vdots Q]\in R^{m\times 2l}$ such that:

$$T(X)=Y\leftrightarrow(PG_1-G_2Q=R \text{ and } PF_1-F_2Q=S).\blacksquare$$

Now we define two norms on $R^{m\times 2l}$ as follows:

Definition (4.2), [1]: $\forall\, R=[P \vdots Q]\in R^{m\times 2l}$, *where P, $Q\in R^{m\times l}$ we define*

$$(4.2)\ \|R\|_2'=\max\{\|P\|_2,\ \|Q\|_2\} \text{ and } \|R\|_F'=\max\{\|P\|_F\ \ \|Q\|_F\}.\blacksquare$$

We can easily see that $\| R \|'_2$, $\| R \|'_F$ are norms on $R^{m \times 2l}$. As we have seen before the operator T is determined by the fixed matrices F_1, G_1, F_2, G_2. Let

(4.3) $$\operatorname{dif}(F_1, G_1; F_2, G_2) \triangleq \inf_{\| X \|'_F = 1} \| T(X) \|'_F.$$

Then dif $(F_1, G_1; F_2, G_2)=0$ iff T is non-singular, that is, iff $\sigma(F_1, G_1) \cap \sigma(F_2, G_2) = \emptyset$, [1].
Also if T is non singular and $T(X)=Y$, then

(4.4) $$\| X \|'_F \leqq \| Y \|'_F / \operatorname{dif}(F_1, G_1; F_2, G_2).$$

The topic examined next is the study of perturbation properties of $\Phi-(F, G)$-e. d. s., when there is uncertainty in the parameters of the pair (F, G). The following analysis is based on the work of Stewart for deflating subspaces.

Let $L=[L_1 \vdots L_2]$ and $K=[K_1 \vdots K_2]$ be orthgonal matrices with L_1, $K_1 \in R^{n \times l}$. Then if $F_{21}=K_2^t F L_1=0$ and $G_{21}=K_2^t G L_1=0$, the columns of L_1 span a Φ-(F, G)-e. d. s. for $sF-G$. If $F_{21}$and G_{21} are small but not exactly zero, it is reasonable to ask whether there exists orthogonal matrices $K'=[K'_1 \vdots K'_2]$ and $L'=[L'_1 \vdots L'_2]$ near K and L, respectively, such that $K'^t_2 F L'_1 = K'^t_2 G L'_1 = 0$. We select K' and L' in the form $K'=[K'_1 \vdots K'_2]$, $L'=[L'_1 \vdots L'_2]$, where

$$K'_1=(K_1+K_2P)(I+P^t P)^{-1/2}, \quad K'_2=(K_2-K_1P^t)(I+PP^t)^{-1/2}$$

(4.5)

$$L'_1=(L_1+L_2Q)(I+Q^tQ)^{-1/2}, \quad L'_2=(L_2-L_1Q^t)(I+QQ^t)^{-1/2},$$

where P, $Q \in R^{(n-l) \times l}$. It is easy shown that K', L' are orthogonal.

By setting $G_{ij}=K_i^t G L_j$, $F_{ij}=K_i^t F L_j$, $\forall\, i, j=1, 2$, then conditions $K'^t_2 F L'_1 = K'^t_2 G L'_1 = 0$ leads to the following system:

(4.6) $$\{PG_{11}-G_{22}Q=G_{21}-PG_{12}Q, \quad PF_{11}F_{22}Q=F_{21}-PF_{21}Q\}.$$

If we define as before $T([P \vdots Q])=[PG_{11}-G_{22}Q \vdots PF_{11}-F_{22}Q]$, the system (4.6) becomes:

(4.7) $$T([P \vdots Q])=[G_{21}-PG_{12}Q \vdots F_{21}-PF_{12}Q].$$

Thus the problem of perturbing K, L into the (deflating) matrices (K', L') is reduced to the following equivalent problem: determine under what conditions the non linear equation (4.7) has a solution. Such conditions are given next.

Proposition (4.2), [1]: *Let $K=[K_1 \vdots K_2]$, $L=[L_1 \vdots L_2]$ be orthogonal matrices with $K_1, L_1 \in \mathrm{R}^{n\times l}$. Let also G_{ij}, F_{ij} be defined as before. Let us also define:*

$$(4.8) \qquad \gamma=\| G_{21} \vdots F_{21} \|'_F, \quad \eta=\| G'_{12} \vdots F'_{12} \|'_2, \quad \delta=\mathrm{dif}(F_{11}, G_{11}; F_{22}, G_{22}).$$

If $\varkappa_1=\gamma\eta/\delta^2$, then there are orthogonal matrices $P, Q \in \mathrm{R}^{(n-l)\times l}$ satisfying $\| P \vdots Q \|'_F \leq \gamma/\delta(1+\varkappa)=\gamma/\delta.\ (1+\sqrt{1-4\varkappa_1}/1-2\varkappa_1+\sqrt{1-4\varkappa_1})<2\gamma/\delta$ such that colspan (L_1+L_2Q) is a deflating and thus a $\Phi-(F, G)$-e. d. s. for the regular pair (F, G).

Proposition (4.3): *If $F_1, G_1, E_1, E'_1 \in \mathrm{R}^{l\times l}$, $F_2, G_2, E_2, E'_2 \in \mathrm{R}^{m\times m}$, then* $\mathrm{dif}(G_1+E_1, F_1+E'_1; G_2+E_2, F_2+E'_2) \geq \mathrm{dif}(G_1, F_1; G_2, F_2)-\max\{\| E_1 \|_2+\| E_2 \|_2; \| E'_1 \|_2+\| E'_2 \|_2\}$.

Proof. By definition, $\mathrm{dif}(G_1+E_1, F_1+E'_1; G_2+E_2, F_2+E'_2)=\inf \| [P(G_1+E_1)-(G_2+E_2)Q,\ P(F_1+E'_1)-(F_2+E'_2)Q] \|'_F$, $\| [P \vdots Q] \|'_F=1$. Let the infimum be obtained for $X=[P \vdots Q]$, then $\| P \|_F, \| Q \|_F \leq 1$. Hence, $\mathrm{dif}(G_1+E_1, F_1+E'_1; G_2+E_2, F_2+E'_2)=\| [P(G_1+E_1)-(G_2+E_2)Q,\ P(F_1+E'_1)-(F_2+E'_2)Q] \|'_F$ $=\max\{\| P(G_1+E_1)-(G_2+E_2)Q \|_F, \| P(F_1+E'_1)-(F_2+E'_2)Q \|_F\} \geq \max$ $\{\| PG_1-G_2Q \|_F-\| E_1 \|_2-\| E_2 \|_2,\ \| PF_1-F_2Q \|_F-\| E'_1 \|_2-\| E'_2 \|_2\} \geq \max$ $\{\| PG_1-G_2Q \|_F, \| PF_1-F_2Q \|_F\}-\max\{\| E_1 \|_2+\| E_2 \|_2, \| E'_1 \|_2+\| E'_2 \|_2 \geq \mathrm{dif}$ $(G_1, F_1; G_2, F_2)-\max\{\| E_1 \|_2+\| E_2 \|_2; \| E'_1 \|_2+\| E'_2 \|_2\}$.■

The question that arises is to investigate whether there exists a $\mathscr{U}'\ \Phi-(F', G')$-e. d. s. of the perturbed pair $(F', G')=(F+E_1, G+E'_1)$ which is close to $\mathscr{U}$. Of course, the essential question is how close such a subspace may be found with respect to the given perturbation (E_1, E'_1). The answer to this is given next.

Theorem (4.1): *Let $K=[K_1 \vdots K_2]$, $L=[L_1 \vdots L_2]$ be orthogonal matrices with $K_1, L_1 \in \mathrm{R}^{n\times e}$, $G_{ij}=K_i^t GL_j$, $F_{ij}=K_i^t FL_j$ $\forall$ i, j=1, 2, and suppose that $G_{21}=F_{21}=0$. In that case $\mathscr{U}$ =*colspan L_1 *is an* $\Phi-(F, G)$-e. d. s. *of the pair (F, G). Assume that $E_1, E'_1 \in \mathrm{R}^{n\times n}$ and $E'_{ij}=K_i^t E'_1 L_j$, $E_{ij}=K_i^t E_1 L_j$ $\forall$ i, j=1, 2, $\varepsilon_{ij}=\max\{\| E'_{ij} \|_F, \| E_{ij} \|_F\}$. Let us also assume that $\gamma=\varepsilon_{21}$, $\eta=\| [G'_{12} \vdots F'_{12}] \|'_2+\varepsilon_{12}$, $\delta=\mathrm{dif}(G_{11}, F_{11}; G_{22}, F_{22})-\varepsilon_{11}-\varepsilon_{12}$. If $\gamma\eta/\delta^2<1/4$, then, there are matrices $P, Q \in \mathrm{R}^{(n-l)\times l}$ satisfying $\| [P \vdots Q] \|'_F<2\gamma/\delta$ such that* colspan (L_1+L_2Q) *is a* $\Phi-(F, G)$-e. d. s. *for* $s(F+E_1)-(G+E'_1)$.

Proof. We will give the proof by using the previous proposition to the problem stated on $s(F+E_1)-(G+E'_1)$. If we now denote by $\tilde{G}_{ij}$, $\tilde{F}_{ij}$ it has been denoted by G_{ij}, F_{ij} in the proposition (4.2), then,

$$\tilde{G}_{ij} = K_i^t(G+E_1')L_j = K_i^t G L_j + K_i^t E_1' L_j = G_{ij} + E_{ij}'$$

(4.9) $\qquad , \; i, j = 1, 2$

$$\tilde{F}_{ij} = K_i^t(F+E_1) L_j = K_i^t F L_j + K_i^t E_1 L_j = F_{ij} + E_{ij}$$

So $\tilde{G}_{21} = G_{21} + E_{21}' = E_{21}'$, $\tilde{F}_{21} = F_{21} + E_{21} = E_{21}$ and $\tilde{\gamma} = \| [\tilde{G}_{21} \vdots \tilde{F}_{21}] \|_F' = \| [E_{21}' \vdots E_{21}] \|_F' = \max\{\| E_{21}' \|_F, \| E_{21} \|_F\} = \varepsilon_{21}$. Also $\tilde{\eta} = \| [\tilde{G}_{12}^t \vdots \tilde{F}_{12}^t] \|_2' = \max \| \tilde{G}_{12}^t \|_2, \| \tilde{F}_{12}^t \|_2\} = \max\{\| G_{12}^t + E_{12}'^t \|_2, \| F_{12}^t + E_{12}^t \|_2\} \leqq \max\{\| G_{12}^t \|_2 + \| E_{12}'^t \|_2, \| F_{12}^t \|_2 + \| E_{12}^t \|_2\} \leqq \max\{\| G_{12}^t \|_2 + \| E_{12}' \|_F, \| F_{12}^t \|_2 + \| E_{12} \|_F\} \leqq \max\{\| G_{12}^t \|_2, \| F_{12}^t \|_2\} + \max\{\| E_{12} \|_F, \| E_{12} \|_F\} = \| [G_{12}^t \vdots F_{12}^t] \|_2' + \varepsilon_{12}$, $\delta = \mathrm{dif}(\tilde{F}_{11}, \tilde{G}_{11}; \tilde{F}_{22}, \tilde{G}_{22}) = \mathrm{dif}(F_{11} + E_{11}, G_{11} + E_{11}'; F_{22} + E_{22}, G_{22} + E_{22}') \geqq \mathrm{dif}(G_{11}, F_{11}; G_{22}, F_{22}) - \max\{\| E_{11} \|_2 + \| E_{22} \|_2; \| E_{11}' \|_2 + \| E_{22}' \|_2\} \geqq \mathrm{dif}(G_{11}, F_{11}; G_{22}, F_{22}) - \max\{\| E_{11} \|_2, \| E_{11}' \|_2\} - \max\{\| E_{22} \|_2, \| E_{22}' \|_2\} = \mathrm{dif}(G_{11}, F_{11}; G_{22}, F_{22}) - \varepsilon_{11} - \varepsilon_{12}$.
If we set now:

$$\gamma = \varepsilon_{21}, \; \eta = \| [G_{12}^t \vdots F_{12}^t] \|_2' + \varepsilon_{12}, \quad \delta = \mathrm{dif}(G_{11}, F_{11}; G_{22}, F_{22}) - \varepsilon_{11} - \varepsilon_{12},$$

then we have $(\tilde{\gamma} = \gamma, \; \tilde{\eta} \leqq \eta, \; \tilde{\delta} \geqq \delta) \Rightarrow \tilde{\gamma} . \tilde{\eta} / \tilde{\delta}^2 < \gamma . \eta / \delta^2$. So if we suppose now that $\gamma . \eta / \delta^2 < 1/4$, then $\tilde{\gamma} . \tilde{\eta} / \tilde{\delta}^2 < 1/4$ and the theorem (4.1) is valid for the pencil $s(F + E_1) - (G + E_1')$; that means that there are matrices $P, Q \in \mathrm{R}^{(n \times l) \times l}$ such that the subspace $\mathcal{U}' = \mathrm{colspan}(L_1 + L_2 Q)$ is a $\Phi - (F, G)$-e. d.s. for $s(F + E_1) - (G + E_1')$, which of course is a subspace close to the $\Phi - (F, G)$-e. d. s. $\mathcal{U} = \mathrm{colspan}\, L_1$. The closeness can be measured by the gap $(\mathcal{U}, \mathcal{U}') = $ gap (colspan L_1, colspan, $(L_1 + L_2 Q)$).

5. Conclusions

In this paper was shown that for a regular matrix pair (F, G) a d-dimensional subspace $\mathcal{U}$ ($\mathcal{U} \subseteq \mathrm{R}^n$) is a deflating subspace iff $\mathcal{U}$ is a $\Phi - (F, G)$-elementary divisor subspace (i.e., the set of strict equivalence invariants of the restriction pencil $(F, G)/\mathcal{U}$ is characterized by elementary divisors and possibly zero row minimal indices). It was seen that in the definition of deflating subspaces given by Stewart [1] the equality sign should be used instead of "$\leqq$", since there is no subspace of R^n for which "$\leqq$" holds true. When the pair (F, G) is not regular it was shown that $\dim(F\mathcal{U} + G\mathcal{U}) \leqq \dim \mathcal{U}$ iff the total number of non zero row minimal indices is less than total number of column minimal indices of the restriction pencil $(F, G)/\mathcal{U}$. Finally, it was shown that for a regular pair (F, G) and for a $\mathcal{U}$ $\Phi - (F, G)$-e. d. s. there exists a $\mathcal{U}'$ $\Phi - (F', G')$-e. d. s. of the perturbed pair $(F', G') = (F + E_1, G + E_1')$ which is close to $\mathcal{U}$.

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