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Lumped Mass Finite Element Method for Parabolic and Eigenvalue Problems

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We consider semidiscrete variant of the lumped mass modification of the standard Galerkin method for parabolic and eigenvalue problems using linear simplicial or bilinear quadrangular elements and prove the corresponding error estimates.

1. Introduction

The diagonalization of the mass matrix (lumped mass matrix) is often an advantageous practical step in the finite element method. This approach was applied for eigenvalue problems by P. Tong, T. H. Pian and L. Bucciarelli [27], where one of the terms was interpreted as resulting from replacing the piecewise linear functions by certain piecewise constant functions. This interpretation was adopted by T. Ushijima [28, 29] and M. Tabata [24] for the case of uniform triangulation. During the last 10 years various types of problems concerning the lumped mass technique have been discussed for second order elliptic eigenvalue problems ([4, 12, 14]), for elasticity theory problems ([13, 18, 19]), semidiscrete and fully discrete versions of the lumped mass modification for linear and nonlinear parabolic equations [25, 26, 7, 20, 6, 10].

In this paper we give a simple technique for establishing rate of convergence for second-order parabolic problems and second-order eigenvalue problems. We establish error estimate of type superconvergence of the gradient for second order parabolic problems.

2. Preliminaries

Let Ω be a bounded domain in R^n, $n=2, 3$ with a polygonal boundary $\partial\Omega$. We denote by $(.,.)$ the scalar product in $L^2(\Omega)$. For positive integer m let $H^m(\Omega)$ denote the Sobolev space provided with the norm [1]

$$\| v \|_{m,\Omega} = \Big(\sum_{s=0}^{m} | v |^2_{s,\Omega}\Big), \text{ where } |v|_{m,\Omega} = \Big(\sum_{|\alpha|=m} \| D^\alpha_v \|^2_{L^2(\Omega)}\Big)^{1/2}$$

and $H_0^1=\{v\in H^1(\Omega): v|_{\partial\Omega}=0\}$.

We introduce the bilinear form $a: H^1\times H^1\to R^1$

$$a(u, v)=\int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx,$$

where the matrix $\{a_{ij}\}_{i,j=1,\dots,n}$ is symmetric and positive definite. Then, $a(u, u)$ is H_0^1 – elliptic.

By $P(k)$ we denote the set of polynomials of degree k and by $Q(k)$ – polynomials of degree k of every variable $x_1,\dots, x_n$.

We construct a 1-strongly regular partition K of $\bar\Omega$ of simplicial and quadrangular finite elements with diameters at most h.

With this partition we associate a space S_h of functions such that: i) S_h is finite dimensional subspace of $H_0^1(\Omega)\cap C^\circ(\bar\Omega)$; ii) the restriction of $v\in S_h$ over every finite element $e\in K$ is linear or polylinear function, i. e., $v\in P(1)$ or $v\in Q(1)$.

For computing the integrals over the finite elements we will use the quadrature formula

$$I_e(v)=\text{meas}(e)\begin{cases}\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n+1} v(w_i) & \text{for simplexes}\\[2ex] \dfrac{1}{n^2}\displaystyle\sum_{i=1}^{n^2} v(w_i) & \text{for quadrangles,}\end{cases}$$

where w_i are the vertices of the finite element.

Note that this quadrature formula is exact for polynomials of $P(1)$ and $Q(1)$ for simplexes and quadrangles, respectively.

We may define an approximation of the inner product and the norm in $L^2(\Omega)$ by

$$(1)\qquad (u, v)_h=\sum_{e\in K} I_e(uv),\quad \|u\|_h=(u, u)_h^{1/2}.$$

3. Parabolic problem

We consider the weak formulation of the following Cauchy boundary value problem in a polygonal domain Ω, [11, 25]: find a function $u(x, t)$ $H_0^1(\Omega)\cdot\times C^1[0, T]$, satisfying the integral identity

$$(2)\qquad \left(\frac{\partial u}{\partial t}, v\right)+a(u,v)=(f,v)\quad \text{for every } v\in H_0^1(\Omega),\ t\in(0,T)$$

$$u(x,0)=u_0(x),\quad x\in\Omega.$$

Let us apply the standart semidiscrete Galerkin method for the problem (2): find $u_h(x,t)=\sum_{j=1}^{N}\alpha_j(t)\varphi_j(x)$, $x\in\Omega$, $t\in(0, T)$, where $\{\varphi_j(x)\}_{j=1,\dots,N}$ is a canonical basis in S_h, which satisfies the integral identity $(\partial u_h/\partial t, v)+a(u_h, v)=(f, v)$, for every $v\in S_h$, $t\in(0, T)$, $u_h(x, 0)$ – given. This will give us the following Cauchy problem for the system of ODE:

$$(3) \qquad M\alpha'(t)+K\alpha(t)=F(t), \quad t\geqq 0, \quad \alpha(0) - \text{given},$$

where $\alpha=(\alpha_1,\ldots, \alpha_N)$, $M=\{(\varphi_i, \varphi_j)\}$ is the mass matrix, $K=\{a(\varphi_i, \varphi_j)\}$ is the stiffness matrix and $F=((f, \varphi_1),\ldots,(f, \varphi_N))$. We shall discuss the lumped mass modification of this method [3, 7, 10, 20, 26]. A simple way to define it is to replace the matrix M in (3) by diagonal matrix $\bar{M}$ obtained by taking for its diagonal elements the numbers $\Sigma_{k=1}^{N}(\varphi_i, \varphi_k)$

$$(4) \qquad \bar{M}\alpha'(t)+K.\alpha(t)=F(t) \quad t\geqq 0.$$

This procedure may also be interpreted as resulting from evaluating the first term in (2) by numerical quadrature (1).

We find that for the considered partition and quadrature formula, (4) is equivalent to the following problem: find $u_h(x, t)\in S_h$ for $t\in(0, T)$ and satisfying the identity

$$(5) \qquad \left(\frac{\partial u_h}{\partial t}, v\right)_h+a(u_h, v)=(f, v), \text{ for every } v\in S_h$$

$$u_h(x, 0)=\sum_{i=1}^{N} u_0(x_i)\varphi_i(x),$$

where $(.,.)_h$ is defined by (1).

Recall that the maximum principles hold for (5) [22].

The use of numerical integration in the finite element method for solving parabolic equations has been investigated by P. Raviart in [22], where $L^2(H^1(\Omega))$ and $L^\infty(L^2(\Omega))$ error estimates are obtained. Error estimates in mean square and maximum norms for discrete solutions are discussed by V. Thomée [26]. The method of lumped masses also is used in combination with different discretization in time by M. El Hatri [10] and C. M. Chen and V. Thomée[7]. All these error estimates are of optimal order in h and require only that the partition is quasiuniform [9].

Our intent here is to produce error estimates of superconvergence type for the lumped mass approximation, similar to the corresponding elliptic and parabolic error estimates from [2, 3, 15, 17, 21].

These estimates are strongly connected with the discrete seminorms which are introduced below.

First, we introduce the seminorm

$$(6) \; |||w|||_h^2=\max_{0\leqq t\leqq T}(w, w)_h+\int_0^T |w|_{1,\Omega}^2 dt\equiv \|w\|_{L^\infty(L_h^2(\Omega))}^2+\|w\|_{L^2(H^1(\Omega))}^2.$$

Next seminorm is connected with the points of superconvergence of the derivatives of the approximate finite element solution.

Define

$$(7) \qquad |||w|||_h^* = (\sum_{e \in K} |w|_{1,e}^{*2})^{1/2},$$

where for the different types of finite elements the seminorm $|w|_{1,e}^*$ is defined as follows:

i) for a polylinear quadrangle $e \in K$ with center x_e^*

$$|w|_{1,e}^* = (\text{meas}(e))^{1/2} |\text{grad } w(x_e^*)|;$$

ii) for a linear simplex $e \in K$ let G_e be the set of the midpoints of the edges of e and for $x \in G_e$ let $\frac{\partial w(x)}{\partial l_x}$ be the derivative along the edge passing by x

$$|w|_{1,e}^* = \{\text{meas}(e) \sum_{x \in G_e} (\frac{\partial w(x)}{\partial l_x})^2\}^{1/2}.$$

In order to prove the superconvergence result we have to suppose that the finite element partition has additional regularity, namely it is quasiuniform (see [5, 15, 17]). For the 2D case this means that every two adjacent finite elements form almost parallelogram [5, 17].

Now we shall formulate the main result concerning semidiscrete lumped mass finite element approximation of the parabolic problem (5).

Theorem 1. *Let the partition K be* 1*-strongly regular and quasiuniform and the solution of the problem* (5) *be sufficiently smooth. Then, the following error estimates are true*

$$(8) \qquad |||u - u_h|||_h^* = 0(h^2)$$

$$(9) \qquad \|u - u_h\|_{L^\infty(L^\infty(\Omega))} = 0(h^2 |\ln h|), \text{ for } n = 2,$$

where $|||\cdot|||_h^*$ *is defined by* (7).

Proof. Let us introduce the finite element interpolate $u_I \in S_h$ of function $u(x) \in C^0(\Omega)$ as follows: $u_I(x) = \Sigma_{i=1}^N u(x_i)\varphi_i(x)$, where $\{\varphi_i\}$ is the canonical basis of S_h.

The estimate (9) is known as a maximum norm error estimate (see, for instance, [7, 26]) and here we present another proof for it. It is an immediate consequence of the estimate (8). We have only to use that [11, 25]

$$\max_{x \in \Omega} |v| \le C|\ln h| \, \|v\|_{1,\Omega} \text{ for every } v \in S_h, \text{and } \max_{x \in e} |u - u_I| \le Ch^2 \max_{\substack{x \in e \\ |\alpha| \le 2}} |D^\alpha u|.$$

The proof of the estimate (8) could be completed in the following steps given in separate lemmas.

Lemma 1. *Under the conditions of Theorem* 1 *the following estimate holds*

$$|a(u-u_I,\ v)| \leqq Ch^2 \|u\|_{1,\Omega} |v|_{1,\Omega} \tag{10}$$

for every $v \in S_h$ *and* $t \in (0,\ T)$.

The proof of the estimate (10) is based on the technique developed for elliptic problems by L. Oganesjan and L. Ruhovec [21] and M. Zlámal [30] (see also [2, 5, 15]). Note that the most restrictive assumptions in the proof are quasiuniformity of the partition, the requirement that it covers exactly the domain Ω and the high regularity of the solution $u(x,t)$.

Lemma 2. *Under the conditions of Theorem* 1 *the following estimate holds:*

$$|||u_h-u_I|||_h \leqq Ch^2(\|u\|_{L^2(H^3(\Omega))}+\|\frac{\partial u}{\partial t}\|_{L^2(H^2(\Omega))}), \tag{11}$$

where the seminorm $|||\ .\ |||_h$ *is defined by* (6).

The following identity is used in the proof:

$$(\frac{\partial z_h}{\partial t},\ v)_h + a(z_h,\ v) = a(u-u_I,\ v) + (\frac{\partial u}{\partial t},\ v) - (\frac{\partial u_I}{\partial t},\ v)_h, \tag{12}$$

for every $v \in S_h$, where $z_h = u_h - u_I \in S_h$, the estimates (10) and $|(w,\ v)-(w_I,\ v)_h| \leqq Ch^2 \|w\|_{2,\Omega} |v|_{1,\Omega}$, $t \in (0, T)$. The last inequality is a consequence of the Bramble-Hilbert lemma and the fact that the quadrature formula involved in (1) is exact for polynomial of P(1).

Putting $v = z_h$ in (12) and using H_0^1-ellipticity of $a(.,.)$, we get for $t \in (0,\ T)$

$$\frac{\partial}{\partial t}\|z_h\|_h^2 + C\|z_h\|_{1,\Omega}^2 \leqq Ch^4(\|u\|_{3,\Omega}^2 + \|\frac{\partial u}{\partial t}\|_{2,\Omega}^2)$$

from where (11) follows.

The last step involves the following simple lemma:

Lemma 3. *The estimates hold*

$$|||u-u_I|||_h^* \leqq Ch^2 \|u\|_{L^2(H^3(\Omega))} \tag{13}$$

$$|||v|||_h^* \leqq C|||v|||_h \ \textit{for every } v \in S_h. \tag{14}$$

The proof of the Theorem 1 ends with the notion that follows immediately from triangle inequality and estimates (11), (13), (14):

$$|||u-u_h|||_h^* \leqq |||u-u_I|||_h^* + |||u_I-u_h|||_h^* \leqq |||u-u_I|||_h^* + C|||u_I-u_h|||_h$$

$$\leqq Ch^2((\|u\|_{L^2(H^2(\Omega))} + \|\frac{\partial u}{\partial t}\|_{L^2(H^2(\Omega))}).$$

Remarks 1. The Theorem 1 is valis also when the integrals in $a(u_h, v)$ and (f, v) from (5) are evaluated using quadrature formula with positive coefficients exact for polynomials at least from $P(1)$.

2. For simplicial finite elements the estimate (8) can be used to produce superconvergence error estimate at the nodes employing the averaging technique from [16] and [15].

4. Elliptic eigenvalue problem

For the bilinear form defined in § 2 we set the following eigenvalue problem: find $\lambda \in R$ and $u \in H_0^1(\Omega)$, such that the following identity is satisfied:

(15) $$a(u, v)=\lambda(u, v) \text{ for every } v \in H_0^1(\Omega).$$

There is a countable set of eigenvalues λ_k and eigenfunctions u_k such that $0<\lambda_1 \leqq \lambda_2 \leqq \ldots$ and $\|u_k\|=(u_k, u_k)^{1/2}=1$.

We set the following finite element approximation of problem (15): find $u_h \in S_h$ and $\lambda_h \in R$, such that the integral identity

(16) $$a(u_h, v)=\lambda_h(u_h, v)_h \text{ holds for every } v \in S_h,$$

where, $(.,.)_h$ is defined by (1) and gives a lumped mass matrix.

Our goal is to produce error estimates for the lumped mass approximation (16) of (15) similar to those obtained in [12, 23] without lumping. Here we systematically use the technique from the parabolic case where quadrature formula is employed. Similar results are given in [14, 27] using different approach.

Let $P: u \rightarrow Pu \in S_h$ be the Ritz projector [23] with the properties

(17) $$a(u-Pu, v)=0 \text{ for every } v \in S_h,$$

(18) $$\|u-Pu\|_{k,\Omega} \leqq Ch^{2-k}\|u\|_{2,\Omega}, \quad k=0, 1, \quad u \in H^2(\Omega).$$

We prove the following theorem:

Theorem 2. *Let λ_h be some eigenvalue of (16) and u_h be its eigenfunction, such that $\|u_h\|_h=1$. If the corresponding eigenfunction u of (15) belongs to $H^2(\Omega) \cap H_0^1(\Omega)$, then*

$$|\lambda-\lambda_h| \leqq C(\lambda)h^2.$$

Proof. It is easy to see that by (15)—(17) $\lambda=a(Pu, u_h)/(u, u_h)$ and $\lambda_h=a(Pu, u_h)/(Pu, u_h)_h$.

Therefore,

(19) $$|\lambda-\lambda_h|=\frac{\lambda_h}{|(u_h, u)|}|(u, u_h)-(Pu, u_h)_h|.$$

Since $(u_h, u)=(u, u)-(u-u_h, u) \geqq 1-\|u-u_h\|$, then the denominator in the right hand side of (19) is bounded away from zero when $h \rightarrow 0$, if $\|u-u_h\| \rightarrow 0$ when $h \rightarrow 0$. This fact is proved in [4].

The second term in (19) is estimated in the following way:

$$|(u, u_h)-(Pu, u_h)_h| \leqq |(u-Pu, u_h)| + |(Pu, u_h)-(Pu, u_h)_h|.$$

The last term is connected with the error of the quadrature formula employed in (1) and by the Bramble-Hilbert argument is estimated by

$$(20) \qquad |(Pu, u_h)-(Pu, u_h)_h| \leqq C(\lambda)h^2.$$

For the first term it is enough to employ (18) and the equivalence of the $L^2(\Omega)$-norm and $\|.\|_h$ in S_h, i.e.,

$$|(u-Pu, u_h)| \leqq \|u-Pu\| \, \|u_h\| \leqq C\|u-Pu\| \, \|u_h\|_h = C\|u-Pu\| \leqq Ch^2 \|u\|_{2,\Omega}.$$

Submitting these estimates in (19) we prove the Theorem 2. •

Having proved the convergence of the eigenvalues, we will prove the convergence of the corresponding eigenfunctions in norm $H^1(\Omega)$.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then,* $|u-u_h|_{1,\Omega} \leqq C(\lambda).h$, *where* u *is an eigenfunction corresponding to the eigenvalue* λ.

Proof.

$$a(u-u_h, u-u_h) = a(u,u) - 2a(u,u_h) + a(u_h, u_h) = \lambda + \lambda_h - 2\lambda(u,u_h)$$
$$\leqq |\lambda-\lambda_h| + \lambda|(u,u) - 2(u,u_h) + (u_h, u_h) - (u_h, u_h) + (u_h, u_h)_h|$$
$$\leqq |\lambda-\lambda_h| + \lambda\|u-u_h\|_{0,\Omega} + \lambda|(u_h, u_h)-(u_h, u_h)_h|.$$

We estimate every member of the right-side of this inequality. From Theorem 2 we have: $|\lambda-\lambda_h| \leqq C(\lambda).\,h^2$.

For the second term we use the result of [14] (Theorem 4) or [4] (Theorem 1) $\|u-u_h\|_{0,\Omega} \leqq C(\lambda)\,.h^2$.

For the last term similarly to (20) we get $|(u_h, u_h)-(u_h, u_h)_h| \leqq C\|u_h\|_{1,\Omega}\,.h^2 \leqq Ch^2$.

At the end it remains to use that $a(.,.)$ is H_0^1-elliptic.

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