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The Best Onesided Algebraic Approximation in $L_p[-1, 1]$ $(1 \leqq p \leqq \infty)$

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Presented by Bl. Sendov

In this paper are proved direct Stečkin's type and converse Salem – Stečkin's type theorems for the best onesided algebraic approximation, using τ-moduli of function.

1. Introduction

Denote by $H_n(T_n$, respectively) the set of all algebraic (trigonometric) polynomials of degree at most n. Let f be a bounded on $[-1, 1]$ function (or 2π-periodic). The best onesided approximation of f by means of algebraic (trigonometric) polinomials of degree at most n in L_p-metric is given by

$$\tilde{E}_n(f)_p = \inf\{\| P-Q \|_{L_p[-1,1]};\quad P, Q \in H_n,\quad P(x) \geqq f(x) \geqq Q(x),\quad -1 \leqq x \leqq 1\};$$

$$(\tilde{E}_n^T(f)_p = \inf\{\| P-Q \|_{L_p[0,2\pi]};\quad P, Q \in T_n,\quad P(x) \geqq f(x) \geqq Q(x),\quad 0 \leqq x \leqq 2\pi\}),$$

where, as usually,

$$\| h \|_{L_p[a,b]} = \{\int_a^b |h(x)|^p dx\}^{1/p},\quad 1 \leqq p < \infty$$

$$\| h \|_{L_\infty[a,b]} = \sup\{|h(x)|;\quad a \leqq x \leqq b\}.$$

Finding estimates for the best onesided algebraic approximation due to G. Freud [6], who proved the first nontrivial result, namely: $\tilde{E}_n(f)_1 = O(n^{-r-1} \overset{b}{\underset{a}{V}} f^{(r)})$, $r = 1, 2, \dots$

Later P. Nevai [1] obtains

$$\tilde{E}_n(f)_1 \leqq c(r) \int_{-1}^{1} (\Delta_n(x))^{r+1} |df^{(r)}(x)|,\quad 1 \leqq r \leqq n,$$

where $\Delta_n(x) = \sqrt{1-x^2}/n + 1/n^2$, $f \in W_1{}^r[-1, 1]$.

In the trigonometric case, A. Andreev and V. Popov [4] proved the following Stečkin's-type theorem:

$$\tilde{E}^T_n(f)_p \leqq c(k)\tau_k(f;\, n^{-1})_p \quad (1 \leqq p \leqq \infty, \quad k \leqq n), \tag{1.1}$$

where $c(k)$ is a constant depending only on k. They use the global modulus of smoothness

$$\tau_k(f;\, \delta)_p := \| \omega_k(f, \cdot\, ;\, \delta) \|_{L_p[0,2\pi]},$$

where $\omega_k(f, x;\, \delta) = \sup\{|\Delta_h^k f(t)|;\ t,\ t+kh \in [x - k\delta/2,\ x + k\delta/2]\}$,

$$\Delta_h^k f(t) = \sum_{\nu=0}^{k} (-1)^{k+\nu} \binom{k}{\nu} f(t+\nu h).$$

V. Popov proves the following inverse theorem [2]:

$$\tau_k(f;\, n^{-1})_p \leqq c(k) n^{-k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} \tilde{E}^T_\nu(f)_p, \quad (1 \leqq p \leqq \infty, \quad k \leqq n). \tag{1.2}$$

The inequalities (1.1) and (1.2) characterize $\tilde{E}^T_n(f)_p$.

In comparison with the trigonometric case, additional difficulties caused by the special role of the end-points of the interval arise. Roughly speaking, the smoothness of the function influences weaker on the degree of the approximating polynomials in the end-points than in the middle of the interval. So the modulus of smoothness characterizing the best onesided algebraic approximation has to take into account this property of the approximating system. We shall use the global modulus of smoothness

$$\tau_k(f;\, \delta)_p = \| \omega_k(f, \cdot\, ;\, \delta(\cdot)) \|_{L_p[-1,1]},$$

where the local modulus of smoothness $\omega_k(f, x, \delta(x))$ is given by

$$\omega_k(f, x;\, \delta(x)) = \sup\{|\Delta_h^k f(t)|;\ t,\ t+kh \in [x - k\delta(x)/2,\ x + k\delta(x)/2] \cap [-1, 1]\}$$

$$\Delta_h^k f(t) = \begin{cases} \sum_{\nu=0}^{k} (-1)^{\nu+k} \binom{k}{\nu} f(t+\nu h) & \text{if } t,\ t+kh \in [-1, 1] \\ 0 & \text{elsewhere.} \end{cases}$$

We shall prove the following statements in this paper:

Theorem 1. *There exists a constant $c>0$, such that $\tilde{E}_n(f)_p \leqq c\tau_1(f;\, \Delta_n)_p$ for any bounded and measurable in $[-1, 1]$ function f.*

Theorem 2. *For any integer number k, $k \geqq 2$ there exists a constant $c(k)>0$, such that if $f(x)$ is bounded and measurable in $[-1, 1]$, then*

$$\tilde{E}_n(f)_p \leqq c(k)\tau_k(f;\, \Delta_n)_p, \quad n \geqq k. \tag{1.3}$$

An application of Theorem 2 for onesided approximation of convex function is given in section 3 of this paper.

Theorem 3. *For any integer number k there exists a constant $c(k)>0$, such that*

$$\tau_k(f;\ \Delta_n)_p \leqq c(k) n^{-k} \sum_{v=0}^{n} (v+1)^{k-1} \tilde{E}_v(f)_p \tag{1.4}$$

for any bounded and measurable in $[-1, 1]$ function f.

Everywhere further c is an absolute constant and $c(A, B, \ldots)$ is a positive number depending only on the marked parameters. These numbers may differ at each occurrence.

2. Jackson's type theorem for the best onesided algebraic approximation

We shall need some auxiliary results in the proof of Theorem 1. We begin with the properties of the τ-moduli.

The global moduli of smoothness were introduced by Bl. Sendov [9]. K. Ivanov [8] generalizes them as follows: $\tau_k(f,\ w;\ \delta)_{p',p} = \| w(\cdot)\omega_k(f,\cdot;\ \delta(\cdot))_{p'} \|_{L_p[-1,1]}$, $\omega_k(f,\ x;\ \delta(x))_{p'} = \{(2\delta(x))^{-1} \int_{-\delta(x)}^{\delta(x)} |\Delta_v^k f(x)|^{p'} dv\}^{1/p'}$. Only few of their properties are mentioned here.

Lemma 1.

$$\tau_k(f+g;\ \delta)_p \leqq \tau_k(f;\ \delta)_p + \tau_k(g;\ \delta)_p; \tag{2.1}$$

$$\tau_k(f+g,\ 1;\ \delta)_{p',p} \leqq \tau_k(f,\ 1;\ \delta)_{p',p} + \tau_k(g,\ 1;\ \delta)_{p',p}; \tag{2.2}$$

$$\tau_k(f;\ \Delta_n)_p \leqq c(k) \| f^{(k)}(\Delta_n)^k \|_{L_p[-1,1]},\ \text{if } k\geqq 1,\ f^{(k)}\in L_p[-1,\ 1]; \tag{2.3}$$

$$\tau_k(f,\ 1;\ \Delta_n)_{p',p} \leqq c(k) \| f^{(k)}(\Delta_n)^k \|_{L_p[-1,1]} \text{ if } 1\leqq p'\leqq p; \tag{2.4}$$

$$\tau_1(f;\ \lambda\Delta_n)_p \leqq c(\lambda)\tau_1(f;\ \Delta_n)_p,\ \lambda=\text{const}\geqq 1; \tag{2.5}$$

$$\tau_k(f;\ \delta_1)_p \leqq \tau_k(f;\ \delta_2)_p \text{ if } \delta_1(x)\leqq\delta_2(x),\ -1\leqq x\leqq 1; \tag{2.6}$$

$$\tau_k(f,\ 1;\ \delta)_{p_1',p} \leqq \tau_k(f,\ 1;\ \delta)_{p_2',p} \text{ if } p_1'\leqq p_2'. \tag{2.7}$$

Inequalities (2.2), (2.4) and (2.7) were proved in [8]. The other ones can be proved by analogy with the corresponding inequalities for $\tau_k(f,\ 1;\ \delta)_{\infty,p}$ ([8]) and $\tau_k(f;\ \delta)_p$ when $\delta=\text{const.}$ ([3]).

Because of the relations $\omega_k(f,\ x;\ \delta(x))_\infty = \sup\{|\Delta_v^k f(x)|;\ |v|\leqq\delta(x)\} \leqq \sup\{|\Delta_v^k f(t)|;\ x-k\delta(x)\leqq t,\ t+kv\leqq x+k\delta(x)\} = \omega_k(f,\ x;\ 2\delta(x))$ we have

$$\tau_k(f,\ 1;\ \Delta_n)_{\infty,p} \leqq \tau_k(f;\ 2\Delta_n)_p. \tag{2.8}$$

Lemma 2. *Let $T(x)=\cos n \arccos x$ be the Chbyshev polynomial of first kind with zeros $x_k=\cos(2k-1)\pi/2n$ $(1\leqq k\leqq n)$. Set $l_k(x) := T(x)/(x-x_k)T'(x_k)$, $1\leqq k\leqq n$, $x_0=1$, $x_{n+1}=-1$. Then*

(2.9) $\quad |l_k(x)| \leqq 2 \quad if \quad x_{k+1} \leqq x \leqq x_{k-1} \quad (1 \leqq k \leqq n);$

(2.10) $\quad |l_k(x)| \leqq v^{-1} \quad if \quad x_{k-v} \leqq x \leqq x_0, \text{ or } x_{n+1} \leqq x \leqq x_{k+v} \quad (1 \leqq k \leqq n, \; v \neq 0).$

Proof. Evidently $l_k(x_k)=1$. Let $x \neq x_k$. We set $x = \cos t$, $0 \leqq t \leqq \pi$, $t_v = (2v-1)\pi/2n$, $(1 \leqq v \leqq n)$. Then

(2.11) $$|l_k(x)| = |\cos nt| \sin t_k/(2n \sin|(t-t_k)/2| \sin((t+t_k)/2)).$$

Note that, for $x_{k+1} \leqq x \leqq x_{k-1}$,

(2.12) $$|(t-t_k)/2| \leqq \pi/2n.$$

The inequalities $\pi/4n \leqq (t+t_k)/2 \leqq \pi - \pi/4n$ give

(2.13) $$\sin((t+t_k)/2) \geqq \sin(\pi/4n) \geqq 1/2n.$$

From (2.11)—(2.13) and the fact that $\sin x$ is a concave function in the interval $[0, \pi]$, we obtain for $x_{k+1} \leqq x \leqq x_{k-1}$ $|l_k(x)| = \{|\cos n(t-t_k)\cos nt_k - \sin n(t-t_k) \sin nt_k|(\sin((t+t_k)/2)\cos((t_k-t)/2) + \cos((t+t_k)/2)\sin((t_k-t)/2))\}/(2n \sin((t+t_k)/2) \sin|(t-t_k)/2|) \leqq |\sin n(t-t_k)| \cos((t_k-t)/2)/(2n \sin|(t-t_k)/2|) + |\sin n(t-t_k)||\cos((t+t_k)/2)|/(2n \sin((t+t_k)/2)) \leqq 2n \sin|(t-t_k)/2|/(2n \sin|(t-t_k)/2|) + 1/(2n(1/2n)) = 2$ and this proves (2.9).

Let $x_{k-v} \leqq x$ or $x \leqq x_{k+v}$. Then $\pi/2 \geqq |(t-t_k)/2| \geqq v\pi/2n$ and $\sin|(t-t_k)/2| \geqq v/n$. From this inequality, (2.11) and concavity of $\sin x$ in $[0, \pi]$ we obtain

$$|l_k(x)| \leqq \frac{\sin t_k}{2n \sin|(t-t_k)/2| \sin((t+t_k)/2)} \leqq \frac{(\sin t_k + \sin t)/2}{n(v/n)\sin((t+t_k)/2)} \leqq v^{-1}$$

if $x \leqq x_{k+v}$ or $x \geqq x_{k-v}$, and (2.10) is proved.

Let us consider the functions:

$$g_k(x) = \begin{cases} 0 & \text{if } -1 \leqq x < x_k \\ 1 & \text{if } x_k \leqq x \leqq 1 \end{cases} \qquad (1 \leqq k \leqq n, \; x_k = \cos((2k-1)\pi/2n)).$$

We construct polynomials $P_k, Q_k \in H_{4n-4}$ $(k=1, 2, \dots n)$, uniquely determined by the conditions

(2.14) $$\begin{cases} P_n(x) \equiv 1; \\ P_k(x_v) = \begin{cases} 0 & k+1 \leqq v \leqq n \\ 1 & 1 \leqq v \leqq k \end{cases} \quad P_k^{(i)}(x_v) = 0, \quad 1 \leqq v \leqq n, \; v \neq k, \; i = 1,2,3 \\ \text{for} \quad 1 \leqq k < n \\ Q_1(x) \equiv 0; \\ Q_k(x_v) = \begin{cases} 0 & k \leqq v \leqq n \\ 1 & 1 \leqq v \leqq k-1 \end{cases} \quad Q_k^{(i)}(x_v) = 0, \quad 1 \leqq v \leqq n, \; v \neq k, \; i = 1,2,3 \\ \text{for} \quad 1 < k \leqq n. \end{cases}$$

Let us note that for the difference $R_k(x)=P_k(x)-Q_k(x)$ we have $R_k \in H_{4n-4}$, $R_k(x_k)=1$, $R_k^{(i)}(x_\nu)=0$, $1 \leqq \nu \leqq n$, $\nu \neq k$, $i=0, 1, 2, 3$, which gives

(2.15)
$$R_k(x)=\prod_{\substack{\nu=1\\ \nu\neq k}}^{n} \left(\frac{x-x_\nu}{x_k-x_\nu}\right)^4=(l_k(x))^4.$$

We shall prove that

(2.16) $\quad P_k(x) \geqq g_k(x) \geqq Q_k(x)$ for $-1 \leqq x \leqq 1$, $k=1, 2, \dots n$.

Lemma 3. $P_k(x) \geqq g_k(x)$ *for* $-1 \leqq x \leqq 1$.

Proof. $P_n(x)=1 \geqq g_n(x)$ for $-1 \leqq x \leqq 1$. Let $1 \leqq k \leqq n-1$. We consider the polynomial $P_k'(x)$. Since $P_k \in H_{4n-4}$, we have $P_k' \in H_{4n-5}$. We see from (2.14) that $P_k(x_{\nu+1})=P_k(x_\nu)$, if $1 \leqq \nu \leqq n-1$, $\nu \neq k$. It follows by Rolle's theorem, that there exist points $y_\nu \in (x_{\nu+1}, x_\nu)$ $1 \leqq \nu \leqq n-1$, $\nu \neq k$, such that $P_k'(y_\nu)=0$, $1 \leqq \nu \leqq n-1$, $\nu \neq k$. We see also that P_k' has triple zeros in the points x_ν, $1 \leqq \nu \leqq n$, $\nu \neq k$. Hence we may write

$$P_k'(x)=C\prod_{\substack{\nu=1\\ \nu\neq k}}^{n}(x-x_\nu)^3\prod_{\substack{\nu=1\\ \nu\neq k}}^{n-1}(x-y_\nu).$$

P_k' changes his sign in each point x_ν, y_μ, $1 \leqq \nu \leqq n$, $\nu \neq k$, $1 \leqq \mu \leqq n-1$, $\mu \neq k$ and only in these points, and therefore P_k is alternatively monotonously increasing and monotonously decreasing in subintervals (x_{k+1}, y_{k-1}), $(x_{\nu+1}, y_\nu)$, (y_μ, x_μ) $0 \leqq \nu \leqq n-1$, $\nu \neq k$, $1 \leqq \mu \leqq n$, $\mu \neq k$ $(y_0=1, y_n=-1)$. Such that $P_k(x_{k+1})=0<1$ $=P_k(x_k)$, $P_k(x)$ increases in (x_{k+1}, y_{k-1}) and from (2.14) by induction we obtain $P_k(x) \geqq 1$, $x_k \leqq x \leqq 1$ and $P_k(x) \geqq 0$, $-1 \leqq x \leqq x_k$ which proves the lemma.

To prove the second inequality in (2.16) it is sufficient to see that $Q_k(x) \equiv 1-P_{n+1-k}(-x)$ and $g_k(x) \equiv 1-g_{n+1-k}(-x)$ and to apply Lemma 3 for g_{n+1-k}.

Consider the function $S(x)=\Sigma_{k=1}^n \delta_k g_k(x)$, where δ_k are real numbers. Define the functions $\varphi_k(S; x)$, $-n+1 \leqq k \leqq n$ in the following way:

$$\varphi_k(S;\, x)=\varphi_k(x)=\begin{cases}|\delta_{k+m}| & \text{if} \quad x_m'<x \leqq x_{m-1}', \quad 1 \leqq m \leqq n\\ 0 & \text{if} \quad -3 \leqq x \leqq -1, \quad 1 \leqq x \leqq 3, \quad \text{for } 0 \leqq m \leqq n\end{cases}$$

where $x_m'=\cos(m\pi/n)$, $x_{-m}'=x_{n-m}'+2$, $x_{n+m}'=x_m'-2$ $(\delta_k=0$ *if* $k>n$ or $k<1)$.

Lemma 4. $\tilde{E}_{4n-4}(S)_p \leqq 256\, \|\varphi_0\|_p$, $1 \leqq p \leqq \infty$.

Proof. Denote $P(x):=\Sigma_{\delta_k>0}\delta_k P_k(x)-\Sigma_{\delta_k<0}\delta_k(1-Q_k(x))$; $Q(x):$ $=\Sigma_{\delta_k>0}\delta_k Q_k(x)-\Sigma_{\delta_k<0}\delta^k(1-P_k(x))(1-P_k(x))$. From (2.15) and (2.16) we obtain $P(x) \geqq S(x) \geqq Q(x)$, $-1 \leqq x \leqq 1$, $R(x):=P(x)-Q(x)=\Sigma_{k=1}^n|\delta_k|R_k(x)$ $=\Sigma_{k=1}^n|\delta_k|(l_k(x))^4$. Using Lemma 2 for $x_m'<x<x_{m-1}'$ $(1 \leqq m \leqq n)$, we get

$$R(x)\leqq\sum_{k=1}^{m-2}|\delta_k|(m-k-1)^{-4}+16(|\delta_{m-1}|+|\delta_m|+|\delta_{m+1}|)$$

$$+\sum_{k=m+2}^{n}|\delta_k|(k-1-m)^{-4}\leqq 16(|\delta_m|+\sum_{\substack{k=1\\k\neq m}}^{n}|\delta_k|(m-k)^{-4})$$

$$\leqq 16(\varphi_0(S\,;\,x)+\sum_{\substack{k=-n+1\\k\neq 0}}^{n}k^{-4}\varphi_k(S\,;\,x)).$$

Hence

$$\tilde{E}_{4n-4}(S)_p\leqq\|R\|_p\leqq 16(\|\varphi_0\|_p+\sum_{\substack{k=-n+1\\k\neq 0}}^{n}k^{-4}\|\varphi_k\|_p). \tag{2.17}$$

But

$$\|\varphi_k\|_p=\{\sum_{\nu=1}^{n}(x'_{\nu-1}-x'_\nu)|\delta_{k+\nu}|^p\}^{1/p},$$

and since $|x'_{\nu-1}-x'_\nu|\leqq 5|k||x'_{\nu+k-1}-x'_{\nu+k}|$ if $k\neq 0$ (see [10], Lemma 2), we get

$$\|\varphi_k\|_p\leqq\{\sum_{\nu=1}^{n}5|k||x'_{k+\nu-1}-x'_{k+\nu}||\delta_{k+\nu}|^p\}^{1/p}=(5|k|^{1/p}\{\sum_{\nu=1}^{n}\int_{x'_{k+\nu}}^{x'_{k+\nu-1}}|\varphi_0(x)|^p dx\}^{1/p}$$

$$=(5|k|)^{1/p}\{\int_{x'_{k+n}}^{x'_k}|\varphi_0(x)|^p dx\}^{1/p}\leqq 5|k|\,\|\varphi_0\|_p.$$

Substituting this inequality in (2.17) we obtain

$$\tilde{E}_{4n-4}(S)_p\leqq 16(\|\varphi_0\|_p+5\|\varphi_0\|_p\sum_{\substack{k=-n+1\\k\neq 0}}^{n}\|k|^{-3})\leqq 256\|\varphi_0\|_p$$

which was to be proved.

P r o o f o f T h e o r e m 1. Let $f(x)$ be a bounded and measurable in $[-1, 1]$ function. Set $x_k=\cos((2k-1)\pi/2N)$, $1\leqq k\leqq N$, $x_0=1$, $x_{N+1}=-1$; $x'_k=\cos(k\pi/N)$, $0\leqq k\leqq N$; $N=10n$, and introduce the functins

$$S(x):=\sup\{f(t)\,;\;x_{k+1}\leqq t\leqq x_k\}\quad\text{if}\quad x_{k+1}\leqq x<x_k,\quad k=0,\,1,\ldots\,N,$$

$$J(x):=\inf\{f(t)\,;\;x_{k+1}\leqq t\leqq x_k\}\quad\text{if}\quad x_{k+1}\leqq x<x_k,\quad k=0,\,1,\ldots\,N.$$

Obviously $S(x)\geqq f(x)\geqq J(x)$ for $-1\leqq x\leqq 1$. Let P_1, P_2, Q_1, $Q_2\in H_{4N-4}$ be such that

$$P_1(x)\geqq S(x)\geqq Q_1(x)\,;\quad \tilde{E}_{4N-4}(S)_p=\|P_1-Q_1\|_p$$

$$P_2(x)\geqq J(x)\geqq Q_2(x)\,;\quad \tilde{E}_{4N-4}(J)_p=\|P_2-Q_2\|_p.$$

From Lemma 4,

$$\tilde{E}_{4N-4}(f)_p\leqq\|P_1-Q_2\|_p\leqq\|P_1-S\|_p+\|S-J\|_p+\|J-Q_2\|_p \tag{2.18}$$

$$\leqq\tilde{E}_{4N-4}(S)_p+\|S-J\|_p+\tilde{E}_{4N-4}(J)_p\leqq 256(\|\varphi_0(S\,;\cdot)\|_p+\|\varphi_0(J\,;\cdot)\|_p)+\|S-J\|_p.$$

Let $x'_k\leqq x\leqq x'_{k-1}$, $1\leqq k\leqq N$ and $x=\cos t$, $(k-1)\pi/N\leqq t\leqq k\pi/N$. Clearly, $x_{k-1}-x_{k+1}=\cos((2k-3)\pi/2N)-\cos((2k+1)\pi/2N)=2\sin((2k-1)\pi/2N)\sin\pi/2N$ $\leqq\frac{\pi}{N}(\sin(\frac{2k-1}{2N}\pi-t+t))=\frac{\pi}{N}(\sin(\frac{2k-1}{2N}\pi-t)\cos t+\cos(\frac{2k-1}{2N}\pi-t)\sin t)$ $\leqq\frac{\pi}{N}(\sin\frac{\pi}{2N}+\sin t)\leqq\frac{\pi}{N}\sqrt{1-x^2}+\frac{\pi^2}{2N^2}<\frac{1}{2}\Delta_n(x)$ if $1<k<N$.

If $k=1$ or $k=N$ then $x_{k-1}-x_{k+1}=1-\cos(3\pi/2N)=2\sin^2(3\pi/4N)$ $\leqq 3\pi\sqrt{1-x^2}/2N+9\pi/8N^2\leqq\Delta_n(x)/2$ and hence

$$x+\Delta_n(x)/2\geqq x'_k+\Delta_n(x)/2\geqq x_{k+1}+\Delta_n(x)/2\geqq x_{k-1}$$

$$x-\Delta_n(x)/2\leqq x'_{k-1}-\Delta_n(x)/2\leqq x_{k-1}-\Delta_n(x)/2\leqq x_{k+1}.$$

For $\varphi_0(S\,;\,x)$ we obtain

$$\varphi_0(S\,;\,x)=|S(x_{k+1})-S(x_k)|\leqq\sup\{|f(t)-f(t')|\,;\,x_{k+1}\leqq t,\ t'\leqq x_{k-1}\}$$

$$\leqq\sup\{|f(t)-f(t')|\,;\,x-\Delta_n(x)/2\leqq t,\ t'\leqq x+\Delta_n(x)/2\}=\omega_1(f,\ x\,;\,\Delta_n(x)).$$

Therefore,

$$\|\varphi_0(S\,;\cdot)\|_p\leqq\tau_1(f\,;\,\Delta_n)_p. \tag{2.19}$$

Similarly we get

$$\|\varphi_0(J\,;\,\cdot)\|_p\leqq\tau_1(f\,;\,\Delta_n)_p. \tag{2.20}$$

Further, for $x_{k+1}\leqq x<x_k$,

$$S(x)-J(x)\leqq\sup\{|f(t)-f(t')|\,;\,x_{k+1}\leqq t,\ t'\leqq x_{k-1}\}\leqq\omega_1(f,\ x\,;\,\Delta_n(x)),$$

which implies

$$\|S-J\|_p\leqq\tau_1(f\,;\,\Delta_n)_p. \tag{2.21}$$

We obtain from (2.18)—(2.21) $\tilde{E}_{40n-4}(f)_p \leqq 513\tau_1(f;\ \Delta_n)_p$. Now this estimate and (2.5) give $\tilde{E}_n(f)_p \leqq c\tau_1(f;\ \Delta_n)_p$. The theorem is proved.

3. Steĉkin's type theorem for the best onesided algebraic approximation

In this section we are going to prove Theorem 2. The following result is used ([7]):

Theorem A. *Let* $n \geqq k$. *There exists a constant* $c(k)>0$, *such that* $E_n(f;\ \Delta_n)_p := \inf\{\ \|\Delta_n(f-Q)\|_p;\ Q\in H_n\} \leqq c(k)\ \|f^{(k)}(\Delta_n)^{k+1}\|_p$.

From Theorem 1 and Theorem A we obtain

Corollary 1. *There exists a constant* $c(k)>0$, *such that*

$$\tilde{E}_{n+1}(f)_p \leqq c(k)\ \|(\Delta_n)^k f^{(k)}\|_p \tag{3.1}$$

for each function $f\in W_p^k\ [-1,\ 1]$.

Proof. Let $P\in H_n$ be the polynomial of best algebraic L_p-approximation with weight $\Delta_n(x)$ for the function $f'(x)$, i. e.

$$E_n(f';\ \Delta_n)_p = \|\Delta_n(f'-P)\|_p. \tag{3.2}$$

Consider the function $F(x)=\int_{-1}^{x}(P(t)-f'(t))dt$. Let $Q_1,\ Q_2\in H_n,\ Q_1(x)\geqq F(x) \geqq Q_2(x)$ be the polynomials of best onesided algebraic approximation of $F(x)$. From Theorem 1 we obtain

$$\|Q_1-Q_2\|_p = \tilde{E}_n(F)_p \leqq c\tau_1(F;\ \Delta_n)_p. \tag{3.3}$$

Define the polynomials $R_1,\ R_2\in H_{n+1}$ in the following manner:

$$R_1(x) := -Q_2(x) + \int_{-1}^{x} P(t)dt + f(-1);$$

$$R_2(x) := -Q_1(x) + \int_{-1}^{x} P(t)dt + f(-1).$$

We have

$$R_1(x)-f(x) = -Q_2(x) + \int_{-1}^{x}(P(t)-f'(t))dt = -Q_2(x)+F(x) \geqq 0$$

$$R_2(x)-f(x) = -Q_1(x) + \int_{-1}^{x}(P(t)-f'(t))dt = -Q_1(x)+F(x) \leqq 0.$$

From Theorem A, Theorem 1, (2.3), (3.3) and (3.2) we obtain

$$\tilde{E}_{n+1}(f)_p \leqq \| R_1 - R_2 \|_p = \| Q_1 - Q_2 \|_p \leqq c\tau_1(F;\ \Delta_n)_p \leqq c \| \Delta_n F' \|_p$$

$$= c \| \Delta_n (f' - P) \|_p = cE_n(f';\ \Delta_n)_p \leqq c(k) \| (\Delta_n)^k f^{(k)} \|_p.$$

P r o o f o f T h e o r e m 2. Let $f(x)$ be bounded and measurable in $[-1, 1]$ and $k \geqq 2$. Then (see [7], Theorem 3.1) there exists a function $G_{k,N}$ with the following properties:

$$(3.4) \qquad G_{k,N} \in W_p^k[-1, 1];$$

$$|G_{k,N}(x) - f(x)| \leqq c(k)\omega_k(f,\ x;\ \Delta_N(x))_1 \leqq c(k)\omega_k(f,\ x;\ \Delta_N^{(x)})\infty;$$

$$(3.5) \qquad \| (\Delta_N)^k G_{k,N}^{(k)} \|_p \leqq c(k)\tau_k(f,\ 1;\ \Delta_N)_{1,p}.$$

We have also (see [5], Lemma 8.3)

$$\tilde{E}_n(f)_p \leqq \tilde{E}_n(g)_p + 2\tilde{E}_n(\varphi)_p + 2 \| \varphi \|_p \qquad \text{if } |f(x) - g(x)| \leqq \varphi(x) \text{ i.e.}$$

$$(3.6) \qquad \tilde{E}_n(f)_p \leqq \tilde{E}_n(G_{k,N})_p + 2c(k)\tilde{E}_n(\omega_k(f,.\,;\ \Delta_N(\cdot))_\infty)_p$$

$$+ 2c(k) \| \omega_k(f,.\,;\ \Delta_N(\cdot))_\infty \|_p.$$

We set $N = 10n$, and obtain from (3.1), (3.4), (3.5), (2.6)—(2.8)

$$(3.7) \qquad \tilde{E}_n(G_{k,N})_p \leqq c(k) \| (\Delta_n)^k (G_{k,N}^{(k)}) \|_p \leqq c(k) \| (\Delta_N)^k G_{k,N}^{(k)} \|_p$$

$$\leqq c(k)\tau_k(f,\ 1;\ \Delta_N)_{1,p} \leqq c(k)\tau_k(f,\ 2\Delta_N)_p \leqq c(k)\tau_k(f,\ \Delta_n)_p.$$

It follows from (2.6), (2.8):

$$(3.8) \quad \| \omega_k(f,\cdot\,;\ \Delta_N(\cdot))_\infty \|_p = \tau_k(f,\ 1;\ \Delta_N)_{\infty,p} \leqq \tau_k(f;\ 2\Delta_N)_p \leqq \tau_k(f,\ \Delta_n)_p.$$

To estimate the second term in the right-hand side of (3.8) we use the following property of the function $\Delta_n(x)$ (see (2.5) in [7]).

$$(3.9) \qquad (4\lambda - 2)^{-1}\Delta_n(x) \leqq \Delta_n(y) \leqq (2\lambda + \frac{3}{2})\Delta_n(x)$$

if $|x - y| \leqq \lambda\Delta_n(x)$, $n \geqq 2\lambda$.

We put $g(x) = \omega_k(f,\ x;\ \Delta_N(x))_\infty \geqq 0$. Then

$$\omega_1(g,\ x;\ \Delta_n(x)) = \sup\{|g(t) - g(t')|;\quad x - \Delta_n(x)/2 \leqq t,\ t' \leqq x + \Delta_n(x)/2\}$$

$$\leqq \sup\{g(t);\quad x - \Delta_n(x)/2 \leqq t \leqq x + \Delta_n(x)/2\}$$

$$=\sup\{\sup\{|\Delta_h^k f(t)|;\quad |h|\leqq\Delta_N(t)\};\quad x-\Delta_n(x)/2\leqq t\leqq x+\Delta_n(x)/2\}.$$

Since $|x-t|\leqq\Delta_n(x)/2$, we obtain from (3.9) $\Delta_n(t)\leqq(5/2)\ \Delta_n(x)$ and $t+kh\leqq x+\Delta_n(x)/2+k\Delta_N(t)\leqq x+\Delta_n(x)/2+(k/10)\ \Delta_n(t)\leqq x+\frac{\Delta_n(x)}{2}(1+\frac{k}{2})\leqq x\ +(k/2)\Delta_n(x)$ and also $t+kh\geqq x-(k/2)\Delta_n(x)$. Then $\omega_1(g,x;\Delta_n(x))\ \leqq\sup\{|\Delta_h^k f(t)|;\ x-(k/2)\Delta_n(x)\leqq t,\ t+kh\leqq x+(k/2)\Delta_n(x)\}=\omega_k(f,\ x;\ \Delta_n(x))$.

From this inequality and Theorem 1 it follows:

$$\tilde{E}_n(\omega_k(f,\cdot;\ \Delta_N(\cdot))_\infty)_p\leqq c\tau_1(g;\ \Delta_n)_p\leqq c\tau_k(f;\ \Delta_n)_p. \tag{3.10}$$

Using (3.7), (3.8) and (3.10) in (3.6) we obtain $\tilde{E}_n(f)_p\leqq c(k)\tau_k(f;\ \Delta_n)_p$, which proves the theorem.

We are going to make an application of Theorem 2. Let us denote by K^M the set of all convex functions in $[-1,\ 1]$ such that $\max\{f(x);\ -1\leqq x\leqq 1\}-\min\{f(x);\ -1\leqq x\leqq 1\}\leqq M$ and $\tilde{E}_n(K^M)=\sup\{\tilde{E}_n(f)_p;\ f\in K^M\}$.

Then

$$\tilde{E}_n(f)_p\leqq cn^{-2/p}\omega(f;\ n^{-(p-1)/p})\qquad \text{if } f \text{ is convex} \tag{3.11}$$

$$\tilde{E}_n(K^M)_p\leqq cMn^{-2/p}. \tag{3.12}$$

Inequalities (3.11) and (3.12) are obtained using Theorem 2 in special case $k=2$ and their proofs do not essentially differ from the proofs of analogous inequalities for the best algebraic approximation given in [14]. The order of n is exact in the estimate (3.12).

4. Salem — Stečkin's type converse theorem for the best onesided algebraic approximation

We use the inequality:

$$\|(n\Delta_n)^k Q^{(k)}\|_p\leqq c(k)n_0^k\|Q\|_p \tag{4.1}$$

if $Q\in H_{n_0}$, $n_0\leqq n$.

Inequality (4.1) is a corollary from Potapov's inequality

$$\|(\sqrt{1-x^2}+n^{-1})^{k+\mu}Q^{(k)}(x)\|_p\leqq c(k,\ \mu,\ p)n^k\|(\sqrt{1-x^2}+n^{-1})^\mu Q(x)\|_p$$

which is proved in [11]. See also [12] and [10].

P r o o f o f T h e o r e m 3. Let $P_\nu Q_\nu\in H_\nu$ be such, that $P_\nu(x)\geqq f(x)\geqq Q_\nu(x)$, $-1\leqq x\leqq 1$ and $\|P_\nu-Q_\nu\|_p=\tilde{E}_\nu(f)_p$ for any integer number ν. Let $x\in[-1,\ 1]$ and t, $t+kh\in[x-(k/2)\Delta_n(x),\ x+(k/2)\Delta_n(x)]\cap[-1,\ 1]$. Then

$$|\Delta_h^k f(t)|=\sigma\Delta_h^k f(t)=\sigma\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}f(t+mh)\leqq\sum_{m=0}^{k}{}'\binom{k}{m}P_n(t+mh)$$

$$-\sum_{m=0}^{k}{}''\binom{k}{m}Q_n(t+mh)=\sigma\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}P_n(t+mh)+\sum_{m=0}^{k}{}''\binom{k}{m}(P_n(t+mh)$$

$$-Q_n(t+mh)-P_n\ (x)\ +Q_n(x))+c(k,\ \sigma)(P_n(x)-Q_n(x))\leqq|\Delta_h^k P_n(t)|$$

$$+c(k)\omega_1(P_n-Q_n,\ x;\ k\Delta_n(x))+c(k)(P_n(x)-Q_n(x)).$$

$(\sigma=\pm 1)$, where Σ' is taken on these m for which $\sigma(-1)^{m+k}=1$, and Σ'' is taken on these m for which $\sigma(-1)^{m+n}=-1$.

We get from here $\omega_k(f,\ x;\ \Delta_n(x))\leqq\omega_k(P_n,\ x;\ \Delta_n(x))+c(k)\omega_1(P_n-Q_n,\ x;\ k\Delta_n(x))+c(k)(P_n(x)-Q_n(x))$ and therefore

$$\text{(4.2)}\qquad \tau_k(f,\ \Delta_n)_p\leqq\tau_k(P_n;\ \Delta_n)_p+c(k)\tau_1(P_n-Q_n;\ k\Delta_n)_p+c(k)\|P_n-Q_n\|_p$$

$$\leqq\tau_k(P_n;\ \Delta_n)_p+c(k)\tau_1(P_n-Q_n;\ \Delta_n)_p+c(k)\widetilde{E}_n(f)_p.$$

Using (4.1) and (2.3), we obtain

$$\text{(4.3)}\qquad \tau_1(P_n-Q_n;\ \Delta_n)_p\leqq c\|(P_n-Q_n)'\Delta_n\|_p=cn^{-1}\|(n\Delta_n)(P_n-Q_n)'\|_p$$

$$\leqq c\|P_n-Q_n\|_p=c\widetilde{E}_n(f)_p.$$

Let us first consider the case $n=2^{S_0}$. From (2.1), (2.3) and (4.1) we obtain

$$\text{(4.4)}\qquad \tau_k(P_n;\ \Delta_n)_p\leqq\sum_{S=1}^{S_0}\tau_k(P_{2^S}-P_{2^{S-1}};\ \Delta_n)_p+\tau_k(P_1-P_0;\ \Delta_n)_p$$

$$\leqq c(k)\sum_{S=1}^{S_0}\|(P_{2^S}-P_{2^{S-1}})^{(k)}(\Delta_n)^k\|_p+c(k)\|(\Delta_n)^k(P_1-P_0)^{(k)}\|_p$$

$$=c(k)n^{-k}\Big(\sum_{S=1}^{S_0}\|(n\Delta_n)^k(P_{2^S}-P_{2^{S-1}})^{(k)}\|_p+\|(n\Delta_n)^k(P_1-P_0)^{(k)}\|_p\Big)$$

$$\leqq c(k)n^{-k}\Big(\sum_{S=1}^{S_0}(2^S)^k\|P_{2^S}-P_{2^{S-1}}\|_p+\|P_1-P_0\|_p\Big)$$

$$\leqq c(k)n^{-k}\Big(\sum_{S=1}^{S_0}2^{Sk}(\|P_{2^S}-f\|_p+\|f-P_{2^{S-1}}\|_p)+\|P_1-f\|_p+\|f-P_0\|_p\Big)$$

$$\leqq c(k)n^{-k}\Big(\sum_{S=1}^{S_0}2^{Sk}\widetilde{E}_{2^{S-1}}(f)_p+\widetilde{E}_0(f)_p\Big)=c(k)n^{-k}(\widetilde{E}_0(f)_p+2^k\widetilde{E}_1(f)_p$$

$$+\sum_{S=2}^{S_0} 2^{2k} \sum_{i=2^{S-2}+1}^{2^{S-1}} (2^{S-2})^{k-1} \tilde{E}_{2^{S-1}}(f)_p) \leqq c(k) n^{-k} (\tilde{E}_0(f)_p + 2^k \tilde{E}_1(f)_p$$

$$+\sum_{S=2}^{S_0} 2^{2k} \sum_{i=2^{S-2}+1}^{2^{S-1}} (i+1)^{k-1} \tilde{E}_i(f)_p) \leqq c(k) n^{-k} \sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_i(f)_p.$$

Using (4.3) and (4.4) in (4.2) we get

$$\tau_k(f;\ \Delta_n)_p \leqq c(k)\tilde{E}_n(f)_p + c(k) n^{-k} \sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_i(f)_p \tag{4.5}$$

$$\leqq c(k) n^{-k} \Big(\sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_n(f)_p + \sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_i(f)_p\Big)$$

$$\leqq c(k) n^{-k} \sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_i(f)_p$$

if $n = 2^{S_0}$.

Now let $2^{S_0} < n < 2^{S_0+1}$. Using (4.5) and (2.6) we have

$$\tau_k(f;\ \Delta_n)_p \leqq \tau_k(f;\ \Delta_{2^{S_0}})_p \leqq c(k)\,(2^{S_0})^{-k} \sum_{i=0}^{2^{S_0}} (i+1)^{k-1} \tilde{E}_i(f)_p$$

$$\leqq c(k) \Big(\frac{n}{2}\Big)^{-k} \sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_i(f)_p = c(k) n^{-k} \sum_{i=0}^{n} (i+1)^{k-1} \tilde{E}_i(f)_p$$

which proves the theorem.

Now Theorem 2 and Theorem 3 give

Corollary 2. *If* k *is an integer number, and* $0 < \alpha < k$, *then*

$$\tilde{E}_n(f)_p = O(n^{-\alpha}) \leftrightarrow \tau_k(f;\ \Delta_n)_p = O(n^{-\alpha}).$$

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