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Local Multiplier Criteria in Banach Spaces

R. J. Nessel, E. van Wickeren

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The main purpose of this paper is to establish those local multiplier criteria, already used in our previous work in connection with a number of applications in approximation theory and numerical analysis. To this end we study the Banach algebra $BV_{j+1}(a, b)$ of functions τ for which $\int_{a+}^{b-} t^j|d\tau^{(j)}(t)|$ is finite for some $0 \leqq a < b \leqq \infty$ and integer $j \geqq 0$. Thereby it is essential to follow up the dependence upon the parameters a, b in order to derive uniform bounds in, e. g., Wiener-type results. The local multiplier criteria are then shown on the basis of an appropriate Riesz summability condition. This is indeed done in terms of a concept for multipliers in Banach spaces which not only generalizes, but simplifies the one developed previously. In fact, the present approach turns out to be more readily accessible to applications.

1. Introduction

The main purpose of this paper is to establish those local multiplier criteria, already used in [13], in connection with a number of applications in approximation theory and numerical analysis.

To this end, Section 2 studies the Banach algebra $BV_{j+1}(a, b)$ of functions τ for which for some integer $j \geqq 0$ (cf. (2.2))

$$(1.1) \qquad \int_{a+}^{b-} t^j|d\tau^{(j)}(t)| < \infty.$$

The treatment may be considered as an extension of results, obtained previously (see [3 ; 4 ; 5 ; 14 ; 17 ; 18] and the literature cited there), for the classes $BV_{j+1}(0, \infty)$, now localized to an arbitrary (small) interval (a, b), $0 \leqq a < b \leqq \infty$. Though the weight t^j in the moment (1.1) only has critical points at $a=0$ and $b=\infty$ (for $j \geqq 1$), it is essentially to follow up the dependence upon the parameters a, b to derive uniform bounds, for example, in the Wiener-type result of Corollary 2.12. In this connection, let us emphasize that we are mainly interested in the situation for $j \geqq 1$ since, from the point of view of applications, the case $j=0$ is a trivial one (though it causes some technical difficulties in our approach, cf. Remark 2.7, 3.2). Let us also mention that extensions to fractional values of $j \geqq 0$ may be worked out along lines, developed in [17] for the classes $BV_{j+1}(0, \infty)$.

On the basis of an appropriate Riesz summability condition (cf. (3.5,6)), the results of Section 2 are then used in Section 3 to derive (local) multiplier criteria (cf. Theorem 3.3). This is indeed done in terms of a concept for multipliers in

Banach spaces which not only generalizes, but simplifies the one developed previously (see [5 ; 14 ; 19, p. 115 ff] and the literature cited there). In fact, the present approach is more readily accessible to applications, the latter aspect being illustrated in Section 4.

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2. The Banach Algebra $BV_{j+1}(a, b)$

Let $BV[c, d]$, $0<c<d<\infty$, be the space of complex-valued functions τ with bounded variation $[\text{Var } \tau]_c^d$ over the (closed) interval $[c, d]$, whereas $B(a, b)$, $0\leqq a<b\leqq\infty$, denotes the space of complex-valued functions τ, defined and bounded on the (open) interval (a, b), equipped with norm

$$\|\tau\|_{B(a,b)} := \sup\{|\tau(t)| : a<t<b\}. \tag{2.1}$$

Let N, P, Z, R, and C be the sets of natural, non-negative integral, integral, real, and complex numbers respectively.

For $j\in$ P let $BV_{j+1}(a, b)\subset B(a, b)$ be the set of functions τ which are j-times differentiable on (a, b) with $\tau^{(j)}\in BV_{\text{loc}}(a, b) := \cap_{a<c<d<b} BV[c, d]$ such that the improper (cf. [20, p. 15]) Riemann-Stieltjes integral

$$\int_a^b t^j |d\tau^{(j)}(t)| := \lim_{\substack{c\to a+\\ d\to b-}} \int_c^d t^j |d\tau^{(j)}(t)| < \infty \tag{2.2}$$

exists as a finite number (cf. (1.1)). In the following, $j\in$ P and $0\leqq a<b\leqq\infty$ are assumed to be fixed.

Lemma 2.1: *For $\tau\in BV_{j+1}(a, b)$ the limits*

$$m_p(\tau ; b) := \lim_{t\to b-} t^p \tau^{(p)}(t), \quad 0\leqq p\leqq j \tag{2.3}$$

exist, and $\tau\in BV_{k+1}(a, b)$ for each $0\leqq k\leqq j$, in fact

$$\frac{1}{k!}\int_a^b t^k |d\tau^{(k)}(t)| \leqq \frac{1}{j!}\int_a^b t^j |d\tau^{(j)}(t)| + \sum_{p=k+1}^{j} \frac{1}{p!} |m_p(\tau ; b)|. \tag{2.4}$$

Moreover, if $b=\infty$, then

$$m_p(\tau ; b)=0, \quad 1\leqq p\leqq j. \tag{2.5}$$

Proof: Let $0\leqq a<r<s<b\leqq\infty$. In view of (2.2) and $|\tau^{(j)}(s)-\tau^{(j)}(r)| \leqq \int_r^s |d\tau^{(j)}(t)| \leqq r^{-j} \int_r^b t^j |d\tau^{(j)}(t)|$, the left-hand side tends to zero as $r\to b-$ so that the limit

$$\delta_j(b) := \lim_{t\to b-} \tau^{(j)}(t) \tag{2.6}$$

exists. This already handles the (trivial) case $j=0$.

For the remaining $j\geqq 1$, consider first $b<\infty$. Then (2.6) also shows the existence of $m_j(\tau\,;\,b)=b^j\delta_j(b)$, thus (2.3) for $p=j$. Moreover,

$$(2.7)\qquad \frac{1}{(j-1)!}\int_a^b t^{j-1}\,|d\tau^{(j-1)}(t)|=\frac{1}{(j-1)!}\int_a^b t^{j-1}\,|\tau^{(j)}(t)|\,dt$$

$$\leqq\frac{1}{(j-1)!}\int_a^b t^{j-1}\int_t^b |d\tau^{(j)}(u)|\,dt+\frac{|\delta_j(b)|}{(j-1)!}\int_a^b t^{j-1}\,dt\leqq\frac{1}{j!}\int_a^b u^j\,|d\tau^{(j)}(u)|+\frac{1}{j!}|m_j(\tau\,;\,b)|,$$

thus $\tau\in BV_{k+1}(a,b)$ and (2.4) for $k=j-1$.

If $b=\infty$, then $\delta_j(\infty)=0$ for $j\geqq 1$ (cf. [17, p. 25]), and therefore instead of (2.7)

$$\frac{1}{(j-1)!}\int_a^\infty t^{j-1}\,|d\tau^{(j-1)}(t)|\leqq\frac{1}{j!}\int_a^\infty u^j\,|d\tau^{(j)}(u)|.$$

Furthermore, for any $s>a$, $|s^j\tau^{(j)}(s)|\leqq s^j\int_s^\infty |d\tau^{(j)}(t)|\leqq\int_s^\infty t^j\,|d\tau^{(j)}(t)|$.
In view of (2.2) one concludes $m_j(\tau\,;\,\infty)=0$, hence (2.5) for $p=j$ as well as $\tau\in BV_{k+1}(a,\infty)$ and (2.4) for $k=j-1$.

Now one may proceed iteratively to complete the proof. □

The linear space $BV_{j+1}(a,b)$ may be equipped with the norm

$$(2.8)\qquad \|\tau\|_{BV_{j+1}(a,b)}:=\frac{1}{j!}\int_a^b t^j\,|d\tau^{(j)}(t)|+\sum_{p=0}^{j}\frac{1}{p!}|m_p(\tau\,;\,b)|.$$

Then (2.4) implies $BV_{j+1}(a,b)\subset BV_{k+1}(a,b)$, $0\leqq k\leqq j$, in the sense of continuous embedding, in particular

$$(2.9)\qquad \|\tau\|_{BV_{k+1}(a,b)}\leqq\|\tau\|_{BV_{j+1}(a,b)},\quad 0\leqq k\leqq j.$$

Lemma 2.2: *Let $\tau\in BV_{j+1}(a,b)$, $\sigma\in BV_{k+1}(a,b)$ for some $j,k\in\mathbf{P}$. Then (cf.* (2.1,8))

$$(2.10)\qquad \frac{1}{p!}\|t^p\tau^{(p)}(t)\|_{B(a,b)}\leqq\|\tau\|_{BV_{j+1}(a,b)},\quad 0\leqq p\leqq j,$$

$$(2.11)\qquad \frac{1}{j!\,k!}\int_a^b t^{j+k}\,|d[\tau^{(j)}\sigma^{(k)}](t)|\leqq 2\|\tau\|_{BV_{j+1}(a,b)}\|\sigma\|_{BV_{k+1}(a,b)}.$$

Proof: One has for any $a<t\leqq d<b$

$$t^p\,|\tau^{(p)}(t)|\leqq\int_t^d u^p\,|d\tau^{(p)}(u)|+d^p\,|\tau^{(p)}(d)|,$$

and therefore by (2.2-4) for any $a<t<b$

$$\frac{1}{p!}t^p|\tau^{(p)}(t)|\leqq\frac{1}{p!}\int_a^b u^p|d\tau^{(p)}(u)|+\frac{1}{p!}|m_p(\tau\,;\,b)|\leqq\frac{1}{j!}\int_a^b u^j|d\tau^{(j)}(u)|+\sum_{q=p}^{j}\frac{1}{q!}|m_q(\tau\,;\,b)|,$$

thus (2.10). To show (2.11), let $a<c<d<b$ and $c=t_0<t_1<\ldots<t_n=d$ be a partition of the interval $[c,\,d]$. Then

$$\frac{1}{j!\,k!}\sum_{i=1}^{n}t_{i-1}^{j+k}[\operatorname{Var}\tau^{(j)}\,\sigma^{(k)}]_{t_{i-1}}^{t_i}\leqq\frac{1}{j!}\|t^j\tau^{(j)}(t)\|_{B(a,b)}\frac{1}{k!}\sum_{i=1}^{n}t_{i-1}^{k}[\operatorname{Var}\sigma^{(k)}]_{t_{i-1}}^{t_i}$$

$$+\frac{1}{k!}\|t^k\sigma^{(k)}(t)\|_{B(a,b)}\frac{1}{j!}\sum_{i=1}^{n}t_{i-1}^{j}[\operatorname{Var}\tau^{(j)}]_{t_{i-1}}^{t_i},$$

so that (2.11) follows by (2.10) since Riemann-Stieltjes sums approximate corresponding integrals (cf. (2.2,8)). □

Lemma 2.3: *For any subinterval* $(c,d)\subset(a,b)$ *one has* $BV_{j+1}(a,b)\subset BV_{j+1}(c,d)$ *in the sense of continuous embedding, in fact*

$$\|\tau\|_{BV_{j+1}(c,d)}\leqq(j+2)\,\|\tau\|_{BV_{j+1}(a,b)},$$

$$\|\tau\|_{BV_{j+1}(a,b)}=\lim_{\substack{c\to a+\\ d\to b-}}\|\tau\|_{BV_{j+1}(c,d)}.$$

Proof: Obviously, $BV_{j+1}(a,b)\subset BV_{j+1}(c,d)$. Moreover, in view of (2.10)

$$\frac{1}{p!}m_p(\tau\,;d)\leqq\|\tau\|_{BV_{j+1}(a,b)},\qquad \lim_{d\to b-}m_p(\tau\,;d)=m_p(\tau\,;b)$$

which imply the assertions. □

Lemma 2.4: *Setting* $\tau_\rho(t):=\tau(t/\rho)$ *(Fejér-type) for* $\tau\in BV_{j+1}(a,b)$, $\rho>0$, *one has* $\tau_\rho\in BV_{j+1}(a\rho,b\rho)$, *in fact*

(2.12) $$\|\tau_\rho\|_{BV_{j+1}(a\rho,b\rho)}=\|\tau\|_{BV_{j+1}(a,b)}.$$

Proof: In view of (2.2,3)

$$\int_{a\rho}^{b\rho}t^j|d\tau_\rho^{(j)}(t)|=\int_{a\rho}^{b\rho}\Big(\frac{t}{\rho}\Big)^j\Big|d\tau^{(j)}\Big(\frac{t}{\rho}\Big)\Big|=\int_a^b u^j|d\tau^{(j)}(u)|,$$

$$m_p(\tau_\rho\,;b\rho)=\lim_{t\to b\rho-}t^p\tau_\rho^{(p)}(t)=\lim_{t\to b\rho-}\Big(\frac{t}{\rho}\Big)^p\tau^{(p)}\Big(\frac{t}{\rho}\Big)=m_p(\tau\,;b).\ \square$$

Lemma 2.5: *If $\tau \in BV_{j+1}(0, \infty)$ is such that $\tau^{(j)}$ is continuously differentiable, then τ_ρ as defined in Lemma 2.4 is a continuous function of $\rho > 0$ with respect to the topology of $BV_{j+1}(0, \infty)$, i. e., for any $\rho_0 > 0$*

$$\lim_{\rho \to \rho_0} \| \tau_\rho - \tau_{\rho_0} \|_{BV_{j+1}(0,\infty)} = 0.$$

Proof: In view of (2.12) it is sufficient to consider $\rho_0 = 1$. Since $\tau^{(j+1)}$ is continuous, one has

$$\int_0^\infty t^j |d(\tau_\rho - \tau)^{(j)}(t)| = \int_0^\infty t^j |\rho^{-j-1} \tau^{(j+1)}(t/\rho) - \tau^{(j+1)}(t)| dt$$

$$\leqq |1 - \rho^{-j-1}| \int_0^\infty t^j |\tau^{(j+1)}(t)| dt + \rho^{-j-1} \int_0^\infty t^j |\tau^{(j+1)}(t/\rho) - \tau^{(j+1)}(t)| dt,$$

which tends to zero as $\rho \to 1$ (dominated convergence). Moreover, $m_0(\tau_\rho; \infty) = m_0(\tau; \infty) := \tau(\infty)$ so that the result follows in view of (2.5). □

The classes $BV_{j+1}(a, b)$ are intimately connected with the Riesz factors $(j \in \mathrm{P}, t > 0)$

$$r_{j,t}(u) := \begin{cases} (1 - u/t)^j, & 0 \leqq u \leqq t \\ 0, & u > t. \end{cases} \tag{2.13}$$

Obviously, $r_{j,t} \in BV_{j+1}(0, \infty)$ for any $t > 0$, in fact

$$\| r_{j,t} \|_{BV_{j+1}(0,\infty)} = 1, \quad t > 0. \tag{2.14}$$

Let $C_{00}^\infty[0, \infty)$ be the set of real-valued functions on $[0, \infty)$, arbitrarily often differentiable with compact support. Obviously, $C_{00}^\infty[0, \infty) \subset BV_{j+1}(0, \infty)$ for any $j \in \mathrm{P}$. Let $\theta_k \in C_{00}^\infty[0, \infty)$, $k \in \mathrm{P}$, be such that

$$|\theta_k(u)| \leqq 1, \quad \theta_k(u) = \begin{cases} (1-u)^k, & 0 \leqq u \leqq 1 \\ 0, & u \geqq 2. \end{cases}$$

Then, setting for $0 \leqq u < \infty$

$$\theta_{k,b}(u) := \begin{cases} \theta_k(u/b), & 0 < b < \infty \\ 1(=\theta_k(0)), & b = \infty, \end{cases}$$

one has $\theta_{k,b} \in BV_{j+1}(0, \infty)$ for any $j, k \in \mathrm{P}$, $0 < b \leqq \infty$, in fact (cf. (2.8,12))

$$\| \theta_{k,b} \|_{BV_{j+1}(0,\infty)} = \begin{cases} \| \theta_k \|_{BV_{j+1}}(0, \infty), & 0 < b < \infty \\ 1(= m_0(\theta_k; \infty)), & b = \infty. \end{cases} \tag{2.15}$$

Whereas Lemma 2.3 deals with restrictions to subintervals, the following result provides suitable extensions to $(0, \infty)$.

Theorem 2.6: *Let $\tau \in BV_{j+1}(a, b)$ for some $j \in \mathrm{P}$, $0 \leqq a < b \leqq \infty$. Then τ may be extended to $(0, \infty)$ via*

$$(2.16)\quad \lambda(u) := \frac{(-1)^{j+1}}{j!} \int_a^b r_{j,t}(u) t^j \, d\tau^{(j)}(t) + \sum_{p=0}^{j} \frac{(-1)^p}{p!} m_p(\tau; b) \theta_{p,b}(u)$$

such that $\lambda \in BV_{j+1}(0, \infty)$ with

$$(2.17)\qquad \| \lambda \|_{BV_{j+1}(0,\infty)} \leqq C_j \| \tau \|_{BV_{j+1}(a,b)},$$

$$C_j := \max \{ \| \theta_p \|_{BV_{j+1}(0,\infty)} : 0 \leqq p \leqq j \} \geqq 1.$$

Remark 2.7 : Concerning the Riemann-Stieltjes integral occurring in (2.16), note that

$$(2.18)\qquad \int_a^b r_{j,t}(u) t^j \, d\tau^{(j)}(t) = \begin{cases} \int_a^b (t-u)^j \, d\tau^{(j)}(t), & 0 \leqq u \leqq a \\ \int_u^b (t-u)^j \, d\tau^{(j)}(t), & a < u < b \\ 0, & u \geqq b \end{cases}$$

which in particular interprets the case $j = 0$, where $\tau_{0,t}(u)$ (for fixed u) and $\tau(t)$ may have a common singularity (jump at $t = u$). Let us mention that this incidentally coincides with the (standard extension of the) definition of a Riemann-Stieltjes integral $\int_a^b f(u) dg(u)$, appropriate for cases where f and g have a common singularity of the first kind (cf. [1], pp. 192, 243 ; [6], p. 210, but see also Remark 3.2).

Proof : Obviously, λ is well-defined, bounded, and j-times differentiable on $(0, \infty)$ with $(0 \leqq k \leqq j)$:

$$(2.19)\quad \lambda^{(k)}(u) = \frac{(-1)^{j+1}}{j!} \int_a^b r_{j,t}^{(k)}(u) t^j \, d\tau^{(j)}(t) + \sum_{p=0}^{j} \frac{(-1)^p}{p!} m_p(\tau; b) \theta_{p,b}^{(k)}(u),$$

$$(2.20)\qquad m_k(\lambda; \infty) = \frac{(-1)^{j+1}}{j!} \int_a^b m_k(r_{j,t}; \infty) t^j \, d\tau^{(j)}(t)$$

$$+ \sum_{p=0}^{j} \frac{(-1)^p}{p!} m_p(\tau; b) m_k(\theta_{p,b}; \infty),$$

$$\frac{1}{j!} \int_0^\infty u^j |d\lambda^{(j)}(u)| \leqq \frac{1}{j!} \int_a^b t^j \Big[\frac{1}{j!} \int_0^\infty u^j |dr_{j,t}^{(j)}(u)|\Big] |d\tau^{(j)}(t)| + \sum_{p=0}^{j} \frac{1}{p!} |m_p(\tau; b)| \frac{1}{j!} \int_0^\infty u^j |d\theta_{p,b}^{(j)}(u)|.$$

Since $C_j \geqq \| \theta_k \|_{BV_{j+1}(0,\infty)} \geqq |\theta_k(0)| = 1$ (cf. (2.10)), one has by (2.8, 14, 15) that

$$(2.21)\qquad \|\lambda\|_{BV_{j+1}(0,\infty)} \leqq \sup_{a<t<b} \|r_{j,t}\|_{BV_{j+1}(0,\infty)} \frac{1}{j!} \int_a^b t^j |d\tau^{(j)}(t)|$$

$$+ \sup_{0\leqq p\leqq j} \|\theta_{p,b}\|_{BV_{j+1}(0,\infty)} \sum_{p=0}^{j} \frac{1}{p!} |m_p(\tau;b)| \leqq C_j \|\tau\|_{BV_{j+1}(0,\infty)}.$$

Thus it remains to show that λ and τ coincide on $a<u<b$. Again this is obvious for $j=0$ whereas for $j\geqq 1$ by partial integration (j-fold, cf. (2.18))

$$\lambda(u) = \frac{(-1)^j}{(j-1)!} \int_u^b (t-u)^{j-1}\, d\tau^{(j-1)}(t) + \sum_{p=0}^{j-1} \frac{(-1)^p}{p!} m_p(\tau;b)\theta_{p,b}(u)$$

$$= \ldots = -\int_u^b d\tau(t) + m_0(\tau;b) = \tau(u). \ \Box$$

Note that formula (2.16) not only provides a suitable extension of a given $\tau\in BV_{j+1}(a,b)$ to some $\lambda\in BV_{j+1}(0,\infty)$, but may also be interpreted as a representation of the elements in $BV_{j+1}(a,b)$ in terms of the basic Riesz factors. Indeed, the classes BV_{j+1} are, roughly speaking, generated by the Riesz factors via partial integration.

Let $D_j[0,\infty)$, $j\in \mathsf{P}$, be the set of real-valued, continuous, strictly increasing functions η on $[0,\infty)$ with

$$\eta(0)=0, \quad \lim_{t\to\infty} \eta(t) = \infty,$$

which in case $j\geqq 1$ are assumed to be $(j+1)$-times differentiable on $(\ \infty)$ such that

$$t^p |\eta^{(p+1)}(t)| \leqq C\eta'(t), \quad 0\leqq p\leqq j, \ t>0,$$

$$\lim_{t\to 0+} t\eta'(t) = 0.$$

Let $\eta\ \ ^1$ denote the inverse of η and $\tau\circ\eta$ the composition $\tau(\eta(t))$ of τ and η. With the aid of the classes $D_j[0,\infty)$ one may formulate the following generalization of Lemma 2.4.

Theorem 2.8: *If* $\tau\in BV_{j+1}(a,b)$, $\eta\in D_j[0,\infty)$, *then* $\tau\circ\eta\in BV_{j+1}(\eta^{-1}(a),\eta^{-1}(b))$ *with* (*for* C_j *see* (2.17))

$$(2.22)\qquad \|\tau\circ\eta\|_{BV_{j+1}(\eta^{-1}(a),\eta^{-1}(b))} \leqq (j+2) C_j C_{j,\eta} \|\tau\|_{BV_{j+1}(a,b)},$$

$$1 \leqq C_{j,\eta} := \sup_{0<t<\infty} \|r_{j,t}\circ\eta\|_{BV_{j+1}(0,\infty)} < \infty.$$

Proof : First of all, $C_{j,\eta}$ exists as a finite number since $r_{j,t}\circ\eta\in BV_{j+1}(0,\infty)$, uniformly for $t>0$ (cf. [17], p. 28). Application of Theorem 2.6 to $\theta_{k,b}\in BV_{j+1}(0,\infty)$ at point $\eta(u)$ delivers for $u\in(0,\infty)$

$$(\theta_{k,b}\circ\eta)(u)=\frac{(-1)^{j+1}}{j!}\int_0^\infty (r_{j,t}\circ\eta)(u)\,t^j\,d\theta_{k,b}^{(j)}(t)+\sum_{p=0}^{j}\frac{(-1)^p}{p!}m_p(\theta_{k,b};\infty).$$

Moreover, following the proof for the estimate (2.17) (see (2.19-21)) one again obtains $\theta_{k,b}\circ\eta\in BV_{j+1}(0,\infty)$ with (cf. (2.15, 21))

$$\|\theta_{k,b}\circ\eta\|_{BV_{j+1}(0,\infty)}\leqq C_{j,\eta}\|\theta_{k,b}\|_{BV_{j+1}(0,\infty)}\leqq C_{j,\eta}C_j.$$

Using this, the preceding arguments as applied to $\tau\in BV_{j+1}(a,b)$ show that the function

$$\lambda(u):=\frac{(-1)^{j+1}}{j!}\int_a^b (r_{j,t}\circ\eta)(u)t^j\,d\tau^{(j)}(t)+\sum_{p=0}^{j}\frac{(-1)^p}{p!}m_p(\tau;b)(\theta_{p,b}\circ\eta)(u)$$

coincides with $\tau\circ\eta$ on $\eta^{-1}(a)<u<\eta^{-1}(b)$ and belongs to $BV_{j+1}(0,\infty)$ with

$$\|\lambda\|_{BV_{j+1}}(0,\infty)\leqq C_jC_{j,\eta}\|\tau\|_{BV_{j+1}(a,b)}.$$

In view of Lemma 2.3 this completes the proof. □

Corollary 2.9: *Setting* $\tau_{\eta,\rho}(t):=\tau(\eta(t)/\rho)$ *(Hardy-type) for* $\tau\in BV_{j+1}(a,b)$, $\eta\in D_j[0,\infty)$, $\rho>0$, *one has* $\tau_{\eta,\rho}\in BV_{j+1}(a_\rho,b_\rho)$ *with* $a_\rho:=\eta^{-1}(a\rho)$, $b_\rho:=\eta^{-1}(b\rho)$ *and, uniformly for* $\rho>0$,

$$\|\tau_{\eta,\rho}\|_{BV_{j+1}(a_\rho,b_\rho)}\leqq(j+2)C_jC_{j,\eta}\|\tau\|_{BV_{j+1}(a,b)}.$$

Proof : In view of (2.22)

$$\|\tau_{\eta,\rho}\|_{BV_{j+1}(a_\rho,b_\rho)}\leqq(j+2)C_jC_{j,\eta}\|\tau(t/\rho)\|_{BV_{j+1}(a\rho,b\rho)}$$

so that the result follows by Lemma 2.4. □

Theorem 2.10: $BV_{j+1}(a,b)$ *is a commutative Banach algebra with unit under the norm (cf. (2.8))*

$$\frac{(j+1)(j+6)}{2}\|\tau\|_{BV_{j+1}(a,b)} \tag{2.23}$$

and the natural pointwise algebra operations.

Proof : Note that (2.23) defines a norm on $BV_{j+1}(a,b)$. In view of Leibniz's rule one has for $\sigma, \tau \in BV_{j+1}(a,b)$ that (cf. (2.9, 11))

$$\frac{1}{j!}\int_a^b t^j |d(\sigma\tau)^{(j)}(t)| \leqq \sum_{k=0}^{j} \frac{1}{k!(j-k)!}\int_a^b t^j |d(\sigma^{(k)}\tau^{(j-k)})(t)|$$

$$\leqq 2(j+1)\|\sigma\|_{BV_{j+1}(a,b)}\|\tau\|_{BV_{j+1}(a,b)},$$

thus $\sigma\tau \in BV_{j+1}(a,b)$. Moreover, in view of (2.10) for any $a<t<b$, $0 \leqq p \leqq j$

$$\frac{1}{p!}t^p|(\sigma\tau)^{(p)}(t)| \leqq \sum_{k=0}^{p}\frac{1}{k!}|t^k\sigma^{(k)}(t)|\frac{1}{(p-k)!}|t^{p-k}\tau^{(p-k)}(t)|$$

$$\leqq (p+1)\|\sigma\|_{BV_{j+1}(a,b)}\|\tau\|_{BV_{j+1}(a,b)}$$

so that finally (cf. (2.3,8))

$$\|\sigma\tau\|_{BV_{j+1}(a,b)} \leqq \frac{(j+1)(j+6)}{2}\|\sigma\|_{BV_{j+1}(a,b)}\|\tau\|_{BV_{j+1}(a,b)}. \tag{2.24}$$

Hence $BV_{j+1}(a,b)$ is a commutative normed algebra under (2.23) with unit $e(t)=1$ for $t \in (a,b)$ satisfying $\|e\|_{BV_{j+1}(a,b)}=1$.

To establish the completeness of the space, consider a Cauchy sequence $\{\tau_n\}$ in $BV_{j+1}(a,b)$. Introducing the Banach space $B_j \subset BV_{\text{loc}}(a,b)$ of all functions σ for which (cf. (2.2))

$$\|\sigma\|_{B_j} := \frac{1}{j!}\int_a^b t^j|d\sigma(t)| + \frac{1}{j!}\|t^j\sigma(t)\|_{B(a,b)} \tag{2.25}$$

is finite, it follows that $\{\tau_n^{(j)}\}$ is a Cauchy sequence in B_j since $\|\tau_n^{(j)}\|_{B_j} \leqq 2\|\tau_n\|_{BV_{j+1}(a,b)}$ of (cf. (2.10)). Since B_j is complete, there exists $\sigma \in B_j$ such that

$$\lim_{n\to\infty}\|\tau_n^{(j)} - \sigma\|_{B_j} = 0. \tag{2.26}$$

Moreover, since $\{m_p(\tau_n; b)\}$ is a Cauchy sequence in C for each $0 \leqq p \leqq j$, there exists $\mu_p \in$ C such that

$$\lim_{n\to\infty} m_p(\tau_n; b) = \mu_p, \quad 0 \leqq p \leqq j. \tag{2.27}$$

In view of $|t^j\sigma(t) - \mu_j| \leqq |t^j[\sigma(t) - \tau_n^{(j)}(t)]| + |t^j\tau_n^{(j)}(t) - m_j(\tau_n; b)| + |m_j(\tau_n; b) - \mu_j|$, one therefore has (cf. (2.3, 25–27))

$$(2.28) \qquad \lim_{t \to b-} t^j \sigma(t) = \mu_j.$$

Following once again the proof of Theorem 2.6 it results that the function τ, well-defined via (cf. (2.16))

$$\tau(u) := \frac{(-1)^{j+1}}{j!} \int_a^b r_{j,t}(u) t^j \, d\sigma(t) + \sum_{p=0}^{j} \frac{(-1)^p}{p!} \mu_p \theta_{p,b}(u),$$

belongs to $BV_{j+1}(0,\infty) \subset BV_{j+1}(a,b)$ (cf. Lemma 2.3) satisfying (cf. (2.21))

$$\|\tau\|_{BV_{j+1}(0,\infty)} \leqq C_j \Big[\frac{1}{j!} \int_a^b t^j |d\sigma(t)| + \sum_{p=0}^{j} \frac{1}{p!} |\mu_p|\Big]$$

as well as (cf. (2.18, 19, 28), with the obvious interpretation for $b=\infty$)

$$\tau^{(j)}(u) = \frac{(-1)^{j+1}}{j!} \int_a^b r_{j,t}^{(j)}(u) t^j \, d\sigma(t) + \sum_{p=0}^{j} \frac{(-1)^p}{p!} \mu_p \theta_{p,b}^{(j)}(u)$$

$$= -\int_u^b d\sigma(t) + \mu_j b^{-j} = -\sigma(b-) + \sigma(u) + \mu_j b^{-j} = \sigma(u)$$

for $a<u<b$ and $m_p(\tau; b) = \mu_p$ for $0 \leqq p \leqq j$ (cf. (2.18, 20)). Thus, in view of (2.25–27), the sequence $\{\tau_n\}$ converges to τ in the $BV_{j+1}(a,b)$-topology. □

For the algebra $BV_{j+1}(a,b)$ we need the following result of Wiener–Levy-type (cf. [11], p. 210) which considers compositions of the kind $F \circ \tau$ instead of $\tau \circ \eta$ of Theorem 2.8.

Theorem 2.11: *Let F be holomorphic on some open $G \subset \mathbb{C}$, and let $K \subset G$ be compact. If $\tau \in BV_{j+1}(a,b)$ is such that $\tau((a,b)) \subset K$, then $F \circ \tau \in BV_{j+1}(a,b)$, and there exists some constant $A>0$, independent of τ, a, b, such that*

$$(2.29) \quad \|F \circ \tau\|_{BV_{j+1}(a,b)} \leqq A\,[\|\tau\|_{BV_1(a,b)} + \max_{0 \leqq m \leqq j} |F^{(m)}(\tau(b-))|]\,[1 + \|\tau\|^j_{BV_{j+1}(a,b)}].$$

Proof: In view of the hypotheses there exists a constant $M \geqq 1$ such that for all $0 \leqq m \leqq j$, $z_1, z_2 \in K$

$$(2.30) \qquad |F^{(m)}(z_2) - F^{(m)}(z_1)| \leqq M |z_2 - z_1|.$$

To show (2.29) for $j=0$, let $a<c<d<b$ and $c=t_0<t_1<\ldots<t_n=d$ be a partition of the interval $[c,d]$. Then by (2.30)

$$\sum_{k=1}^{n} |(F \circ \tau)(t_k) - (F \circ \tau)(t_{k-1})| \leqq M \sum_{k=1}^{n} |\tau(t_k) - \tau(t_{k-1})| \leqq M \int_a^b |d\tau(t)|$$

so that $F\circ\tau\in BV_1(a,b)$ with

$$\|F\circ\tau\|_{BV_1(a,b)}\leqq M\int_a^b|d\tau(t)|+|m_0(F\circ\tau\,;b)|\leqq M\|\tau\|_{BV_1(a,b)}+|F(\tau(b-))|,$$

establishing (2.29) for $j=0$. If $j\geqq 1$, then again $F^{(m)}\circ\tau\in BV_1(a,b)$ for each $0\leqq m\leqq j$ (cf. (2.9, 30)) and

$$\|F^{(m)}\circ\tau\|_{BV_1(a,b)}\leqq M\|\tau\|_{BV_1(a,b)}+|F^{(m)}(\tau(b-))|. \tag{2.31}$$

Since $F\circ\tau$ is j-times differentiable on (a,b), one has (cf. [9, p. 45])

$$(F\circ\tau)^{(p)}=\sum_{m=0}^{p}\frac{F^{(m)}\circ\tau}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\tau^k(\tau^{m-k})^{(p)},\quad 1\leqq p\leqq j,$$

thus, in particular, $(F\circ\tau)^{(j)}\in BV_{\mathrm{loc}}(a,b)$. Consider for $0\leqq k\leqq m\leqq j$

$$I_{k,m}:=\frac{1}{j!}\int_a^b t^j|d[(F^{(m)}\circ\tau)\tau^k(\tau^{m-k})^{(j)}](t)|.$$

In view of (2.9) and Theorem 2.10 one has $(F^{(m)}\circ\tau)\tau^k\in BV_1(a,b)$, $\tau^{m-k}\in BV_{j+1}(a,b)$ and, using (2.24, 31),

$$\|(F^{(m)}\circ\tau)\tau^k\|_{BV_1(a,b)}\leqq 3^k[M\|\tau\|_{BV_1(a,b)}+F^{(m)}(\tau(b-))|]\,\|\tau\|^k_{BV_1(a,b)},$$

$$\|\tau^{m-k}\|_{BV_{j+1}(a,b)}\leqq\Big[\frac{(j+1)(j+6)}{2}\Big]^{m-k-1}\|\tau\|^{m-k}_{BV_{j+1}(a,b)}.$$

Therefore by (2.9,11) for all $0\leqq k\leqq m\leqq j$

$$\begin{aligned}I_{k,m}&\leqq 2\cdot 3^k[M\|\tau\|_{BV_1(a,b)}+|F^{(m)}(\tau(b-))|]\,\|\tau\|^k_{BV_1(a,b)}\cdot\\&\quad\cdot\Big[\frac{(j+1)(j+6)}{2}\Big]^{m-k-1}\|\tau\|^{m-k}_{BV_{j+1}(a,b)}\\&\leqq A_1[\|\tau\|_{BV_1(a,b)}+\max_{0\leqq m\leqq j}|F^{(m)}(\tau(b-))|][1+\|\tau\|^j_{BV_{j+1}(a,b)}].\end{aligned}$$

Analogously, one has by (2.10) for each $0\leqq p\leqq j$

$$\frac{1}{p!}t^p|[(F^{(m)}\circ\tau)\tau^k(\tau^{m-k})^{(p)}](t)|\leqq\|(F^{(m)}\circ\tau)\tau^k\|_{BV_1(a,b)}\|\tau^{m-k}\|_{BV_{j+1}(a,b)}$$

$$\leqq \frac{A_1}{2}[\|\tau\|_{BV_1(a,b)} + \max_{0\leqq m\leqq j} |F^{(m)}(\tau(b-))|][1+\|\tau\|^j_{BV_{j+1}(a,b)}]$$

from which the result follows. □

Corollary 2.12: *For a parameter ρ varying over some set J, let $0\leqq a_\rho < b_\rho \leqq \infty$ and $\tau_\rho \in BV_{j+1}(a_\rho, b_\rho)$ be such that*

$$C_1 := \sup_{\rho\in J} \|\tau_\rho\|_{BV_{j+1}(a_\rho, b_\rho)} < \infty, \tag{2.32}$$

$$C_2 := \inf\{|\tau_\rho(t)| : a_\rho < t < b_\rho, \rho\in J\} > 0. \tag{2.33}$$

Then $1/\tau_\rho \in BV_{j+1}(a_\rho, b_\rho)$ and

$$\sup_{\rho\in J} \|1/\tau_\rho\|_{BV_{j+1}(a_\rho, b_\rho)} < \infty. \tag{2.34}$$

Proof: Consider $F(z)=1/z$, $G=\mathbb{C}\setminus\{0\}$, $K=\{z\in\mathbb{C} : C_2\leqq|z|\leqq C_1\}$ (note that $C_2\leqq C_1$). Then $\tau_\rho((a_\rho, b_\rho))\subset K$ by (2.10, 32, 33) so that the result follows by an immediate application of Theorem 2.11. □

For the Banach algebra $BV_{j+1}(a,b)$ one may work out further details (maximal ideal space, etc.), envisaged by abstract harmonic analysis. Here, however, we confined ourselves to those facts, used in [13] and needed for the following approach to a general multiplier theory.

3. Multipliers in Banach Spaces and Riesz Summability

For a measure space on Ω let $L^\infty = L^\infty(\Omega)$ be the Banach algebra of complex-valued functions, essentially bounded on Ω with essential bound $\|\cdot\|_\infty$. Let X be a Banach space (with norm $\|\cdot\|_X$) and $[X]$ be the space of bounded linear operators of X into itself.

A subalgebra $M=M(X,\Omega)\subset L^\infty(\Omega)$ with unit $e(x)=1$ for all $x\in\Omega$ is called a multiplier space for X if there exists an injective homomorphism $T: M\to[X]$ such that $T(e)=I$, the identity operator, and T is closed in the sense that for $\mu_n\in M$, $\nu\in L^\infty$, and $S\in[X]$ the relations

$$\sup_{n\in\mathbb{N}} \|\mu_n\|_\infty < \infty, \quad \lim_{n\to\infty} \mu_n(x)=\nu(x) \text{ (a.e.)}, \tag{3.1}$$

$$\lim_{n\to\infty} \|T(\mu_n)f - Sf\|_X = 0, \quad f\in X \tag{3.2}$$

imply

$$\nu\in M \text{ with } T(\nu)=S. \tag{3.3}$$

Then M becomes a normed algebra under the norm

$$\|\mu\|_M := \|T(\mu)\|_{[X]}. \tag{3.4}$$

In this framework functions of $BV_{j+1}(a,b)$ give raise to multipliers on X if X satisfies the following Riesz summability condition: Let $j\in\mathsf{P}$ and a function $\psi:\Omega\to(0,\infty)$ be given. Then X is called R^j_ψ-bounded if (cf. (2.13)) $r_{j,t}\circ\psi\in M$ such that $(t>0)$

$$\sup_{t>0}\|r_{j,t}\circ\psi\|_M =: A_j<\infty, \tag{3.5}$$

$$T(r_{j,t}\circ\psi)f \text{ is } (X\text{-}) \text{ continuous in } t>0 \text{ for each } f\in X \tag{3.6}$$

((3.6) should be added in [7; 14; 15], cf. [5] and Remark 3.2).

Theorem 3.1: *Let X be R^j_ψ-bounded and $\tau\in BV_{j+1}(0,\infty)$. Then $\tau\circ\psi\in M$ such that*

$$T(\tau\circ\psi)f=\frac{(-1)^{j+1}}{j!}\int_0^\infty T(r_{j,t}\circ\psi)ft^j\,d\tau^{(j)}(t)+\tau(\infty)f,\quad f\in X, \tag{3.7}$$

$$\|\tau\circ\psi\|_M\leqq A_j\|\tau\|_{BV_{j+1}(0,\infty)}. \tag{3.8}$$

Proof: For each $x\in\Omega$ one has the representation (cf. (2.2, 16))

$$(\tau\circ\psi)(x)=\frac{(-1)^{j+1}}{j!}\int_0^\infty (r_{j,t}\circ\psi)(x)t^j\,d\tau^{(j)}(t)+\tau(\infty)$$

$$=\lim_{n\to\infty}\Big[\frac{(-1)^{j+1}}{j!}\sigma_{1/n,n}(x)+\tau(\infty)\Big]=:\lim_{n\to\infty}v_n(x),$$

$$\sigma_{c,d}(x):=\int_c^d (r_{j,t}\circ\psi)(x)t^j\,d\tau^{(j)}(t),\quad 0<c<d<\infty.$$

In view of (3.6) the vector-valued Riemann-Stieltjes integral

$$S_{c,d}f:=\int_c^d T(r_{j,t}\circ\psi)ft^j\,d\tau^{(j)}(t)$$

is well-defined for each $f\in X$. Let a sequence $\{t_{kn}\}$ of partitions $c=t_{on}<\ldots<t_{nn}=d$ be given such that $\max_{1\leqq k\leqq n}(t_{kn}-t_{k-1,n})$ tends to zero. Then by the definition of $S_{c,d}$

$$\lim_{n\to\infty}\|T(\mu_n)f-S_{c,d}f\|_X=0,\quad f\in X,$$

(3.9) $$\mu_n(x) := \sum_{k=1}^{n} (r_{j,t_{kn}} \circ \psi)(x) t_{kn}^j [\tau^{(j)}(t_{kn}) - \tau^{(j)}(t_{k-1,n})].$$

Moreover, if $j \geqq 1$, $\mu_n(x)$ converges to $\sigma_{c,d}(x)$ for each $x \in \Omega$ satisfying

$$\| \mu_n \|_\infty \leqq \int_c^d t^j |d\tau^{(j)}(t)|.$$

Thus (3.3) delivers

(3.10) $$\sigma_{c,d} \in M \text{ with } T(\sigma_{c,d}) = S_{c,d}.$$

In case $j=0$ the integral $\sigma_{c,d}(x)$ need not be the (pointwise) limit of $\mu_n(x)$ (note that the sequence $\{t_{kn}\}$ of partitions is given in connection with the integral $S_{c,d}$, independent of x, cf. Remark 2.7). Therefore, to establish (3.10), one has to argue separately. First consider the (right-hand continuous) Riesz factor $(u, t > 0)$

$$r_{0,t}^+(u) := \begin{cases} 1, & u < t \\ 0, & u \geqq t. \end{cases}$$

Since $\lim_{s \to t-} r_{0,s}(u) = r_{0,t}^+(u)$, one has by (3.1—3,6) that

(3.11) $$r_{0,t}^+ \circ \psi \in M \text{ with } T(r_{0,t}^+ \circ \psi) = T(r_{0,t} \circ \psi).$$

Thus (3.5,6) are fulfilled for $r_{0,t}^+ \circ \psi$, too.

Let $J \subset [c,d]$ be the set of (jump) discontinuities of $\tau \in BV_1(0,\infty)$ in $[c,d]$. Following [6, p. 224], consider the corresponding step-function

$$\tau_s(u) := \sum_{\substack{c \leqq t < u \\ t \in J}} [\tau(t+) - \tau(t)] + \sum_{\substack{c < t \leqq u \\ t \in J}} [\tau(t) - \tau(t-)],$$

so that $\tau_c := \tau - \tau_S$ is continuous on $[c,d]$. Then (cf. (2.18) for $j=0$)

$$\sigma_{c,d}(x) = \zeta_1(x) + \zeta_2(x), \quad \zeta_1(x) := \int_c^d (r_{0,t} \circ \psi)(x) d\tau_c(t),$$

$$\zeta_2(x) := \sum_{\substack{c \leqq t < d \\ t \in J}} (r_{0,t} \circ \psi)(x)[\tau(t+) - \tau(t)] + \sum_{\substack{c < t \leqq d \\ t \in J}} (r_{0,t}^+ \circ \psi)(x)[\tau(t) - \tau(t-)].$$

Since τ_c is continuous, the integral is again the limit of the Riemann-Stieltjes sums (cf. (3.9))

$$\tilde{\mu}_n(x) := \sum_{k=1}^{n} (r_{0,t_{kn}} \circ \psi)(x)\,[\tau_c(t_{kn}) - \tau_c(t_{k-1,n})]$$

so that similarly as above $\zeta_1 \in M$ with (cf. (3.10))

$$T(\zeta_1)f = \int_c^d T(r_{0,t} \circ \psi) f\, d\tau_c(t).$$

Moreover, ζ_2 is an absolutely and uniformly convergent series so that (cf. (3.1 – 3, 11))

$$T(\zeta_2)f = \sum_{\substack{c \leq t < d \\ t \in J}} T(r_{0,t} \circ \psi) f[\tau(t+) - \tau(t)] + \sum_{\substack{c < t \leq d \\ t \in J}} T(r_{0,t}^{+} \circ \psi) f[\tau(t) - \tau(t-)],$$

where the right-hand side exists since it is absolutely convergent in X and X is complete. Therefore $\sigma_{c,d} \in M$ with (cf. (3.6) and [6], p. 225)

$$T(\sigma_{c,d})f = \int_c^d T(r_{0,t} \circ \psi) f\, d\tau(t).$$

Summarizing, since v_n satisfies (3.1) and for each $f \in X$ (cf. (3.5))

$$Sf := \frac{(-1)^{j+1}}{j!} \int_0^{\infty} T(r_{j,t} \circ \psi) f t^j\, d\tau^{(j)}(t) + \tau(\infty) f = (X-) \lim_{n \to \infty} T(v_n) f,$$

$$\| Sf \|_X \leq A_j \, \| \tau \|_{BV_{j+1}}(0, \infty) \, \| f \|_X,$$

the assertions (3.7, 8) now follow in view of (3.3). □

Remark 3.2 : Note that in case $j=0$ the strong continuity condition (3.6) ensures not only the existence of the integral $T(\zeta_1)f$ but also $r_{0,t}^{+} \circ \psi \in M$. But for discrete biorthogonal systems (cf. Section 4.1) condition (3.6) fails. However, if one requires (3.5) also for $r_{0,t}^{+} \circ \psi$ and substitutes (3.6) by the condition that $T(r_{0,t} \circ \psi)\, f$ is of bounded variation (cf. [10, p. 63]), then assertion (3.8) is still preserved. On the other hand, if $S_{c,d} f$ is understood as a Bochner integral, it is even sufficient to postulate that $T(r_{0,t} \circ \psi) f$ is merely (Lebesgue-) measurable (cf. [5 ; 19, p. 115]). Here we preferred a Riemann approach to the integrals, allowing a pointwise interpretation of the multipliers at each $x \in \Omega$.

Replacing the (global) interval $(0, \infty)$ by a (local) one (a, b), one finally arrives at the following local version of Theorem 3.1 which was essentially used in [13].

Theorem 3.3: *Let X be R_ψ^j-bounded and τ a complex-valued function, defined on $(0, \infty)$, such that $\tau \in BV_{j+1}(a, b)$ for some $0 \leq a < b \leq \infty$. Then for each $\sigma \in M$ satisfying*

$$(3.12) \qquad \sigma(x)=0 \quad \textit{for } x\in\Omega \ (a.\ e.) \textit{ with } \psi(x)\notin(a,b)$$

one has $\sigma(\tau\circ\psi)\in M$, *in fact* (*for* A_j, C_j *see* (2.17), (3.5))

$$(3.13) \qquad \|\sigma(\tau\circ\psi)\|_M \leqq A_j C_j \|\sigma\|_M \|\tau\|_{BV_{j+1}(a,b)}.$$

Proof: In view of Theorem 2.6 the function τ has an extension λ in $BV_{j+1}(0,\infty)$. Thus $\lambda\circ\psi\in M$ by Theorem 3.1 such that (cf. (3.12))

$$\sigma(x)(\lambda\circ\psi)(x)=\sigma(x)(\tau\circ\psi)(x) \qquad \text{(a. e.)}.$$

Thus $\sigma(\tau\circ\psi)\in M$. To establish (3.13) apply (2.17), (3.8) to derive

$$\|\sigma(\tau\circ\psi)\|_M \leqq \|\sigma\|_M \|\lambda\circ\psi\|_M \leqq A_j \|\sigma\|_M \|\lambda\|_{BV_{j+1}(0,\infty)} \leqq A_j C_j \|\sigma\|_M \|\tau\|_{BV_{j+1}(a,b)}. \Box$$

4. Particular Multiplier Spaces

In this section concrete instances of multiplier spaces are studied to illustrate the direct approach of Section 3. In particular, the abstract definition of multipliers via spectral measures as developed in [5] is shown to be subsumed under the present frame.

4.1. Biorthogonal Systems. Let X be a Banach space with dual space X^*, and let $\Omega=\mathsf{K}$ be an infinite countable index set. Consider a total biorthogonal system $\{f_k\}_{k\in\mathsf{K}}\subset X$, $\{f_k^*\}_{k\in\mathsf{K}}\subset X^*$, i. e., $f_k^*(f_j)=\delta_{jk}$ (Kronecker symbol), and $f_k^*(f)=0$ for all $k\in\mathsf{K}$ implies $f=0$. In this situation the usual procedure is that a bounded sequence $\mu=\{\mu(k)\}_{k\in\mathsf{K}}$ is called a multiplier if for each $f\in X$ there exists $f^\mu\in X$ such that $f_k^*(f^\mu)=\mu(k)f_k^*(f)$ for all $k\in\mathsf{K}$. But then $T(\mu)f:=f^\mu$ is a well-defined operator of $[X]$ such that T is an injective homomorphism of M into $[X]$. Moreover, T is closed in the sense that (3.1,2) imply

$$f_k^*(Sf)=\lim_{n\to\infty} f_k^*(T(\mu_n)f)=\lim_{n\to\infty}\mu_n(k)f_k^*(f)=\nu(k)f_k^*(f).$$

Concerning the Riesz summability condition let us just mention the many results known for, e. g., $\psi(x)=|x|$ in connection with Fourier expansions into (multivariate) trigonometric, Laguerre, Hermite, or Jacobi polynomials, respectively (cf. [17], p. 74 and the literature cited there).

4.2. Multipliers in Connection with Spectral Measures. Let Σ be a σ-field over a set Ω (e. g., $\Omega=\mathsf{R}^n$, the Euclidean n-space, and Σ, the family of Borel sets). For a Hilbert space H let E be a (countably additive, selfadjoint, bounded linear) spectral measure on Ω, and let $L^\infty(\Omega,E)$ be the space of ($E-$) essentially bounded functions μ so that the integral

$$T_H(\mu):=\int_\Omega \mu(x)\,dE(x)$$

is well-defined as an element in $[H]$ (for basic properties and further details see [8], p. 900). Then $M = L^{\infty}(\Omega, E)$ is a multiplier space for H (with $\|\mu\|_M = \|\mu\|_{\infty}$) as well as R_{ψ}^{j}-bounded for any ψ and j since (3.1 — 3) as well as (3.6) follow by dominated convergence (cf. [8], p. 901).

In [5] this definition of multipliers was extended to Banach spaces X for which H and X are continuously embedded in some linear Hausdorff space Y (this hypothesis should be added in [5; 14; 15], see [13]; [19], p. 116) and for which $H \cap X$ is dense in H and X, i. e.,

$$\overline{H \cap X}^{\|\circ\|_H} = H, \quad \overline{H \cap X}^{\|\circ\|_X} = X. \tag{4.1}$$

Then $\mu \in L^{\infty}(\Omega, E)$ is called a multiplier on X if for each $f \in H \cap X$

$$T_H(\mu) f \in H \cap X, \quad \|T_H(\mu) f\|_X \leqq C \|f\|_X. \tag{4.2}$$

In view of (4.1,2) the $(X-)$ closure $T_X(\mu)$ of $T_H(\mu)$ belongs to $[X]$, defining an injective homomorphism T_X such that (cf. (3.4))

$$\|\mu\|_M = \|T_X(\mu)\|_{[X]} = \sup\{\|T_H(\mu) f\|_X : f \in H \cap X, \ \|f\|_X \leqq 1\}.$$

To establish (3.3) let (3.1,2) be valid and $f \in H \cap X$. Then $T_X(\mu_n) f$ converges to $T_H(v) f$ in H by (3.1) and to Sf in X by (3.2) so that (3.3) is fulfilled since $H, X \subset Y$ continuously. Of course, the Riesz summability condition for X is now nontrivial.

As an application (apart from Section 4.1), let us consider the (multivariate, trigonometric) Fourier spectral measure over $\Omega = \mathrm{R}^n$. To this end, let $L^p = L^p(\mathrm{R}^n)$ denote the space of Lebesgue measurable functions f for which the norm

$$\|f\|_p := \begin{cases} \{(2\pi)^{-n/2} \int_{\mathrm{R}^n} |f(u)|^p \, du\}^{1/p}, & 1 \leqq p < \infty \\ \operatorname{ess\,sup}_{u \in \mathrm{R}^n} |f(u)|, & p = \infty, \end{cases} \tag{4.3}$$

respectively, is finite. Let $\mathscr{F}$ be the Fourier-Plancherel transform (cf. (4.4)) on the Hilbert space $H = L^2$ and $\mathscr{F}^{-1}$ the inverse transform. For a Borel measurable set $B \subset \mathrm{R}^n$ let $\mathscr{P}_B$ be the multiplication projection : $(\mathscr{P}_B f)(x) := f(x)$ for $x \in B$ and $:= 0$ for $x \notin B$. Then $E(B) := \mathscr{F}^{-1} \mathscr{P}_B \mathscr{F}$ is a spectral measure for $H = L^2$ (cf. [8, p. 1989]). Furthermore, for the spaces $X = L^p$, $1 \leqq p < \infty$, condition (4.1) is satisfied, and (4.2) coincides with the classical definition of Fourier multipliers (cf. Section 4.3). Concerning a Riesz summability condition see (4.6).

4.3. Fourier Multipliers. The approach of Section 3 is flexible enough to incorporate Fourier multipliers more directly. Indeed, let $\mathscr{S}$ denote the Schwartzian space of infinitely differentiable functions, rapidly decreasing at infinity, and

$$\mathscr{F} f(x) := (2\pi)^{-n/2} \int_{\mathrm{R}^n} f(u) e^{-ixu} \, du \tag{4.4}$$

be the Fourier transform of $f \in \mathscr{S}$ and $\mathscr{F}^{-1}$ be its inverse. In case $1 \leqq p < \infty$ an element $\mu \in L^{\infty}$ (cf. (4.3)) is then called a multiplier on L^p if for $f \in \mathscr{S}$

$$\mathscr{F}^{-1}(\mu \mathscr{F} f) \in L^p, \quad \|\mathscr{F}^{-1}(\mu \mathscr{F} f)\|_p \leqq C \|f\|_p. \tag{4.5}$$

Since $\mathscr{S}$ is dense in L^p, $1 \leqq p < \infty$, the (L^p-) closure $T_p(\mu)f$ of $\mathscr{F}^{-1}(\mu\mathscr{F}f)$ is in $[L^p]$ defining an injective homomorphism T_p of M in $[L^p]$. Moreover, if (3.1,2) are valid, then for (fixed) $f \in \mathscr{S}$

$$\lim_{n\to\infty} \mathscr{F}^{-1}(\mu_n \mathscr{F} f)(u) = \mathscr{F}^{-1}(\nu\mathscr{F} f)(u)$$

in view of $\mathscr{F} f \in L^1$ and dominated convergence. On the other hand, (3.2) implies for a subsequence $\{n_k\} \subset \mathrm{N}$

$$\lim_{k\to\infty} \mathscr{F}^{-1}(\mu_{n_k} \mathscr{F} f)(u) = (Sf)(u) \qquad \text{(a. e.)}$$

so that $\mathscr{F}^{-1}(\nu\mathscr{F} f) = Sf$ on the dense set $\mathscr{S}$ establishing (3.3).

Analogously to (4.5) multipliers are defined for the space $C_0 = C_0(\mathrm{R}^n)$ of continuous functions on R^n vanishing at infinity. But this approach fails for L^∞ since $\mathscr{S}$ is not dense. However, a multiplier μ on C_0 can be identified by a bounded Borel measure m via

$$\mu(x) = (2\pi)^{-n/2} \int_{\mathrm{R}^n} e^{-ixu}\, dm(u),$$

$$T_{C_0}(\mu)f(v) = (2\pi)^{-n/2} \int_{\mathrm{R}^n} f(v-u)\, dm(u).$$

In view of this representation, $T_{C_0}(\mu)$ can be extended to $T_\infty(\mu) \in [L^\infty]$ so that $M(L^\infty, \mathrm{R}^n) := M(C_0, \mathrm{R}^n)$ is an appropriate choice of a multiplier space on L^∞ (see also [12, p. 74]).

Concerning the Riesz summability condition, it is a classical result (cf. [16], p. 114, see also [5]) that L^p is R_ψ^j-bounded for

(4.6) $$\psi(x) = |x|, \quad j > (n-1)|1/p - 1/2|.$$

Other admissible choices of ψ can be obtained by using the fact that any surjective affine transformation A from R^n to R^m induces an isometry from $M(L^p, \mathrm{R}^m)$ to $M(L^p, \mathrm{R}^n)$ via $\mu(Ax)$, $x \in \mathrm{R}^n$, $\mu \in M(L^p, \mathrm{R}^m)$ (cf. [2, p. 15]). For example, take $m=1$, $0 \neq h \in \mathrm{R}^n$, and $\psi(x) = |Ax| = |\Sigma_{j=1}^n h_j x_j|$. Then R_ψ^j-boundedness follows by (4.6) (on R^1) for each $j > 0$.

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Lehrstuhl A. für Mathematik
RVTH-Aachen
Templergraben 55
5100 Aachen
GERMANY

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