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Inverse Systems of *R*-closed Spaces

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In the present paper it is proved that a limit of an inverse system of non-empty R-closed spaces and closed irreducible bonding mappings is a non-empty space.

If $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system of R-closed spaces X_α with open-closed mappings $f_{\alpha\beta}$, then $X=\lim \underline{X}$ is R-closed.

1. Introduction

Our main purpose is to prove non-emptyness and R-closedness of the limit of an inverse system of R-closed spaces.

For the basic definitions and properties on inverse systems and their limits the reader is refered to R. Engelking [7].

We say that an inverse system $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ is well-ordered if the set A is well-ordered.

In the sequel we use the next theorem from [3].

1.1. Theorem. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of non-empty spaces* X_α *such that for each* $\alpha\in A$ *there exists a family* $\mathscr{I}_\alpha$ *of the subsets of* x_α *with the following properties :*

(I) *If* $\mathscr{P}_\alpha$ *is centred subfamily of* $\mathscr{I}_\alpha$, *then* $\cap\{P : P\in\mathscr{P}_a\}\neq\emptyset$,

(II) *The intersection of a members of* $\mathscr{I}_\alpha$ *is a member of* $\mathscr{I}_\alpha$,

(III) *If* $S_\beta\in\mathscr{I}_\beta, \beta\geqq\alpha$, *then* $f_{\alpha\beta}(S_\beta)\in\mathscr{I}_\alpha$,

(IV) $f_{\alpha\beta}^{-1}(x_\alpha)\in\mathscr{I}_\beta$ *for each* $x_\alpha\in X_\alpha$ *and each* $\beta\geqq\alpha$.

Then $X=\lim \underline{X}$ *is non-empty and for each* $\alpha\in A$ *the relation* $f_\alpha(X)=\cap\{f_{\alpha\beta}(X_\beta) : \beta\geqq\alpha\}$ *holds.*

A subset A of a space X is regularly closed (open) if $A=\overline{\operatorname{int} A}$ $(A=\operatorname{int}\overline{A})$.

A mapping $f: X\to Y$ means a continuous function.

A mapping $f: X\to Y$ is regularly closed if $f(A)$ is closed for every regularly closed $A\subseteq X$.

2. Inverse systems of R-closed spaces

A regular space X is R-closed [6] if X is closed in every regular space containing X as subspace.

An open filter $\mathscr{F}$ is a filter [7 :76] such that if $F\in\mathscr{F}$, then F is open.

An open filter $\mathscr{F}$ is **regular** if for each $U\in\mathscr{F}$ there is a $V\in\mathscr{F}$ such that $\overline{V}\subseteq U$.

A regular space X is R-closed [6] iff for each regular filter $\mathscr{F}$ its adherence $\cap\{\overline{U} : U\in\mathscr{F}\}$ is non-empty.

A mapping $f:X\to Y$ is an irreducible mapping if the set $f^{\#}(U)=\{y\in Y : f^{-1}(y)\subseteq U\}$ is a non-empty set for each open non-empty set $U\subseteq X$. If $f:X\to Y$ is a closed irreducible mapping, then $f^{\#}(U)$ is open and non-empty [7 :52].

We shall now show the following theorem.

2.1. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with the surjective closed irreducible mappings $f_{\alpha\beta}$. If the spaces X_α are non-empty R-closed, then $X=\lim\underline{X}$ is non-empty. Moreover, the projections $f_\alpha : X\to X_\alpha$, $\alpha\in A$, are surjective.*

Proof. We shall show that for each point $x_\alpha\in X_\alpha$ there exists a point $x\in X$ such that $f_\alpha(x)=x_\alpha$, where $f_\alpha : X\to X_\alpha$ is the natural projection. Let $\mathscr{U}_\alpha=\{U_{\mu_\alpha} : \mu_\alpha\in M_\alpha\}$ be a regular ultrafilter of open sets $U_{\mu_\alpha}\subseteq X_\alpha$ such that $\cap\{\overline{U}_{\mu_\alpha} : U_{\mu_\alpha}\in\mathscr{U}_\alpha\}=\{x_\alpha\}$. The existence of $\mathscr{U}_\alpha$ follows from regularity of X_α. For every $\beta>\alpha$ we consider the family $\mathscr{U}'_\beta=\{f_{\alpha\beta}^{-1}(U_{\mu_\alpha}) : U_{\mu_\alpha}\in\mathscr{U}_\alpha\}$. The family $\mathscr{U}'_\beta$ is a filter-base since $f_{\alpha\beta}^{-1}(U_{\mu_{\alpha_1}}\cap U_{\mu_{\alpha_2}})=f_{\alpha\beta}^{-1}(U_{\mu_{\alpha_2}})\cap f^{-1}(U_{\mu_{\alpha_2}})$.

The regularity of $\mathscr{U}'_\beta$ follows from the regularity of $\mathscr{U}_\alpha$ and from the implication $\overline{V}\subseteq U\to f_{\alpha\beta}^{-1}(V)\subseteq\overline{f_{\alpha\beta}^{-1}(\overline{V})}\subseteq f^{-1}(\overline{V})\subseteq f^{-1}(U)$. Now, we prove that there exists a single ultrafilter $\mathscr{U}_\beta$ which contains $\mathscr{U}'_\beta$. Let U_β be an open subset of X_β such that every intersection $U_\beta\cap f_{\alpha\beta}^{-1}(U_{\mu_\alpha})=U_{\beta,\mu_\alpha}$ is non-empty. Then the set $f^{\#}_{\alpha\beta}(U_{\beta,\mu_\alpha})$ is non-empty and open because $f_{\alpha\beta}$ is a closed irreducible mapping. Hence, all the sets $f^{\#}_{\alpha\beta}(U_\beta)\cap U_{\mu_\alpha}$, $U_{\mu_\alpha}\in\mathscr{U}_\alpha$, are non-empty. By maximality of $\mathscr{U}_\alpha$ it follows that $f^{\#}_{\alpha\beta}(U_\beta)\in\mathscr{U}_\alpha$. This means that $f_{\alpha\beta}^{-1}f^{\#}_{\alpha\beta}(U_\beta)\in\mathscr{U}'_\beta$. Moreover, we have the inclusion $f_{\alpha\beta}^{-1}f^{\#}(\overline{U}_\beta)\subseteq U_\beta$. This fact implies that there exists a single ultrafilter $\mathscr{U}_\beta$ containing $\mathscr{U}'_\beta$ since from $U_\beta\cap V_\beta=\emptyset$ it follows $f^{\#}_{\alpha\beta}(U_\beta)\cap f^{\#}_{\alpha\beta}(V_\beta)=\emptyset$. Now, it follows that for each $\beta>\alpha$ there is a single point $x_\beta\in X_\beta$ such that $x_\beta\in\{\overline{U}_{\mu_\beta} : U_{\mu_\beta}\in\mathscr{U}_\beta\}$. Moreover, from the fact that for $\gamma\geqq\beta\geqq\alpha$ $f_{\alpha\gamma}^{-1}(\mathscr{U}_\alpha)=f_{\beta\gamma}^{-1}(\mathscr{U}'_\beta)$, it follows that $f_{\beta\gamma}(x_\gamma)=x_\beta$. This means that the obtained points $x_\beta\in X_\beta$ determine a point x of $X=\lim\underline{X}$. The proof is complete.

2.2. Problem. Is it true that the limit X in Theorem 2.1. is R-closed?

2.3. Remark. If the space X in Theorem 2.1. are H-closed, then X is H-closed [13].

In the sequel we use the following

2.4. Lemma. *Let $f: X \to Y$ be a closed mapping into regular Y. If X is R-closed, then $f^{-1}(y)$ is R-closed for each $y \in Y$.*

Proof. Let $\mathcal{U}'$ be a regular filter on $f^{-1}(y)$. For each open $U' \in \mathcal{U}'$ there is an open U such that $U' = U \cap f^{-1}(y)$. Let $V' \in \mathcal{U}'$ such that closure $[V']_{f^{-1}(y)} \subset U'$. Let V be an open set in X such that $V' = V \cap f^{-1}(y)$. Suppose that $\overline{V} \nsubseteq U$. Then $X \setminus (\overline{V} \setminus U)$ is the neighborhood of $f^{-1}(y)$. From the closedness of f it follows that there is an open set $V_y \ni y$ such that $f^{-1}(V_y) \subseteq X \setminus (\overline{V} \setminus U)$. From the regularity of Y it follows that there exists an open $W_y \ni y$ such that $\overline{W}_y \subseteq V_y$. Then $\overline{f^{-1}(W_y)} \subseteq f^{-1}(V_y)$. Consider the sets $U_1 = U \cap f^{-1}(V_y)$ and $V_1 = V \cap f^{-1}(W_y)$. Clearly, $U_1 \cap f^{-1}(y) = U \cap f^{-1}(y) = U'$ and $V_1 \cap f^{-1}(y) = V'$. Moreover, $\overline{V}_1 \subseteq \overline{V} \cap \overline{f^{-1}(W_y)} \subseteq \overline{V} \cap f^{-1}(\overline{W}_y) \subseteq \overline{V} \cap f^{-1}(V_y) \subseteq U \cap f^{-1}(V_y) = U_1$ since $f^{-1}(V_y) \subseteq X \setminus (V \setminus U)$. We infer that each regular filter $\mathcal{U}'$ on $f^{-1}(y)$ can be extended to a regular filter $\mathcal{U}$ on X such that for each $U' \in \mathcal{U}'$ there is $U \in \mathcal{U}$ with $U' = U \cap f^{-1}(y)$. Moreover, each $U \in \mathcal{U}$ is the intersection $U'' \cap f^{-1}(V_y)$ for some open $U'' \subseteq X$ and some open $V_y \ni y$. Clearly we may assume that $\mathcal{U}$ contains all the sets $f^{-1}(V_y)$, where V_y is a neighborhood of y. Now, by R-closedness of X there exists a point $x \in X$ such that $x \in \cap \{\overline{U} : U \in \mathcal{U}\}$. From the hypothesis that $\mathcal{U}$ contains the family $\{f^{-1}(V_y) : V_y$ is a neighborhood of $y\}$ it follows that $f(x) = y$, i.e. $x \in f^{-1}(y)$. Clearly, x is an adherence point of $\mathcal{U}'$ in $f^{-1}(y)$. The proof is complete.

2.5. Question: Is it true that closedness of f in Theorem 2.4. can be omitted?

Let X and Y be the regular spaces. We say that a mapping $f: X \to Y$ is R-perfect if f is closed and $f^{-1}(y)$ is R-closed for each $y \in Y$.

2.6. Lemma. *If $f: X \to Y$ is an R-perfect open mapping of a regular space X into an R-closed space Y, then X is R-closed.*

Proof. Let $\mathcal{U}$ be a regular filter on X. Then $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ is the regular filterbase on Y since f is open and closed. By R-closedness of Y there exists a $y \in Y$ such that $y \in \cap \{\overline{f(U)} : U \in \mathcal{U}\}$. We now prove that for each $U \in \mathcal{U}$ the intersection $f^{-1}(y) \cap U \neq \emptyset$. Namely, if $f^{-1}(y) \cap U = \emptyset$ for some $U \in \mathcal{U}$, then by regularity of $\mathcal{U}$ it follows that there exists a $V \in \mathcal{U}$ such that $\overline{V} \subseteq U$ and $f^{-1}(y) \cap \overline{V} = \emptyset$.

By closedness of f it follows that there is an open set $V_y \ni y$ such that $f^{-1}(V_y) \cap \overline{V} = \emptyset$. This means that $V_y \cap \overline{f(V)} = \emptyset$. This is in contradiction with $y \in \cap \{\overline{f(U)} : U \in \mathcal{U}\}$. Now, the family $\mathcal{U}' = \{U \cap f^{-1}(y) : U \in \mathcal{U}\}$ is a regular filterbase on $f^{-1}(y)$. Since $f^{-1}(y)$ is R-closed, there exists an $x \in f^{-1}(y)$ such that $x \in \overline{\{U \cap f^{-1}(y) : U \in \mathcal{U}\}}$. Clearly, $x \in \cap \{\overline{U} \cap f^{-1}(y) : U \in \mathcal{U}\}$. This means that $\cap \{\overline{U} : U \in \mathcal{U}\} \neq 0$. The proof is complete.

For an inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ with open and regularly closed projections $f_\alpha: X \to X_\alpha$, $\alpha \in A$, we prove

2.7. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of R-closed spaces X_α such that the projections $f_\alpha:\lim\underline{X}\to X_\alpha$, $\alpha\in A$, are open and regularly closed, then the limit $X=\lim\underline{X}$ is R-closed.*

Proof. Firstly, X is regular since X_α, $\alpha\in A$, are regular. Secondly, let $\mathscr{U}=\{U_\mu:\mu\in M\}$ be a regular ultrafilter of open sets in X. For each $\alpha\in A$ let $\mathscr{U}_\alpha$ be a family of open sets U_{μ_α} in X_α, $\mu_\alpha\in M$, such that $U_{\mu_\alpha}\supseteq f_\alpha(U_\mu)$ for some $U_\mu\in\mathscr{U}$. This family is a regular filter since $\overline{V}\subseteq U\subseteq X$ implies $\overline{f_\alpha(V)}\subseteq f_\alpha(U)$ (f_α, $\alpha\in A$, are open and regularly closed). By virtue of R-closedness of X_α it follows that $Y_\alpha=\cap\{\overline{U}_{\mu_\alpha}:U_{\mu_\alpha}\in\mathscr{U}_\alpha\}\neq 0$. Let x_α be a point of Y_α, and let V be an open set about x_α. We prove that $\mathscr{U}_\alpha$ converges to x_α. Namely, if there is an $U\in\mathscr{U}$ such that $f_\alpha^{-1}(V)\cap U=\emptyset$, then $\overline{f_\alpha^{-1}(V)}\cap U=\emptyset$. Since for open f_α equality $f_\alpha^{-1}(\overline{V})=\overline{f_\alpha^{-1}(V)}$ [7 :57] holds, we have $f_\alpha^{-1}(\overline{V})\cap U=\emptyset$. This means that $f_\alpha(U)\cap\overline{V}=\emptyset$, i.e. $X_\alpha-\overline{V}\in\mathscr{U}_\alpha$. This is in a contradiction with $x_\alpha\in\cap\overline{\mathscr{U}}_\alpha$. This means that $f_\alpha^{-1}(V)\cap U\neq 0$ and $f_\alpha^{-1}(V)\in\mathscr{U}$, i.e. $V\in\mathscr{U}_\alpha$. Now, for every $\alpha\in A$ we have the point x_α such that $\mathscr{U}_\alpha$ converges to x_α. It is readily seen that $f_{\alpha\beta}(x_\beta)=x_\alpha$ for $\beta\geqq\alpha$. This means that $x=(x_\alpha)$ is a point of $X=\lim\underline{X}$. Clearly, $\mathscr{U}$ converges to x. The proof is complete. The next theorem is closely related to the corresponding theorem for inverse system of H-closed spaces [20].

2.8. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of R-closed spaces X_α such that $f_{\alpha\beta}$ are open and closed surjective mappings. Then the limit $X=\lim\underline{X}$ is non-empty iff all spaces X_α, $\alpha\in A$, are non-empty.*

Proof. Let $\mathscr{F}_\alpha$, $\alpha\in A$, be the family of all regular filter-base of open subsets of X_α. For each $F_\alpha\in\mathscr{F}_\alpha$ the set ad $F_\alpha=$ ad $\{\overline{U}:U\in F_\alpha\}$ is a non-empty closed set since X_α is R-closed. Moreover, ad $F_\alpha=\cap\{U:U\in F_\alpha\}$. Let $\mathscr{I}_\alpha$ be a family $\{\emptyset\}\cup\{\text{ad } F_\alpha:F_\alpha\in\mathscr{U}_\alpha\}$. We prove that $\mathscr{I}_\alpha$ satisfies the conditions (I)–(IV) of Theorem 1.1. If $\{\text{ad } F_\mu:\mu\in M\}$ is centred family, then $\mathscr{U}=\{U:U\in F_\mu, \mu\in M\}$ is regular filter-base. By virtue of R-closedness of X_α it follows that ad is non-empty and ad $\mathscr{U}\in\varphi_\alpha$ since ad $\mathscr{U}=\cap\{U:U\in\mathscr{U}\quad\}=\cap\{\text{ad } F_\mu:\mu\in M\}$. The family $\mathscr{I}_\alpha$ satisfies property (I). We prove property (II) as follows. Let $\{\text{ad } F_{\alpha_\mu}:F_{\alpha_\mu}\in\mathscr{F}_a\}$ be a subfamily of $\mathscr{I}_\alpha$. If $\cap\{\text{ad } F_{\alpha_\mu}:F_{\alpha_\mu}\in\mathscr{I}_a\}=\emptyset$, then property (II) is proved since $\{\emptyset\}\in\mathscr{I}_\alpha$. If $\cap\{\text{ad } F_{\alpha_\mu}:F_{\alpha_\mu}\in\quad_\alpha\}=Y\neq 0$, then from $y\in Y$ it follows that $y\in U$ for each $U\in F_{\alpha_\mu}$, $F_{\alpha_\mu}\in\mathscr{F}_a$, since ad $F_\alpha=\cap\{\overline{U}:U\in F_{\alpha_\mu}\}=\cap\{U:U\in F_{\alpha_\mu}\}$. This means that $\{\text{ad } F_{\alpha_\mu}:F_{\alpha_\mu}\in\mathscr{F}_a\}$ is centred family. Arguing as in the proof of property (I) prove property (II). In order to prove that property (III) is satisfied we recall that $f_{\alpha\beta}$ are closed. Let $S_\beta\in\mathscr{I}_\beta$ and let F_β by a regular filter-base such that $S_\beta=\cap\{U:U\in F_\beta\}\;=\{\overline{U}:U\in F_\beta\}$. We prove that $f_{\alpha\beta}(S_\beta)=\cap\{f_{\alpha\beta}(U):U\in F_\beta\}$ $=\cap\{\overline{f_{\alpha\beta}(U)}:U\in F_\beta\}$. Clearly, $f_{\alpha\beta}(S_\beta)\subseteq\{f_{\alpha\beta}(U):U\in F_\beta\}$. Let $x_\alpha\in\cap\{f_{\alpha\beta}(U)$ $:U\in F_\beta\}$ and let $\mathscr{V}_\alpha$ be a family of all neighborhoods of x_α. Then $\mathscr{W}=F_\beta\cap f_{\alpha\beta}^{-1}(\mathscr{I}_\alpha)$ is regular filter-base. Since X_β is R-closed, we have ad $\mathscr{W}\neq 0$

and ad $\mathscr{W} \subseteq S_\beta$, $f_{\alpha\beta}(\text{ad}\,\mathscr{W}) = x_\alpha$. This means that $x_\alpha \in f_{\alpha\beta}(S_\beta)$. It follows that $f_{\alpha\beta}(S_\beta) = \cap \{f_{\alpha\beta}(U) : U \in F_\beta\} \in \mathscr{I}_\alpha$. Hence, $\mathscr{I}_\alpha$ satisfies the property (III) of Theorem 1.1. Property (IV) follows from the relations $f_{\alpha\beta}^{-1}(x_\alpha) = \cap \{f_{\alpha\beta}^{-1}(U) : U \in \mathscr{V}_a\}$. From Theorem 1.1. it follows that $X = \lim \underline{X} \neq \emptyset$. The proof is complete.

Let us recall that it is proved in [9] that $\lim \underline{X} \neq \emptyset$ if $\underline{X}$ is a well-ordered inverse system of non-empty R-closed spaces and surjective bonding mappings.

If the mappings $f_{\alpha\beta}$ are open, then $Y = \cap \{f_{\alpha\beta}(X_\beta) : \beta \geqq \alpha\}$ is non-empty since X_α is R-closed. Moreover, $f_{\alpha\beta}(Y_\beta) = Y_\alpha$, $\beta \geqq \alpha$. The proof is similar to the proof of the property (IV) in Theorem 2.8. From this it follows

2.9. Theorem. *If $\underline{X}$ is an inverse system of non-empty R-closed spaces and open perfect bonding mappings, then $X = \lim \underline{X} \neq 0$.*

2.10. Corollary. *Let $\mathscr{U} = \{U_\mu : \mu \in M\}$ be a centred family of closed and open R-closed sets in X. If $Y = \{U_\mu : \mu \in M\}$ is open, then Y is R-closed. In particular, each closed and open set of an R-closed space X is R-closed.*

By virtue of transfinite induction we have

2.11. Theorem. *Let $\underline{X}$ be a well-ordered inverse system of non-empty R-closed spaces and open bonding mappings, then $X = \lim \underline{X} \neq \emptyset$.*

We shall show some theorems concerning the connectedness of the space $X = \lim \underline{X}$.

Let us begin with the following simple

2.12. Lemma. *Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with open and regularly closed projections. If the spaces X_α, $\alpha \in A$, are connected, then $X = \lim \underline{X}$ is connected.*

P r o o f. If X is not connected, then there exists closed and open set F such that $\emptyset \neq F \neq X$. For each $\alpha \in A f_\alpha(F)$ is closed and open. From the connectedness of X_α it follows that $f_\alpha(F) = X_\alpha$. It follows that $F = X$ since $F = \overleftarrow{\lim} \{f_\alpha(F), f_{\alpha\beta}/f_\beta(F), A\}$ [7 : 137]. The contradiction $F \neq X \wedge F = X$ completes the proof.

2.13. Theorem. *Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of regular connected spaces X_α. If the projections $f_\alpha X \to X_\alpha$, $\alpha \in A$, are surjective monotone mappings and if X is R-closed, then X is connected.*

P r o o f. If X is not connected, then there exist two non-empty closed sets F_1, F_2 such that $F_1 \cap F_2 = \emptyset$, $F_1 \cup F_2 = X$. For each $\alpha \in A$ we have $f_\alpha(F_1) \overline{\cup} f_\alpha(F_2) = X_\alpha$ and $f_\alpha(F_1) \cup f_\alpha(F_2) = X_\alpha$. Suppose that $x_\alpha \in \overline{f_\alpha(F_1)} \cap \overline{f_\alpha(F_2)}$. Let $\mathscr{U}_\alpha$ be a filter of all open neighborhoods of x_α. Then the traces of $f_\alpha^{-1}(\mathscr{U}_\alpha)$ on F_1, F_2 are the regular filterbase. Since F_1, F_2 are R-closed (2.10. Corollary), there exist points y_1 and y_2 which are in the adherences of these filter-trace. Moreover, $y_1 \in F_1$ and $y_2 \in F_2$. On the other hand, we have $f_\alpha(y_1) = x_\alpha$, $f(y_2) = x_\alpha$. This means that $f_\alpha^{-1}(x_\alpha) \cap F_1 \neq 0$, and $f_\alpha^{-1}(x_\alpha) \cap F_2 \neq \emptyset$. This is impossible since $f_\alpha^{-1}(x_\alpha)$ is connected. The proof is complete.

From the proof it follows that, in fact, the following theorem is proved.

2.14. Theorem. *Let X be R-closed and $f: X \to Y$ a monotone mapping onto regular Y. The space X is connected iff Y is connected.*

Theorem 2.14. can be proved by virtue of Lemma 2.12. and the next

2.15. Lemma. *Let $f: X \to Y$ be a monotone regularly closed mapping of space X onto a regular connected space Y. Then X is connected.*

Proof. Suppose that X is not connected. Then there exist two disjoint closed sets $F_1, F_2 \subseteq X$ such that $F_1 \cup F_2 = X$. Clearly, $f(F_1) \cup f(F_2) = Y$. We now prove that $f(F_1) \cap f(F_2) = \emptyset$. Namely, if $y \in f(F_1) \cap f(F_2)$, then $f^{-1}(y) \cap F_1 \neq 0$ and $f^{-1}(y) \cap F_2 \neq \emptyset$. This is impossible since $f^{-1}(y)$ is connected. Hence, $f(F_1) \cap f(F_2) = \emptyset$ and $f(F_1) \cup f(F_2) = Y$. The last two relations are in contradiction with connectedness of Y. The proof is complete.

We say that a space X is strongly minimal regular (shortly, SMR) [6] iff there is a base $\mathscr{B}$ for X such that each member of $\mathscr{B}$ has an R-closed complement.

2.16. Theorem. *Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of SMR spaces X_α such that the projections are open and closed. Then $X = \lim \underline{X}$ is SMR.*

Proof. Let $\mathscr{B}_\alpha$, $\alpha \in A$, be a base for X_α such that $X_\alpha - V_\alpha$ is R-closed for each $V_\alpha \in \mathscr{B}_\alpha$. Let $\mathscr{B}$ be the standard base for $X = \lim \underline{X}$. This means that for each $V \in \mathscr{B}$ there is an open $V_\alpha \in \mathscr{B}_\alpha$ such that $f_\alpha^{-1}(V_\alpha) = V$. Clearly, $X - V = \lim \{X_\beta - f_{\alpha\beta}^{-1}(V_\alpha), f'_{\beta\gamma}, \alpha \leqq \beta \leqq \gamma\}$, where $f'_{\alpha\gamma}$ is the restriction of $f_{\alpha\gamma}$ onto $X_\gamma - f_{\alpha\gamma}^{-1}(V_\alpha)$.

The mappings $f'_{\alpha\gamma}$ are open and closed [7 :95]. From Lemmas 2.4. and 2.6. it follows that $X_\beta - f_{\alpha\beta}^{-1}(V_\alpha)$ is R-closed, $\beta \geqq \alpha$. Finally, from 2.7. Theorem it follows that $X - V = \lim \{X_\beta - f_{\alpha\beta}^{-1}(V_\alpha), f'_{\alpha\gamma}, \alpha \leqq \beta \leqq \gamma\}$ is R-closed. This completes the proof.

2.17. Theorem. *Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of MR spaces X_α such that the projections $f_\alpha : X \to X_\alpha$, $\alpha \in A$, are open and regularly closed. Then $X = \lim \underline{X}$ is MR.*

Proof. Since X is R-closed (2.7. Theorem), it suffices to prove that each regular filter-base on X which has a unique adherent point is convergent. Let $\mathscr{U}$ be a regular filter-base of open sets of X which has a unique adherent point $x \in X$. For each $\alpha \in A$ let $\mathscr{U}_\alpha = \{f_\alpha(U) : U \in \mathscr{U}\}$ be a filter-base of open sets of X_α. The filter-base $\mathscr{U}_\alpha$ is regular since f_α, $\alpha \in A$, are open and regularly closed. This means that $f_\alpha(x)$ is an adherent point of $\mathscr{U}_\alpha$. We now prove that $f_\alpha(x)$ is the unique adherent point of $\mathscr{U}_\alpha$. Let $x_\alpha \neq f_\alpha(x)$ be an adherent point of $\mathscr{U}_\alpha$. We consider a filter-base $\mathscr{U} \cup f_\alpha^{-1}(\mathscr{N}_\alpha)$, where $\mathscr{N}_\alpha$ is the filter-base of all neighborhoods of x_α. Clearly, x is the unique adherent point of $\mathscr{U} \cup f_\alpha^{-1}(\mathscr{N}_\alpha)$ since X is R-closed. This means that x is an adherent point of $f_\alpha^{-1}(\mathscr{N}_\alpha)$ and, consequently, $f_\alpha(x) = x_\alpha$. This is in contradiction with $x_\alpha \neq f_\alpha(x)$. It follows that $\mathscr{U}_\alpha$ is convergent. Thus, each open $U_\alpha \ni f_\alpha(x)$ is a member of $\mathscr{U}_\alpha$, i. e., there is an open $U \in \mathscr{U}$ such that $f_\alpha(U) \subseteq U_\alpha$. We infer that $f_\alpha^{-1}(U_\alpha) \supseteq U$. Since each neighborhood of x at least contains a set of the form $f_\alpha^{-1}(U_\alpha)$, it follows that $\mathscr{U}$ converges to x. The proof is complete.

A regular T_1-space X is said to be R-functionally compact [9 : 446] if for each T_1 regular space Y and each mapping f from X onto Y, f is closed. It is known that a regular T_1 space X is R functionally compact iff for each closed set F in X and each regular filterbase $\mathscr{U}$ for which $F=\cap\mathscr{U}=\cap\overline{\mathscr{U}}$, $\mathscr{U}$ is a neighborhood filterbase for F.

2.18. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of R-functionally compact X_α such that the projections are open and closed. Then $X=\lim\underline{X}$ is R-functionally compact iff the following condition (S) is satisfied: (S) For each pair of disjoint closed sets $F_1, F_2\subseteq X$ such that $F_1=\cap\mathscr{U}=\cap\overline{\mathscr{U}}$, for some regular filterbase $\mathscr{U}$, there exists an $\alpha\in A$ such that $f_\alpha(F_1)\cap f_\alpha(F_2)=\emptyset$.*

P r o o f. If (S) is satisfied, then for every regular filterbase $\mathscr{U}$ such that $F=\cap\mathscr{U}=\cap\overline{\mathscr{U}}$ and an open $U\supset F$ there is an $\alpha\in A$ such that $f_\alpha(F)\cap f_\alpha(X-U)=\emptyset$. This means that $Y_\alpha=X_\alpha-f_\alpha(X-U)$ is a neighborhood of $f_\alpha(F)$. Let $f_\alpha(\mathscr{U})=\{f_\alpha(U) : U\in\mathscr{U}\}$ be a regular filterbase (f_α is open and closed). We now prove that $f_\alpha(F)=\cap f_\alpha(\mathscr{U})=\cap\overline{f_\alpha(\mathscr{U})}$. Clearly, $f_\alpha(F)\subseteq\cap f_\alpha(\mathscr{U})$. For each $x_\alpha\in\cap f_\alpha(\mathscr{U})$ we consider the filterbase $\mathscr{N}_\alpha$ of all neighborhoods of x_α. The family $\mathscr{V}=\mathscr{U}\cup f_\alpha^{-1}(\mathscr{N}_\alpha)$ is regular filterbase. Since X is R-closed (Theorem 2.7), it follows that there exists an $x\in X$ such that $x\in\cap\overline{\mathscr{V}}$. This means that $x\in F$ and $f_\alpha(x)=x_\alpha$. Now we have that $f_\alpha(F)=\cap f_\alpha(\mathscr{U})=\cap\overline{f_\alpha(\mathscr{U})}$. Since X_α is R-functionally compact, there is a $U_\alpha\in f_\alpha(\mathscr{U})$ such that $U_\alpha\subseteq Y_\alpha$. This means that $f_\alpha^{-1}(U_\alpha)\subseteq U$. Since $U_\alpha\in f_\alpha(\mathscr{U})$, there is $V\in\mathscr{U}$ such that $f_\alpha(V)=U_\alpha$. Finally, we have $V\subseteq f_\alpha^{-1}(U_\alpha)\subseteq U$, i.e., $\mathscr{U}$ is a neighborhood filterbase for F.

Conversely, let X be R-functionally compact. Let F_1 and F_2 be as in (S). We consider the family $\mathscr{V}=\{f_\alpha^{-1}f_\alpha(\mathscr{U}) : \alpha\in A\}$. $\mathscr{V}$ is a regular filter-base such that $F_1=\cap\mathscr{V}=\cap\overline{\mathscr{V}}$. Since X is R-functionally compact, there is an $\alpha\in A$ and an $U_\alpha\in f_\alpha^{-1}f_\alpha(\mathscr{U})$ such that $U_\alpha\subset X\backslash F$. There exists a $U\in\mathscr{U}$ such that $U_\alpha=f_\alpha^{-1}f_\alpha(U)$. From $f_\alpha^{-1}f_\alpha(U)\subseteq X\backslash F_2$ it follows $f_\alpha(U)\cap f_\alpha(F_2)=\emptyset$. Since $U\supseteq F_1$, we have $f_\alpha(F_1)\cap f_\alpha(F_2)=\emptyset$. The proof is complete.

A topological space X is called a lightly compact space if every countable centred family $\{U_n : U_n$ is open in X, $n\in N\}$ has non-empty intersection $\{\bar{U}_n : n\in N\}$.

Every R-closed space is lightly compact [15].

We say that a mapping $f: X\to Y$ is semi-open if f is continuous and Int $f(U)\neq\emptyset$ for every non-empty open subset U of X.

The proof of the following Lemma is routine.

2.19. Lemma. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with surjective projections. The projections $f_\alpha : X=\lim\underline{X}\to X_\alpha$ are semi-open iff the bonding mappings are semi-open.*

Now we prove

2.20. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of R-closed spaces X_α with perfect semi-open mappings $f_{\alpha\beta}$. Then $X=\lim \underline{X}$ is lighty compact.*

Proof. Let $\mathcal{U}=\{U_n : n\in N\}$ be a centred family of non-empty open subsets U_n of X. By virtue of Lemma 2.19. it follows that $\{\mathrm{Int} f_\alpha(U_n) : n\in N\}$ is a centred family of non-empty open subsets of X_α. Since X_α is lightly compact, it follows that the set

$$(1) \qquad Y_\alpha=\cap\{\overline{\mathrm{Int} f_\alpha(U_n)} : n\in N\}$$

is non-empty for every $\alpha\in A$. Let us prove that

$$(2) \qquad f_{\alpha\beta}(Y_\beta)=Y_\alpha, \beta\geqq\alpha.$$

For every point $y_\alpha\in Y_\alpha$ the sets $Z_{\beta,n}=f_{\alpha\beta}^{-1}(y_\alpha)\cap\overline{\mathrm{Int} f_\beta(U_n)}$ are non-empty sets since from the closedness and semi-openness of $f_{\alpha\beta}$ it follows that $f_{\alpha\beta}(\overline{\mathrm{Int} f_\beta(U_n)})=\overline{\mathrm{Int} f_\alpha(U_n)}$, $n\in N$. The family $\{Z_{\beta_1,n} : n\in N\}$ is a centred family of closed sets in the compact space $f_{\alpha\beta}^{-1}(y_\alpha)$. It follows that there exists some point $y_\beta\in f_{\alpha\beta}^{-1}(y_\alpha)\cap(\cap\{Z_{\beta,n} : n\in N\})$. This means that $y_\beta\in Y_\beta=\cap\{\overline{\mathrm{Int} f_\beta(U_n)} : n\in N\}$ and that $f_{\alpha\beta}(y_\beta)=y_\alpha$ i.e. (2) is proved. The inverse system

$$(3) \qquad \underline{Y}=\{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$$

has non-empty limit Y since $f_{\alpha\beta}/Y_\beta$ are perfect surjective mappings. It is easy to prove that $Y\subseteq\{\bar{U}_n : n\in N\}$, *i.e.* $\cap\{\bar{U}_n : n\in N\}\neq\emptyset$. Hence, $X=\lim\limits_{\leftarrow}\underline{X}$ is lightly compact space. The proof is complete.

2.21. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with perfect irreducible mappings $f_{\alpha\beta}$ and R-closed spaces X_α. Then $X=\lim \underline{X}$ is lightly compact.*

Proof. The closed irreducible mappings are semi-open mappings. Hence, Theorem 2.21. follows from Theorem 2.20. Let us observe that (2) holds under the weaker assumption that $f_{\alpha\beta}^{-1}(y_\alpha)$ is m-compact, $m\geqq\aleph_0$, for every $y_\alpha\in X_\alpha$.

From [11] and [14] follows that $\lim \underline{Y}$, is non-empty, where $\underline{Y}$ is the inverse system (3), if $f_{\alpha\beta}^{-1}(y_\alpha)$ is countably compact and $A=N$, or $A=W_0=$the set of all countable ordinals.

Let as recall that a mapping $f: X\to Y$ is pseudo-perfect if f is closed and $f^{-1}(y)$ is countably compact for each $y\in Y$.

From the above observations we shall deduce the following theorems.

2.22. Theorem. *Let $\underline{X}=\{X_n, f_{nm}, N\}$ be an inverse sequence with pseudo-perfect semi-open* (irreducible) *mappings f_{nm}. The limit space $X=\lim \underline{X}$ is lightly compact if the spaces X_n, $n\in N$, are R-closed.*

2.23. Theorem. *If* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, W_0\}$ *is an inverse system with pseudo-perfect semi-open* (irreducible) *mappings* $f_{\alpha\beta}$, *then* $X=\lim\limits_{\leftarrow}\underline{X}$ *is lightly compact if the spaces* X_α, $\alpha\in A$, *are R-closed.*

2.24. Theorem. *Let* $\underline{X}$ *be an inverse system* $\{X_\alpha, f_{\alpha\beta}, \omega_\tau\}$ *or* $\{X_\alpha, f_{\alpha\beta}, \omega_{\tau+1}\}$ *with closed semi-open* (irreducible) *mappings* $f_{\alpha\beta}$ *such that the fibers* $f_{\alpha\beta}^{-1}(x_\alpha)$, $x_\alpha\in X_\alpha$, $\beta>\alpha$, *are* $\mathscr{A}_\tau$*-compact. The limit space* $X=\lim\underline{X}$ *is lightly compact if the spaces* X_α, $\alpha\in A$, *are R-closed.*

A mapping $f:X\rightarrow Y$ has the inverse property if $f^{-1}(\overline{V})=\overline{f^{-1}(V)}$ for each open set $V\subseteq Y$.

A set A is σ-directed if for every countable family $\{\alpha_i : i\in N, \alpha_i\in A\}$ there exists an $\alpha\in A$ with the property $\alpha\geqq\alpha_i, i\in N$. We close this section with the following theorem.

2.25. Theorem. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be a σ-directed inverse system of R-closed spaces* X_α. *If* $f_{\alpha\beta}$ *are mappings with the inverse property, then* $X=\lim\limits_{\leftarrow}\underline{X}$ *is lightly compact.*

Proof. Let $\mathscr{U}=\{U_n : n\in N\}$ be a centred family of non-empty open subsets of X. For each U_n there exists a maximal non-empty open subset $U_{\alpha,n}\subseteq X_{\alpha(n)}$ with the property $f_{\alpha(n)}^{-1}(U_{\alpha,n})\subseteq U_n$. Since A is σ-directed, there exists an $\alpha\in A$ such that $\alpha\geqq\alpha(n)$, $n\in N$. It follows that we can assume that the sets $U_{\alpha,n}$ are the subsets of some X_α. Since X_α is lightly compact (as an R-closed space), it follows that $\cap\{\overline{U}_{\alpha,n} : n\in N\}\neq\emptyset$. Applying the inverse property of f_α, we have $\cap\{\overline{f_\alpha^{-1}(U_{\alpha,n})} : n\in N\}=\cap\{f_\alpha^{-1}(\overline{U}_{\alpha,n}) n\in N\}=f_\alpha^{-1}(\cap\{\overline{U}_{\alpha,n} : n\in N\})\neq 0$. Since $\cap\{\overline{U}_n : n\in N\}\supseteq\{\overline{f_\alpha^{-1}(U_{\alpha,n})} : n\in N\}$, it follows that $\cap\{\overline{U}_n : n\in N\}\neq 0$. Hence X is a lightly compact space. The proof is complete.

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