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Semi-Infinite Optimization. Existence and Uniqueness of the Solution

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Necessary and sufficient conditions for the existence and uniqueness of the solution of an optimization problem with linear goal function are found. The structure of the set of linear semi-infinite optimization problems for which the solution is unique is also discussed.

0. Introduction

Let R^n be the usual n-dimensional Eucledean space, $X\subset R^n$ be a closed convex set and $p\in R^n$ be a vector.

For each pair (p, X) we define the optimization problem :

$$(*)\qquad \min\{M\langle p, x\rangle : x\in X\}. \qquad *$$

In the present paper we have found necessary and sufficient conditions for a solvability in the case of this general setting as well as a sufficient condition for the uniqueness of the solution of $(*)$.

Let T be a Hausdorff compact space. Consider the set of all triples $\sigma=(B, b, p)\in\theta$, where $B: T\to R^n$ and $b: T\to R$ are continuous mappings; i.e., $\theta=\{C(T)^n\times C(T)\times R^n\}$.

$C(T)$ is the space of continuous functions over the compact T with the usual sup-norm.

For each $\sigma=(B,b,p)\in\theta$ as in [1, 2] we consider the linear semi infinite optimization problem :

$$\sigma:\min\{\langle p,x\rangle : \langle B(t),x\rangle\geqq b(t) \text{ for every } t\in T\}.$$

The problem σ is obviously a special case of the problem $(*)$.

In the set θ we introduce the norm $\|\sigma\|=\|B\|_\infty+\|b\|_\infty+\|p\|_{R^n}$, where $\|B\|_\infty=\max_{t\in T}\|B(t)\|_{R^n}$, $\|b\|_\infty=\max_{t\in T}|b(t)|$ and $\|p\|_{R^n}$ is the usual norm in R^n. This norm turns θ into a Banach space and generates in θ the natural Cartesian product topology.

The main results in this paper concern some topological properties of the sets :

$L=\{\sigma\in\theta:\sigma$ has a solution$\}$
$LU=\{\sigma\in L:\sigma$ has an unique solution$\}$.

It turns out that the set LU contains an open and dense subset of L. This assertion is a generalization of a theorem given in [3] and completes (in some sense) G. Nurnberger's investigations [4, 5] of this topic.

1. Existence and uniqueness of the solution

We consider the optimization problem (*).

$$\min\{\langle p,x\rangle : x\in X\}.$$

Some definitions follow :

With $\operatorname{ext} X$ we denote the set of extremal points of X, with $\operatorname{rec} X$ — the recession cone, and with $\operatorname{rint} X$ — the set of relatively interior points of X.

If K is a cone, then K^* is the dual cone of K.

1.1. Proposition. *Let X be a closed convex cone. If $X\neq\emptyset$ and $p\in\operatorname{int}(\operatorname{rec} X)^*$, then the problem (*) has a solution.*

Proof. It is easy to prove that for every $\alpha\in R$ the set $Z=X\cap\{x:\langle p,x\rangle\leqq\alpha\}$ is compact. Then an assertion follows immediately. The proposition is proved.

1.2. Theorem. *Let X be closed convex set. $X\neq\emptyset$ and $K=\operatorname{rec} X$. Then*

a) *If the problem (*) has a solution, then $p\in K^*$.*

b) *If $p\in\operatorname{rint} K^*$, then the problem (*) has a solution.*

Proof. a) Let (*) have a solution then $\langle p,z\rangle\geqq 0$ for every $z\in K$, i.e. $p\in K^*$.

b) Let $p\in\operatorname{rint} K^*$. If $\operatorname{rint} K^*=\operatorname{int} K^*$, then from the proposition 1.1 follows that the problem (*) has a solution. Let $\operatorname{int} K^*=\emptyset$, i.e. $\operatorname{ext} X=\emptyset$, $S=K\cap(-K)$ is a recession subspace of X. $X_1=X\cap S^\perp$ and $K_1=K\cap S^\perp$. Since $p\in\operatorname{rint} K^*$, then $\langle p,s\rangle\geqq 0$ for all $s\in S$. Consequently $\langle p,s\rangle=0$ for every $s\in S$, i.e. the problem (*) is equivalent to the problem

$$(**)\qquad \min\{\langle p,x\rangle : x\in X_1\}.$$

$K_1^*=K^*+S$ and $\operatorname{int} K_1^*=\operatorname{rint} K^*+S$ so that $p\in\operatorname{int} K_1^*$. By a proposition 1.1 the problem (**) has a solution, i.e. the problem (*) has a solution as well. The theorem is proved.

Remark. The condition $p\in K^*$ is not sufficient for a solvability of the problem (*).

1.3. Definition. *Let X be a closed convex set and $x\in X$. The set $D(X,x)=\{z:$ there exists $\alpha>0$ such that $x+\alpha z\in X\}$ is called a cone of the feasible directions.*

This cone has interesting properties listed below :

1. $D(X,x)=\{z:z=\lambda(y-x)\,;\ y\in X,\ \lambda\geqq 0\}$.
2. $D(X,x)$ is a convex cone.
3. $\dim D(X,x)=\dim X$.
4. $x\in\operatorname{ext} X$ iff $D(X,x)\cap\{-D(X,x)\}=\emptyset$.

As a consequence of the well-known, "Pshenichnii's condition" [6], we obtain the following.

1.4. Theorem. $\bar{x}\in$ Argmin $\{\langle p,x\rangle\ x\in X\}$ *if and only if* $\bar{x}\in X$ *and* $p\in D^*(X,x)$.

Proof. Let $\bar{x}$ be a solution of (*). Then for each $z\in D(X,x)$ holds $\langle p, z\rangle \geqq 0$,i. e. $p\in D^*(X,x)$. Conversely, let $p\in D^*(X,\bar{x})$. If we assume that there is $y\in X$ with $\langle p,y\rangle < \langle p,\bar{x}\rangle$, we obtain $y-\bar{x}\in D(X,x)$ and $\langle p,z\rangle=\langle p,y\rangle-\langle p,\bar{x}\rangle<0$ which is a contradiction. The proof is completed.

Now we are able to give a promised characterization of the unicity in (*).

1.5. Theorem. *If* $\bar{x}\in X$ *and* $p\in\operatorname{int} D^*(X,\bar{x})$, *then* $\{\bar{x}\}=\operatorname{Argmin}\ \{\langle p,x\rangle : x\in X\}$.

Proof. By a theorem 1.4. holds that $\bar{x}$ is a solution of the problem (*). Let there exists a solution $y\neq\bar{x}$ of the problem (*), then $z=y-\bar{x}\in D(X,\bar{x})$ and $\langle p,z\rangle=\langle p,y\rangle-\langle p,\bar{x}\rangle=0$

The assumption $p\in\operatorname{int} D^*(X,x)$ implies $\langle p,z\rangle>0$ for all $z\in D(X,\bar{x})$ which leads to a contradiction. The theorem is proved.

Remark. In general the converse statement of a theorem 1.5. is not true.

2. Infinite polyhedral sets

In the context with the investigations in the linear semi-infinite optimization we have to work with the feasible sets of the optimization problems. Therefore we discuss various properties of them.

Sets of the following type are considered :

$$\Omega=\{x:\langle B(t),x\rangle\geqq b(t)\quad\text{for every } t\in T\}.$$

Obviosly Ω is a closed convex subset of R^n.

We will need some notations.

1. $(x|\varepsilon)=\{y:\|y-x\|<\varepsilon\}=O_\varepsilon(x)$
2. $[x|\varepsilon]=\{y:\|y-x\|\leqq\varepsilon\}=O_\varepsilon(x)$
3. $W=\{B(t):t\in T\}$
4. $T_A=\{t\in T:\text{there exists } x\in A \text{ with } \langle B(t),x\rangle=b(t)\}$
5. $W_A=\{B(t):t\in T_A\}$.

2.1. Lemma. *Let A be a compact subset of R^n. Then T_A and W_A are compact.*

Proof. Let $\{t_\alpha\}_\alpha\subset T_A$ and $\lim t_\alpha=t_0$. Since $t_\alpha\in T_A$, then there exists $x_\alpha\in A$ such that $\langle B(t_\alpha),x_\alpha\rangle=b(t_\alpha)$ for all α.

By the compactness of A we obtain that $\lim x_\alpha=x_0$ and $x_0\in A$. Having in mind the latter, we conclude $\langle B(t_0),x_0\rangle=b(t_0)$, i.e. $t_0\in T_A$. The assumption $B\in C^n(T)$ completes the proof.

The following assertions are evident :

6. For an arbitrary subset X of R^n we have $X^*=(\operatorname{ray} X^*)$ and $X^{**}=\overline{\operatorname{cone} X}$.
7. Let $K=\operatorname{rec}\Omega$, then

1.° $z\in K$ iff $\langle B(t), z\rangle\geqq 0$ for all $t\in T$, i.e. $K=W^*$.

2.° K is a pointed cone iff rank $W=n$.

Next we consider some useful properties of the cone of feasible directions. If $d \in D(\Omega, x)$, then $\langle B(t), d\rangle \geqq 0$ for each $t \in T_{\{x\}}$ but the opposite assertion is not valid, i.e.,

$$D(\Omega, x) \subset \{B(t) : t \in T_{\{x\}}\}^* = W^*_{\{x\}}.$$

2.2. Lemma. *If $x \in \Omega$, $y \in (x|\varepsilon)$ and $\langle B(t), y\rangle \geqq b(t)$ for all $t \in T_{(x/\varepsilon)}$, then $y \in \Omega$.*

Proof. Let $y \notin \Omega$. Then there exists $t_1 \in T$ such that $\langle B(t_1), y\rangle < b(t_1)$. But $\langle B(t_1), x\rangle \geqq b(t_1)$, so that there exists a point z from the segment $[x, y]$ with the property $\langle B(t_1), z\rangle = b(t_1)$, $z \in (x/\varepsilon)$, i.e. $t_1 \in T_{(x/\varepsilon)}$. This is, however, a contradiction to the assumption. The proof is completed.

2.3. Proposition. *If there exists $\delta > 0$ such that $\langle B(t), d\rangle \geqq 0$ for all $t \in T_{(x/\delta)}$, then $d \in D(\Omega, x)$ i.e.,*

$$W^*_{(x/\delta)} = \{B(t) : t \in T_{(x/\delta)}\} \subset D(\Omega, x).$$

Proof. Let $\|d\| = 1$ and $0 < \delta_1 < \delta$, then $x + \delta_1 d \in (x/\delta)$ $\langle B(t), x + \delta_1\rangle = \langle B(t), x\rangle + \delta_1 \langle B(t), d\rangle \geqq b(t)$ for every $t \in T_{(x/\delta)}$.

Following lemma 2.2. we get that $x + \delta_1 d \in \Omega$ i.e. $d \in D(\Omega, x)$. The proposition is proved.

2.4. Lemma. *If $\langle B(t), d\rangle > 0$ for all $t \in T_{\{x\}}$, then there exists $\delta > 0$ such that $\langle B(t), d\rangle \geqq 0$ whenever $t \in T_{(t/\delta)}$.*

Proof. Let us assume that for each $k = 1, 2, \ldots$ we can find $t_k \in T(x, 1/k)$ so that $\langle B(t_k), d\rangle < 0$.

Let $x_k \in (x, 1/k)$ be such that $\langle B(t_k), x_k\rangle = b(t_k)$. Since T is a compact space, there exists a net $\{t_\alpha\}_\alpha \subset \{t_k\}_{k=1}^\infty$ which converges to some $\bar{t} \in T$. Now we get immediately $\langle B(\bar{t}), d\rangle \leqq 0$ and $\langle B(\bar{t}), x\rangle = b(\bar{t})$, i.e. $\bar{t} \in T_{\{x\}}$. This contradiction leads to the assertion. The next theorem is a direct consequence of lemma 2.4 and proposition 2.3.

2.5. Theorem. *If $\langle B(t), d\rangle > 0$ for all $t \in T_{\{x\}}$, then $d \in D(\Omega, x)$.*

A characterisation of the extremal points of the set Ω is given by

2.6. Theorem. *If $x \in \text{ert}\,\Omega$, then for each $\varepsilon > 0$ holds* rank $W_{(x/\varepsilon)} = n$.

Proof. We assume there exists $\varepsilon > 0$ such that rank $W_{(x/\varepsilon)} < n$. Then we can find $h \in R^n$, $\|h\| = 1$ so that $\langle B(t), h\rangle = 0$ for all $t \in T_{(x/\varepsilon)}$.

By the continuity of the mappings B and b and by the compactness of the set $T \setminus T_{(x/\varepsilon)}$ we conclude that there exists $\tau > 0$ such that $\langle B(t), x\rangle \geqq b(t) + \tau$ for each $t \in T \setminus T_{(x/\varepsilon)}$. $/\tau = \min\{\langle B(t), x\rangle - b(t), t \in T \setminus T_{(x/\varepsilon)}\}/$.

Next if we restrict $0 < \delta < \tau/\max_{t \in T} \|B(t)\|$, then for all $t \in T \setminus T_{(x/\varepsilon)}$ we obtain

$$\langle B(t), \pm\delta h\rangle = \langle B(t), x\rangle + \delta\langle B(t), \pm h\rangle \geqq b(t) + \tau - \delta \max_{t \in T} \|B(t)\| \geqq 0,$$

i.e. $\pm h \in D(\Omega, x)$. Thus we get a contradiction. The theorem is proved.

3. Slater condition

For various profound results in the linear semi-infinite optimization the Slater condition turns out to be the most important. In this section we present some properties of the problems for which the Slater condition is fulfilled.

First we give some useful definitions :

1. Let $\beta=(B,b)$. We put

$$\Omega_\beta=\{x\in R^n : \langle B(t),x\rangle \geqq b(t) \quad \text{for every } t\in T\}.$$

2. $\psi=\{\beta=(B,b) : B\in C^n(T),\ b\in C(T)\}$.

3. In ψ we consider the norm $\|\beta\|=\|B\|_\infty+\|b\|_\infty$. This norm generates in ψ the natural Cartesian product topology.

3.1. Definition. *The Slater condition is fulfilled for $\beta\in\psi$, i.e. $\beta\in S$ if there exists $x\in R^n$ such that $\langle B(t),x\rangle > b(t)$ for all $t\in T$.*

3.2. Lemma. *If $\beta\in S$, then there exists $x\in R^n$ and $\delta>0$ such that $\langle B(t),x\rangle \geqq b(t)+\delta$ for every $t\in T$.*

Proof. This lemma is a consequence of the definition 3.1 and because T is a compact and (B,b) are continuous mappings.

3.3. Proposition. *If $\beta\in\psi$ and $\Omega_\beta\neq\emptyset$, then $\beta_\varepsilon=(B,b-\varepsilon)\in S$ for each $\varepsilon>0$.*

Proof. Let $x\in\Omega_\beta$ i.e. $\langle B(t),x\rangle\geqq b(t)$ for all $t\in T$. Then $\langle B(t),x\rangle \geqq b(t)-\varepsilon$ for every $t\in T$. Consequently, $\beta_\varepsilon\in S$. The proof is completed.

3.4. Corollary. *S is a dense subset of the set $\{\beta\in\psi : \Omega_\beta\neq\emptyset\}$.*

3.5. Theorem. $\beta\in S$ iff $\operatorname{int}\Omega_\beta\neq\emptyset$ and $O\bar{\in}\{[B(t),b(t)] : t\in T\}$.

Proof. a) Let $\beta\in S$. By lemma 3.2 there exists $x\in R^n$ and $\delta>0$ with $\langle B(t),x\rangle\geqq b(t)+\delta$ for all $t\in T$, from what follows that $[B(t),b(t)]\neq O^{n+1}$ for every $t\in T$.

Let $y=x+\varepsilon h$, where $h\in R^n$, $\|h\|=1$ and $\varepsilon>0$. Then

$$\langle B(t),y\rangle=\langle B(t),x\rangle+\varepsilon\langle B(t),h\rangle\geqq b(t)+\delta-\varepsilon\|B\|.$$

For $\varepsilon\in[o,\delta\|B\|]$ we have $\langle B(t),y\rangle\geqq b(t)$ for all $t\in T$, i.e. $y\in\Omega_\beta$. Thus $x\in\operatorname{int}\Omega_\beta$.

b) Let $x\in\operatorname{int}\Omega_\beta$ and $O\bar{\in}\{[B(t),b(t)]\, t\in T\}$. We assume that for some $t\in T$ holds $\langle B(t),x\rangle=b(t)$. Then for every $\varepsilon>0$ $\langle B(t),x-\varepsilon B(t)\rangle=\langle B(t),x\rangle-\varepsilon\langle B(t),\ B(t)\rangle=b(t)-\varepsilon\|B(t)\|^2<b(t)$, i.e., $x\bar{\in}\operatorname{int}\Omega_\beta$. But this is a contradiction. The theorem is proved.

An analogous result is reported in [7].

3.6. Theorem. *If $\beta\in S$ and $x\in\Omega_\beta$, then* $\operatorname{co} W_{\{x\}}$ *is a compact set and* $O\bar{\in}\operatorname{co} W_{\{x\}}$.

Proof. If $W_{\{x\}}=\emptyset$, then the assertion is evident.

Let $W_{\{x\}}\neq\emptyset$. Since $\beta\in S$, there is $x_0\in R^n$ with $\langle B(t),x_0\rangle>b(t)$ for every $t\in T$. Consequently, for the direction $d=x_0-x$ the relations $\langle B(t),d\rangle=\langle B(t),x_0\rangle-\langle B(t),x\rangle>b(t)-b(t)=0$ for all $t\in T_{\{x\}}$ hold.

If $z=\Sigma_{i=1}^{s}\alpha_i B(t_i)$, where $\alpha_i \geqq 0$, $\Sigma_{i=1}^{s}\alpha_i=1$ and $B(t_i)\in W_{\{x\}}\ i=\overline{1}, \overline{s}$, then $\langle z, d\rangle=\Sigma_{i=1}^{s}\alpha\langle B(t_i), d\rangle>0$. Thus $z\neq 0$, i.e. $O\bar{\in}\operatorname{co} W_{\{x\}}$.

Since $W^i_{\{x\}}$ is a compact set, we conclude that co $W_{\{x\}}$ is also a compact set. The proof is completed.

The following assertion is an evident consequence of the previous theorem.

3.7. Corollary. *If $\beta\in S$ and $x\in\Omega_\beta$, then* cone $W_{\{x\}}$ *is a closed, convex and pointed cone.*

3.8. Corollary. *If $\beta\in S$ and $x\in\Omega_\beta$, then $\bar{D}(\Omega_\beta, x)=\underline{W^*_{\{x\}}}$.*

Proof. With theorem 2.5. and corollary 3.7. we get $\operatorname{int} W^*_{\{x\}}\subset D(\Omega_\beta, x)\subset W^*_{\{x\}}$ and $\operatorname{int} W^*_{\{x\}}\subset\bar{D}(\Omega_\beta, x)\subset\bar{W}^*_{\{x\}}$ respectively. But $\operatorname{int} W^*_{\{x\}}=\bar{W}^*_{\{x\}}=W^*_{\{x\}}$ which implies $\bar{D}(\Omega_\beta, x)=W^*_{\{x\}}$. This completes the proof.

3.9. Corollary. $D^*(\Omega_\beta, x)=\overline{\text{cone}}\, W_{\{x\}}$ *and if $\beta\in S$, then* $D^*(\Omega_\beta, x)=\overline{\text{cone}}\, W_{\{x\}}$.

3.10. Corollary. *Let $K_\beta=\operatorname{rec}\Omega_\beta$. Then $K^*_\beta=\overline{\text{cone}}\, W$.*

3.11. Theorem. *S is an open subset of ψ.*

Proof. It has to be proved that for any $\beta\in S$ there exists $\delta>0$ so that $O_\delta(\beta)\subset S$.

Let $\beta\in S$ and $\bar{\beta}=(\bar{B}, b)\in O_\delta(\beta)$ for some $\delta>0$. There exists $x\in R^n$ and $\delta_1>0$ such that $\langle B(t), x\rangle-b(t)>\delta_1$ for all $t\in T$.

Whenever $0<\delta<\delta_1/(\|x\|+1)$ is fulfilled, then the relations $\langle\bar{B}(t), x\rangle-b(\bar{t}) =\langle\bar{B}(t)-B(t), x\rangle+\langle B(t), x\rangle-b(t)+b(t)-b(\bar{t})\geqq\delta_1-\delta\|x\|-\delta>0$ are valid. The proof is completed.

With the corollary 3.4. and the preceding theorem we obtain immediately that S is an open and dense subset of the set $\{\beta\in\psi:\Omega_\beta\neq\emptyset\}$.

4. Linear semi-infinite optimization problems

In this section we present several important results, concerning linear semi-infinite optimization, which are consequences from these given in paragraph 1.

For each $\sigma=(B, b, p)\in\theta$ we put

$\Omega_\sigma=\{x\in R^n:\langle B(t), x\rangle\geqq b(t)$ for every $t\in T\}$

$W_\sigma=\{B(t):t\in T\}$

$W_\sigma(A)=\{B(t):$ there exists $x\in A$ with $\langle B(t), x\rangle=b(t)\}$

$K_\sigma=\operatorname{rec}\Omega_\sigma$.

We say that $\sigma=(B, b, p)\in S$ if $\beta=(B, b)\in S$, i.e. the Slater condition is fulfilled.

Furthermore, we know that

$K_\sigma=\{B(t):t\in T\}^*=W^*_\sigma$

$K^*_\sigma=\overline{\text{cone}}\{B(t):t\in T\}=\overline{\text{cone}}\, W_\sigma$

$D(x)\subset\{B(t):t\in T_{\{x\}}\}^*=W^*_\sigma(x)$

$\overline{\text{cone}}\, W_\sigma(x)=\overline{\text{cone}}\{B(t):t\in T_{\{x\}}\}\subset D^*_\sigma(x)$.

If $\sigma\in S$,

$\bar{D}_\sigma(x)=W_\sigma^*(x)$ and $D_\sigma^*(x)=\text{cone}\, W_\sigma(x)$ respectively.

Using the theorem 1.2., we derive directly the following :

4.1. Theorem. *Let* $\Omega_\sigma \neq \emptyset$.

a) *A necessary condition for a solvability of the problem is* $p \in \overline{\text{cone}\, W_\sigma}$.

b) *A sufficient condition for a solvability of the problem* σ *is* $p \in \text{intcone}\, W_\sigma$.

Next, we formulate the Brosowski's theorem from

4.2. Theorem. *For* $\sigma \in S$ *the following two statements are equivalent:*

a) $x \in \text{Argmin}\, \sigma$

b) *There exist* $t_i \in T_{\{x\}}$, $\alpha_i \geqq 0$, $i=1,\dots,q$, $q \leqq n$ *such that*

$$p=\sum_{i=1}^{q} < \alpha_i B(t_i).$$

A similar result can be obtained using a theorem 1.4.

4.3. Theorem. *Let* $x \in \Omega_\sigma$. *If* $p \in \overline{\text{cone}}\, W_0(x)$, *then* $x \in \text{Argmin}\, \sigma$. *Conversely, if* $x \in \text{Argmin}\, \sigma$ *and in addition* $\sigma \in S$, *then* $p \in \text{cone}\, W_\sigma(x)$.

P r o o f. Since $\overline{\text{cone}}\, W_\sigma(x) \subset D_\sigma^*(x)$, then $p \in D_\sigma^*(x)$ and with theorem 1.4. we obtain $\sigma \in L$. To see the inverse statement we make use of the fact that if $\sigma \in S$, then cone $W_\sigma(x)=D_\sigma^*(x)$. This completes the proof.

The theorem 1.5. in the terms of σ reads as follows :

4.4. Theorem. *If* $x \in \Omega_\sigma$ *and* $p \in \text{intcone}\, W_\sigma(x)$, *then* $x=\text{Argmin}\, \sigma$.

5. Uniqueness of the solution in the linear semi-infinite optimization

In the last paragraph we prove that the majority (in the sense of Baire category) of the linear semi-infinite optimization problems have at most one solution. In this section we require that $|T| \geqq n$.

Let us denote

$$\Pi=\{B \in C^n(T). \quad \text{rank}\, W=n\}.$$

5.1. Proposition. *The set* Π *is an open and dense subset of* $C^n(T)$.

P r o o f. Let B be an arbitrary element of $C^n(T)$. We fix both $\varepsilon>0$ and the different points $\{t_i\}_{i=1}^n \subset T$.

Next we consider the vectors $\{B(t_i)\}_{i=1}^n$. Let $B(t_1),\dots,B(t_k)$ be the maximum number of linearly independent vectors in the previous system and N be the linear subspace generated of them.

The vectors $B_{k+1},\dots,B_n$ are basis of the orthogonal complement N. Now we define the continuous vector function B^0 in the following way :

$$B^0(t_i)=0, \quad i=1,\dots,k\ ; \quad B^0(t_i)=B_i, \quad i=k+1,\dots,n.$$

Furthermore, we put $B_0 = B + \varepsilon B^0 / \| B^0 \|$.

Evidently rank $\{B_0(t_i) : \; i = 1, \ldots, n\} = n$, i.e. $B_0 \in \Pi$.

Having in mind that $\| B - B_0 \| = \| \varepsilon B^0 / \| B \|$, we obtain that Π is a dense subset of $C^n(T)$.

With the continuity considerations we conclude that there exists a neighbourhood Y of B_0 such that for each $B_y \in Y$ holds $\{B_y(t_i) : i = 1, \ldots, n\} = n$. The proposition is proved.

We define the sets

$$\Gamma = \{\sigma \in \theta : B \in \Pi\}$$

$$S = \{\sigma \in \theta : \beta = (B, b) \in S\}$$

$$S\Gamma = S \cap \Gamma$$

$$\Phi = \{\sigma \in \theta : \Omega_\sigma \neq \emptyset\}.$$

By a proposition 5.1. and a theorem 3.11. we get that Γ is an open and dense subset of θ and S is an open and dense subset of Φ.

5.2. Theorem. *The set $S\Gamma$ is dense in L.*

Proof. Let $\sigma \in L$. We fix $\varepsilon > 0$. According to theorem 4.1 it follows that $\overline{p \in \text{cone}\, W_\sigma}$. Consequently, there exists $P^\varepsilon = \Sigma_{i=1}^{q'} \alpha_i B(t_i)$, where $\alpha_i > 0$, $i = 1, \ldots, q$, $q \leqq n$ and $\| p - p^\varepsilon \| < \varepsilon/8$.

Since $S\Gamma$ is an open and dense subset of Φ, $L \subset \Phi$ and with continuity considerations, we can find $\sigma_\varepsilon = (B_\varepsilon, b_\varepsilon, p_\varepsilon) \in S\Gamma$ such that $\| B - B_\varepsilon \| < \varepsilon / (4a \max_i \alpha_i)$, where $a \geqq q$ and $a \max_i \alpha_i \geqq 1$. $\| b - b_\varepsilon \| < \varepsilon/4$.

$$p_\varepsilon = \Sigma_{i=1}^{q} \alpha_i B_\varepsilon(t_i) + \varepsilon p / (8 \| p \|), \text{ where } p \in \text{intcone}\, W_{\sigma_\varepsilon} (\text{rank}\, W_{\sigma_\varepsilon} = n).$$

Obviously, $p_\varepsilon \in \text{intcone}\, W_{\sigma_\varepsilon}$; therefore by theorem 4.1. $\sigma_\varepsilon \in L$.

$$\| \sigma - \sigma_\varepsilon \| = \| B - B_\varepsilon \| + \| b - b_\varepsilon \| + \| p - p_\varepsilon \| < \varepsilon / (4a \max_i \alpha_i) + \varepsilon/4$$

$$+ \| p - p^\varepsilon \| + \| p^\varepsilon - p_\varepsilon \| < \varepsilon/4 + \varepsilon/4 + \varepsilon/8 + \Sigma_{i=1}^{q} \alpha_i \| B(t_i) - B_\varepsilon(t_i) \|$$

$$+ \| \varepsilon / (8 \| p \|) \| \leqq 5\varepsilon/8 + \Sigma_{i=1}^{q} \alpha_i \| B - B_\varepsilon \| + \varepsilon/8 \leqq 3\varepsilon/4 + \Sigma_{i=1}^{q} \alpha_i \varepsilon / (4a \max_i \alpha_i)$$

$$\leqq 3\varepsilon/4 + \Sigma_{i=1}^{q} \varepsilon/(4a) = 3\varepsilon/4 + \varepsilon q/(4a) \leqq 3\varepsilon/4 + \varepsilon/4 = \varepsilon.$$

The theorem is proved.

5.3. Definition. *Let $\sigma \in \theta$. We say that the extremal point $x \in \Omega_\sigma$ is of first kind, respectively second, if* rank $W_\sigma(x) = n$ *resp.* rank $W_\sigma(x) < n$.

5.4. Theorem. *Let $\sigma=(B,b,p)\in S\Gamma\cap L$ and the solution $x\in\operatorname{ext}\Omega_\sigma$ be an extremal point of the second kind. Then for every $\varepsilon>0$ there exists $\sigma_\varepsilon\in S\Gamma\cap L$ such that $x\in\operatorname{ext}\Omega_{\sigma_\varepsilon}$ is a solution of σ_ε, x is an extremal point of the first kind, and $\|\sigma_\varepsilon-\sigma\|<\varepsilon\|B\|$.*

Proof. We define the function

$$\varphi_\varepsilon(t)=\begin{vmatrix}1, & t\in T_\sigma[x/\varepsilon/2]\\ 0, & t\in T\setminus T_\sigma(x/\varepsilon)\end{vmatrix} \quad \text{and } 0<\varphi_\varepsilon(t)<1, \quad t\in T_\sigma(x/\varepsilon)\setminus T_\sigma[x/\varepsilon/2]$$

and the point

$$\sigma_\varepsilon=(B,b_\varepsilon,p)\in S\Gamma, \quad \text{where } b_\varepsilon=b+\varphi_\varepsilon[\langle B,x\rangle-b].$$

With theorem 4.3. we obtain that there exists $t_i\in T_\sigma(x)$ and $\alpha_i\geqq 0$, $i=1,\dots,q, q\leqq n$ such that $p=\Sigma_{i=1}^q\alpha_i B(t_i)$ (*).

It is clear that

$$T_\sigma(x)\subset T_{(x,\varepsilon)}=T_{\sigma_\varepsilon}(x).$$

By theorem 2.6. we have rank $W_\sigma(T_\sigma(x/\varepsilon))=n$, therefore $x\in\Omega_{\sigma_\varepsilon}$ is an extremal point of the first kind.

Furthermore, with theorem 4.3. and the condition (*) we conclude that x is a solution of the problem.

1. Let $t\in T_\sigma[x/\varepsilon/2]$, then for some $y\in[x/\varepsilon/2]$ holds $\langle B(t),y\rangle=b(t)$. $|b_\varepsilon(t)-b(t)|=|\varphi_\varepsilon(t)[\langle B(t),x\rangle-b(t)]|=|\langle B(t),x-y\rangle|\leqq\|B\|\,\|x-y\|\leqq\varepsilon\|B\|/2$.

2. Let $t\in T_\sigma(x/\varepsilon)\setminus T_\sigma[x/\varepsilon/2]$ then in the same way we get $|b(t)-b_\varepsilon(t)|\leqq\varepsilon\|B\|$.

But this means that $\|\sigma-\sigma_\varepsilon\|=\|b-b_\varepsilon\|\leqq\varepsilon\|B\|$. This completes the proof.

In the next theorem G. Nurnberger [4] gives a characterization of the so-called "strong unicity" of the solution of the linear semi-infinite optimization problems.

5.5. Theorem. *For $\sigma=(B,b,p)\in S\cap L$ the following two statements are equivalent:*

a) $\sigma\in\operatorname{int} LU$

b) *There exist $\{t_i\}_{i=1}^n\subset T_\sigma(x)$ and $\alpha_i\geqq 0$, $i=1,\dots,n$, so that $p=\Sigma_{i=1}^n\alpha_i B(t_i)$ and for every such points the vectors $p=B(t_0)$, $B(t_1)\dots B(t_{i-1})$, $B(t_{i+1}),\dots,B(t_n)$ are linearly independent.*

The main result in this paper states

5.6. Theorem. *The set LU contains an open and dense subset of L.*

Proof. Let $\sigma\in S\Gamma\cap L$ and the solution $x\in\operatorname{ext}\Omega_\sigma$ be an extremal point of first kind.

Since $\sigma\in L$, by theorem 4.3, there exist $\{t_i\}_{i=1}^n\subset T_\sigma(x)$, $\alpha_i>0$, $i=1,\dots,q, q\leqq n$, so that $p=\Sigma_{i=1}^q\alpha_i B(t_i)$.

Because of rank $W_\sigma(x)=n$, we can complement the system $B(t_1),\dots,B(t_q)$ to n linearly independent vectors $B(t_i)$, $i=1,\dots,n$.

Now, we put

$$p_\varepsilon=\sum_{i=1}^{q}\alpha_i B(t_i)+\frac{\varepsilon}{2(n-q)\|B\|}\sum_{i=q+1}^{n}B(t_i)\quad b_\varepsilon=b+c_\varepsilon\quad\text{where}\quad c_\varepsilon(t)=0,$$

$$t\in\{t_1,\ldots,t_n\};\quad 0<c_\varepsilon(t)\leqq\frac{\varepsilon}{2}t\in T\setminus\{t_1,\ldots,t_n\}\quad\text{and}\quad\sigma_\varepsilon=(B,b_\varepsilon,p_\varepsilon)$$

$$\|\sigma-\sigma_\varepsilon\|=\|p-p_\varepsilon\|+\|b-b_\varepsilon\|=\|c_\varepsilon\|+\frac{\varepsilon}{2(n-q)\|B\|}\|\sum_{i=q+1}^{n}B(t_i)\|$$

$$\leqq\varepsilon/2+\frac{\mathrm{e}(n-q)\|B\|}{(n-q)\|B\|}=\varepsilon/2+\varepsilon/2=\varepsilon.$$

Obviously $x\in\operatorname{ext}\Omega_{\sigma_\varepsilon}$ is a solution of σ_ε and for every $i=0,1,\ldots,n$ the vectors $p_\varepsilon=B(t_0)$, $B(t_1),\ldots,B(t_{i-1})$, $B(t_{i+1}),\ldots,B(t_n)$ are linearly independent, i.e. with theorem 5.5 the problem $\sigma_\varepsilon\in\operatorname{int}LU$.

To complete the proof we use the density given in the theorems 5.2 and 5.4. The theorem is proved.

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