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Periodic and Almost-periodic Solutions for a Nonlinear Operator Equation in Hilbert Space

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1. Introduction

In this paper, we shall study the existence and uniqueness of the periodic and almost-periodic solutions of the operator equation

$$(1.1)\qquad E_t+E_x A+A^* E+E_x E=G$$

in the class of the monotone operators on a real Hilbert space H.

The main results are contained in Theorems 3.1 and 4.1 and they are followed by applications to the Riccati equation (Remarks 3.1 and 4.1).

The existence and uniqueness of the solution of the Cauchy problem associated to the equation (1.1) are proved in [1, 2], by considering the Hamilton-Jacobi equation

$$(1.2)\quad \varphi_t(t,x)+\frac{1}{2}|\varphi_x(t,x)|^2+(Ax,\varphi_x(t,x))=g(t,x)\ ;\ x\in D(A), t\in[0,T].$$

As we shall see later (Remark 3.2) there is a close relationship between Equations (1.1) and (1.2).

2. Notations

The following spaces and mappings will be used :

1. — Let X, Y be two Banach spaces. For every $R>0$ denote by Σ_R the closed ball $\{x\in X\ ;\ |x|\leqq R\}$. We shall denote by the same symbol $|\cdot|$ the norms in X, Y and in the space $L(X,Y)$ of linear continuous from X to Y.

By $C^k(\Sigma_R\ ;\ Y)$ we shall denote the space of all mappings $E:\Sigma_R\to Y$ which are Fréchet differentiable up to order k on Σ_R such that $E^{(j)}$, $j=0, 1,\dots,k$ are continuous and bounded on Σ_R.

By $C^k_{\mathrm{Lip}}(\Sigma_R\ ;\ Y)$ we shall denote the space of all $E\in C^k(\Sigma_R\ ;\ Y)$ such that $E^{(k)}$ is Lipschitz on Σ_R, i.e.

$$\| E \|_{k,R} = \sup\left\{ \frac{| E^{(k)}(x) - E^{(k)}(y) |}{|x-y|} \; ; \; \begin{matrix} x \neq y, \\ x,y \in \Sigma_R \end{matrix} \right\} < +\infty. \tag{2.1}$$

The spaces $C^k(\Sigma_R\,;Y)$ and $C^k_{\text{Lip}}(\Sigma_R\,;Y)$ are endowed with the norms :

$$|E|_{C^k(\Sigma_R\,;Y)} = \sum_{j=0}^{k} |E|_{j,R}, \tag{2.2}$$

$$|E|_{C^k_{\text{Lip}}(\Sigma_R\,;Y)} = \sum_{j=0}^{k} |E|_{j,R} + \| E \|_{k,R}, \tag{2.3}$$

where

$$|E|_{j,R} = \sup\{ |E^{(j)}(x)| \; ; \; x \in \Sigma_R \}. \tag{2.4}$$

2. – If k is a natural number or zero, then $C^k(X, Y)$ is the space of all mappings $E : X \to Y$ which are k-times Fréchet differentiable and their restrictions to every Σ_R belong to $C^k(\Sigma_R\,;Y)$. The space $C^k_{\text{Lip}}(X\,;Y)$ is defined as the set of all $E \in C^k(X\,;Y)$ such that the restrictions of E at every Σ_R belong to $C^k_{\text{Lip}}(\Sigma_R\,;Y)$.

3. – We shall denote by $B([0,T]\,;C^k(X\,;Y))$ the space of all continuous mappings $E : [0,T] \times X \to Y$ such that

(i) The mapping

$$[0,T] \times X \to Y, \; (t,x) \to E^{(j)}(t,x)(z_1, \ldots, z_j) \text{ is}$$

continuous $j = 1, \ldots, k$, $z_1, \ldots, z_j \in X$.

(ii) We have

$$\sup\{ |E(t,.)|_{j,R} \; ; t \in [0,T] \} < +\infty \text{ for } j = 0, 1, \ldots, k, \; R > 0. \tag{2.5}$$

The space $B([0,T]\,;C^k_{\text{Lip}}(X, Y))$ is defined as the set of all $E \in B([0,T]\,;C^k(X, Y))$ such that

$$\sup\{ \| E(t,.) \|_{k,R} \; ; \; t \in [0,T] \} < +\infty, \; R > 0. \tag{2.6}$$

In the same way we define the spaces :

$$B(\mathsf{R}\,;C^k(X\,;Y)) \text{ and } B(\mathsf{R}\,;\, C^k_{\text{Lip}}(X\,;Y)).$$

4. – If $k = 0$ we shall simply write $C(X\,;Y)$, $C_{\text{Lip}}(X\,;Y)$ instead of $C^0(X\,;Y)$ and $C^0_{\text{Lip}}(X\,;Y)$ respectively.

3. Periodic solutions

We shall study the operator equation

$$(3.1)\qquad \begin{aligned} &E_t + E_x A + A^* E + E_x E = G, \qquad t \in [0, T] \\ &E(0, x) = E(T, x) \end{aligned}$$

in the class of convex functions on a real Hilbert space H.

First we assume that :

(a) H is a real Hilbert space with inner product $(.,.)$ and norm $|\cdot|$.

(b) The linear operator $-A$ is the infinitesimal generator of a C_0-semigroup of contractions e^{-tA} on H, verifying the estimate :

$$(3.2)\qquad |e^{-tA}| \leqq e^{-\omega t} \text{ with } \omega > 0, \ \forall\, t \geqq 0.$$

We shall denote by A^* the adjoint of A and by $D(A)$ the domain of A.

We introduce the following convex cone of $C(H;H)$

$$(3.3)\qquad \Pi = \{E \in C(H;H);\ E \text{ monotone and } E(0) = 0\}.$$

For every $E \in \Pi$ we denote by 1 the identity operator in H and by

(3.4) $\quad E_\varepsilon = E(1 + \varepsilon E)^{-1}$ the Yosida approximation of E and notice that

$$(3.5)\qquad E'_\varepsilon(x) = E'((1+\varepsilon E)^{-1}x)(1+\varepsilon E')(1+\varepsilon E)^{-1}x))^{-1}$$

(By E' we shall denote the Fréchet derivative of the operator E.)

Let $E \in \Pi$. Then for all $\varepsilon > 0$ and $R > 0$ one has (see e. g. [1], p. 43) :

$$(3.6)\qquad |E_\varepsilon|_{0,R} \leqq |E|_{0,R},$$

$$(3.7)\qquad |E_\varepsilon|_{1,R} \leqq |E|_{1,R}, \text{ if } E \in C^1(H;H),$$

$$(3.8)\qquad \| E_\varepsilon \|_{1,R} \leqq \| E \|_{1,R} + \varepsilon |E|_{1,R}, \text{ if } E \in C^1_{\text{Lip}}(H;H).$$

Consider the approximating equation

$$(3.9)\qquad E^\varepsilon(t,x) = e^{-t/\varepsilon}\ e^{-tA^*}\ E_0(e^{-tA}x) + \int_0^t e^{-(t-s)/\varepsilon}\ e^{-(t-s)A^*}[\varepsilon^{-1}\ E^\varepsilon_\varepsilon + G](s, e^{-(t-s)A}x)\,ds\ ;$$

$$t \in [0,T],\ x \in H,\ \varepsilon > 0,$$

and assume that

$$(c)\qquad \begin{cases} G \in B([0,T];\ \ C^1_{\text{Lip}}(H;H)), \\ G(t,.) \in \Pi, \qquad \forall\ t \in [0,T]. \end{cases}$$

For $E_0 \in C^1_{\text{Lip}}(H;H) \cap \Pi$, the equation (3.9) has a unique solution $E^\varepsilon \in B([0,T];\ C^1_{\text{Lip}}(H;H))$ with $E^\varepsilon(t,.) \in \Pi$, $\forall\ t \in [0,T]$ (see e. g. [1], p. 43 and [2], p. 274).

Moreover, one has

$$E^{\varepsilon}(0,x)=E_0(x) \quad \text{for all } x\in H.$$

We look for a T-periodic solution E^{ε} for equation (3.9) in $B([0,T]; C^1_{\text{Lip}}(H;H))$ with $E^{\varepsilon}(t,.)\in\Pi$, $\forall\ t\in[0,T]$.
To this purpose, define the operator $\Gamma : C(\Sigma_R;H)\ C(\Sigma_R;H)$

$$\Gamma E_0(\mathbf{x})=E^{\varepsilon}(T,x). \tag{3.10}$$

Hence

$$E^{\varepsilon}(T,x)=e^{-T/\varepsilon}\quad e^{-TA^*}E_0(e^{-TA}x) \tag{3.11}$$
$$+\int_0^T e^{-(T-s)/\varepsilon}\quad e^{-(T-s)A^*}[\varepsilon^{-1}E^{\varepsilon}_{\varepsilon}+G](s,e^{-(T-s)A}x)ds.$$

$E^{\varepsilon}(T,x)$ defined by (3.11) belongs to $C(\Sigma_R;H)$.
Since

$$|E_{\varepsilon}-\tilde{E}_{\varepsilon}|_{0,R}\leqq|E-\tilde{E}|_{0,R}$$

the operator Γ is contractant. Indeed, for all $E_0,\tilde{E}_0\in C(\Sigma_R;H)$ one has

$$|\Gamma E_0-\Gamma\tilde{E}_0|_{0,R}=|E^{\varepsilon}(T)-\tilde{E}^{\varepsilon}(T)|_{0,R}$$
$$\leqq e^{-T(\varepsilon-1+\omega)}|E_0-\tilde{E}_0|_{0,R}+|E^{\varepsilon}(0)-\tilde{E}^{\varepsilon}(0)|_{0,R}\,\varepsilon^{-1}\int_0^T e^{-(T-s)(\varepsilon^{-1}+\omega)}ds$$
$$=\frac{\varepsilon^{-1}+\omega e^{-T(\varepsilon^{-1}+\omega)}}{\varepsilon^{-1}+\omega}|E_0-\tilde{E}_0|_{0,R}\leqq\rho_{\varepsilon}|E_0-\tilde{E}_0|_{0,R},$$

where $0<\rho_{\varepsilon}<1$.
Applying the Banach fixed-point theorem it follows that for every $\varepsilon>0$, equation (3.9) has a unique T-periodic solution

$$E^{\varepsilon}\in B([0,T];\qquad C^1_{\text{Lip}}(H;H)),$$

$$E^{\varepsilon}(t,.)\in\Pi,\qquad\forall\ t\in[0,T]$$

which verifies $E^{\varepsilon}(0,x)=E_0(x)$ and hence

$$E^{\varepsilon}(t,x)=e^{-t/\varepsilon}\quad e^{-tA^*}\ E^{\varepsilon}(0,e^{-tA}x) \tag{3.9'}$$
$$+\int_0^t e^{-(t-s)/\varepsilon}\quad e^{-(t-s)A^*}[\varepsilon^{-1}E^{\varepsilon}_{\varepsilon}+G](s,e^{-(t-s)A}x)ds.$$

We shall show now that this solution is bounded in the norms :

$$|\cdot|_{0,R},\ \|\cdot\|_{0,R},\ |\cdot|_{1,R} \text{ and } \|\cdot\|_{1,R},\ \forall\ t\in[0,T].$$

By (3.9′) it follows via (3.2) and (3.6)

$$(3.12)\qquad |E^{\varepsilon}(t)|_{0,R}\leqq|E^{\varepsilon}(0)|_{0,R}\cdot e^{-t(\varepsilon^{-1}+\omega)}+\sup_{0\leqq t\leqq T}|G(t)|_{0,R}\cdot\int_0^t e^{-(t-s)(\varepsilon^{-1}+\omega)}ds$$

$$+\varepsilon^{-1}\int_0^t e^{-(t-s)(\varepsilon^{-1}+\omega)}|E^{\varepsilon}(s)|_{0,R}ds,\ t\in[0,T].$$

We set

$$P^{\varepsilon}(t)=|E^{\varepsilon}(t)|_{0,R}\cdot e^{t(\varepsilon^{-1}+\omega)}$$

and (3.12) becomes

$$(3.13)\ P^{\varepsilon}(t)\leqq|E^{\varepsilon}(0)|_{0,R}+\frac{e^{t(\varepsilon^{-1}+\omega)}-1}{\varepsilon^{-1}+\omega}\sup_{0\leqq t\leqq T}|G(t)|_{0,R}+\varepsilon^{-1}\int_0^t P^{\varepsilon}(s)ds.$$

Applying the Gronwall lemma, (3.13) yields

$$(3.14)\qquad |E^{\varepsilon}(t)|_{0,R}\leqq|E^{\varepsilon}(0)|_{0,R}\cdot e^{-\omega t}+\frac{1}{\omega}\sup_{0\leqq t\leqq T}|G(t)|_{0,R},\ t\in[0,T].$$

On the other hand, since $E^{\varepsilon}(0,x)=\mathsf{E}(T,x), x\in H$, by (3.14) it follows that :

$$(3.15)\qquad |E^{\varepsilon}(0)|_{0,R}\leqq\frac{\frac{1}{\omega}}{1-e^{-\omega T}}\sup_{0\leqq t\leqq T}|G(t)|_{0,R}.$$

Using (3.14) and (3.15) we deduce the estimate

$$(3.16)\qquad |E^{\varepsilon}(t)|_{0,R}\leqq C\sup_{0\leqq t\leqq T}|G(t)|_{0,R},\ t\in[0,T],$$

where $C=\dfrac{2}{\omega(1-e^{-\omega T})}$.

Since

$$\|E_{\varepsilon}\|_{0,R}\leqq\|E\|_{0,R},$$

in the same way we may show that :

$$(3.17)\qquad \|E^{\varepsilon}(t)\|_{0,R}\leqq C\sup_{0\leqq t\leqq T}|G(t)|_{0,R},\ t\in[0,T].$$

By (3.9′) we have

$$(3.18)\qquad E^{\varepsilon}_x(t,x)=e^{-t/\varepsilon}.e^{-tA^*}E^{\varepsilon}_x(0,e^{-tA}x)e^{-tA}$$

$$+\int_0^t e^{-(t-s)/\varepsilon}\,.\,e^{-(t-s)A^*}[\varepsilon^{-1}(E^\varepsilon_\varepsilon)_x+G_x](s,e^{-(t-s)A}x)\,.\,e^{-(t-s)A}ds.$$

By (3.18), using (3.2) and (3.7) it follows

$$(3.19)\qquad |E^\varepsilon(t)|_{1,R}\leqq|E^\varepsilon(0)|_{1,R}\cdot e^{-t(\varepsilon^{-1}+2\omega)}$$
$$+\sup_{0\leqq t\leqq T}|G(t)|_{1,R}\cdot\int_0^t e^{-(t-s)(\varepsilon^{-1}+2\omega)}ds$$
$$+\varepsilon^{-1}\int_0^t e^{-(t-s)(\varepsilon^{-1}+2\omega)}|E^\varepsilon(s)|_{1,R}ds,\quad t\in[0,T].$$

Using the same technics (based on the Gronwall lemma) as in the proof of (3.14), it follows that

$$(3.20)\qquad |E^\varepsilon(t)|_{1,R}\leqq|E^\varepsilon(0)|_{1,R}\cdot e^{-2\omega t}+\frac{1}{2\omega}\sup_{0\leqq t\leqq T}|G(t)|_{1,R}.$$

On the other hand, since $E^\varepsilon(0,x)=E^\varepsilon(T,x)$, $x\in H$, we obtain

$$(3.21)\qquad |E^\varepsilon(0)|_{1,R}\leqq\frac{1/2\omega}{1-e^{-2\omega T}}\sup_{0\leqq t\leqq T}|G(t)|_{1,R}.$$

By (3.20) and (3.21), it follows

$$(3.22)\qquad |E^\varepsilon(t)_{1,R}\leqq C'\,.\,\sup_{0\leqq t\leqq T}|G(t)|_{1,R},\quad t\in[0,T],$$

where

$$C'=\frac{1}{2\omega(1-e^{-2\omega T})}.$$

Reverting to (3.18) and using (3.2) and (3.8), we get

$$(3.23)\qquad \|E^\varepsilon(t)\|_{1,R}\leqq\|E^\varepsilon(0)\|_{1,R}\cdot e^{-t(\varepsilon^{-1}+2\omega)}$$
$$+\sup_{0\leqq t\leqq T}\|G(t)\|_{1,R}\int_0^t e^{-(t-s)(\varepsilon^{-1}+2\omega)}ds$$
$$+\varepsilon^{-1}\int_0^t e^{-(t-s)(\varepsilon^{-1}+2\omega)}[\|E^\varepsilon(s)\|_{1,R}+\varepsilon|E^\varepsilon(s)|_{1,R}]ds,\quad t\in[0,T].$$

Using (3.22), this yields as above

$$(3.24)\ \|E^\varepsilon(t)\|_{1,R}\leqq C_R(\sup_{0\leqq t\leqq T}\|G(t)\|_{1,R}+C'_R\sup_{0\leqq t\leqq T}|G(t)|_{1,R}),\quad t\in[0,T].$$

Now we define the mapping Φ from $B([0,T];C(H;H))$ to $B([0,T];C(H;H))$, by $G \xrightarrow{\Phi} E^{\varepsilon}$, where E^{ε} is the T-periodic solution to the approximating equation (3.9′).

Using the estimate (3.16), we get

$$(3.25)\qquad |\Phi(G_1)(t)-\Phi(G_2)(t)|_{0,R}=|E_1^{\varepsilon}(t)-E_2^{\varepsilon}(t)|_{0,R}$$

$$\leqq C_R\cdot \sup_{0\leqq t\leqq T}|G_1(t)-G_2(t)|_{0,R},\quad t\in[0,T].$$

On the other hand, we have (see e.g. [2], p.274)

$$(3.26)\ \varepsilon^{-1}(E(x)-E_{\varepsilon}(x))-E_x(x)E(x)=\int_0^1(E'(sx+(1-s)y_{\varepsilon}(x))E(y_{\varepsilon}(x))$$

$$-E'(x)E(x))ds,\quad x\in H,$$

where

$$y_{\varepsilon}(x)=(1+\varepsilon E)^{-1}x.$$

Using the estimates (3.16), (3.17), (3.22) and (3.24), (3.26) yields

$$(3.27)\qquad |\varepsilon^{-1}(E^{\varepsilon}(t)-E_{\varepsilon}^{\varepsilon}(t))-E_x^{\varepsilon}(t)E^{\varepsilon}(t)|_{0,R}$$

$$\leqq(\|E^{\varepsilon}(t)\|_{1,R}\cdot|E^{\varepsilon}(t)|_{0,R}+|E^{\varepsilon}(t)|_{1,R}\cdot\|E^{\varepsilon}(t)\|_{0,R})\varepsilon\leqq C_R''\cdot\varepsilon,\quad t\in[0,T].$$

Now coming back to equation (3.9′) we observe that for each $x\in D(A)$, the T-periodic solution $E^{\varepsilon}(t,x)$ is differentiable on $[0,T]$ and satisfies the equation

$$(3.28)\qquad E_t^{\varepsilon}+A^*E^{\varepsilon}+E_x^{\varepsilon}A+\varepsilon^{-1}(E^{\varepsilon}-E_{\varepsilon}^{\varepsilon})=G\ \text{ in }\ [0,T]\times D(A).$$

Using (3.27), we see that the equation (3.28) is equivalent to

$$E_t^{\varepsilon}+A^*E^{\varepsilon}+E_x^{\varepsilon}A+E_x^{\varepsilon}E^{\varepsilon}=G+R^{\varepsilon},$$

where

$$|R^{\varepsilon}(t)|_{0,R}\leqq\varepsilon C_R'',\quad \forall\ t\in[0,T].$$

Similarly we have

$$E_t^{\nu}+A^*E^{\nu}+E_x^{\nu}A+E_x^{\nu}E^{\nu}=G+R^{\nu},$$

where

$$|R^{\nu}(t)|_{0,R}\leqq\nu C_R'',\quad \forall\ t\in[0,T].$$

Using (3.25), we get

$$|E^{\varepsilon}(t)-E^{\nu}(t)|_{0,R}\leqq C_R\cdot C_R''(\varepsilon+\nu)\ ;\ t\in[0,T]\ ;\ \varepsilon,\nu>0$$

and therefore

$$\lim_{\varepsilon\to 0} E^\varepsilon = E \text{ exists in } B([0,T]\,;\,C(H\,;H)).$$

Definition 3.1. *The function E is called a* weak-*T*-periodic solution *to equation* (3.1) *if there exists a sequence*

$\{E^\varepsilon\}\subset B([0,T]\,;\,C^1_{\mathrm{Lip}}(H\,;H))$, with $E^\varepsilon(t,.)\in\Pi$, $\forall\ t\in[0,T]$, *such that for* $\varepsilon\to 0$
(i) $E^\varepsilon\to E$ in $B([0,T]\,;\,C(H\,;H))$;
(ii) $E^\varepsilon\in C^1([0,T]\,;\,C(D(A)\,;H))$; $E^\varepsilon(0,x)=E^\varepsilon(T,x)$;
(iii) $E^\varepsilon_t+A^*E^\varepsilon+E^\varepsilon_xA+E^\varepsilon_xE^\varepsilon\to G$ *in* $B([0,T]\,;\,C(H\,;H))$;
(iv) $\{E^\varepsilon(t,.),\ t\in[0,T],\ \varepsilon>0\}$ *is bounded in* $C^1_{\mathrm{Lip}}(H\,;H)$.
Summarising, we have proved the following theorem

Theorem 3.1. *Suppose that assumptions* (a), (b) *and* (c) *are satisfied. Then the equation* (3.1) *has a unique weak T-periodic solution E. Moreover,* $E\in B([0,T]\,;\,C_{\mathrm{Lip}}(H\,;H))$ *and for every* $x\in D(A)$ *the function* $E(t,x)$ *is absolutely continuous on* $[0,T]$. *In addition, the map* $G\to E^\varepsilon$ *is Lipschitz from* $B([0,T]\,;\,C(H\,;H))$ *to* $B([0,T]\,;\,C(H\,;H))$.

Remark 3.1. In the particular case in which G is a linear continuous self-adjoint operator on H, equation (3.1) reduces to the Riccati equation

$$\begin{cases} P'(t)+P(t)A+A^*P(t)+P^2(t)=Q(t),\\ P(0)=P(T).\end{cases}\tag{3.29}$$

In consequence, if

$$G(t,x)=\frac{1}{2}(Q(t)x,x),$$

where

$$Q(t)=Q^*(t),\ \ Q\geqq 0$$

and suppose that $Q(\cdot)x$ is continuous on $[0,T]$ for each $x\in H$, then assumptions (a), (b) and (c) are satisfied and Theorem 3.1 gives existence and uniqueness of a weak T-periodic solution P to Riccati equation (3.29).

Remark 3.2. If the operators : E^ε_x and G_x are self-adjoint, then

$$e^{-tA^*}E^\varepsilon_x(0,e^{-tA}x)e^{-tA},$$

$$e^{-(t-s)A^*}G_x(s,e^{-(t-s)A}x)e^{-(t-s)A} \text{ and}$$

$$e^{-(t-s)A^*}(E^\varepsilon_\varepsilon)_x(s,e^{-(t-s)A}x)e^{-(t-s)A} \text{ are self-adjoint.}$$

This implies that the operator E^ε_x (E^ε is the solution to (3.9′)) is self-adjoint and for $\varepsilon\to 0$ we obtain that E_x is self-adjoint.
Hence, for $G=\nabla g$ and $E=\nabla\varphi$, by using the relation

$$\varphi(t,x)=\int_0^1(E(t,\lambda x),x)\,d\lambda,$$

we see φ verifies the Hamilton-Jacobi equation

$$\varphi_t + \frac{1}{2}|\varphi_x|^2 + (Ax, \varphi_x) = g.$$

4. Almost-periodic solutions

We shall study here the existence and uniqueness of a weak almost-periodic solution to operator equation

$$(4.1) \qquad E_t + E_x A + A^* E + E_x E = G, \ t \in \mathsf{R},$$

in the class of convex functions on a real Hilbert space H. Throughout this section we shall assume that conditions (a), (b) are satisfied and we shall use the notations of Section 3.

Consider the approximating equation

$$(4.2) \qquad E^\varepsilon(t,x) = \int_{-\infty}^{t} e^{-(t-s)/\varepsilon} e^{-(t-s)A^*} [\varepsilon^{-1} E_\varepsilon^\varepsilon + G](s, e^{-(t-s)A} x) ds ;$$

$$t \in \mathsf{R}, \ x \in H, \ \varepsilon > 0,$$

where the function G satisfies the following conditions :

$$(4.3) \qquad G \in B(\mathsf{R} ; C^1_{\mathrm{Lip}}(H ; H)) ;$$

$$(4.4) \qquad G(t,.) \in \Pi, \ \forall \ t \in \mathsf{R} ;$$

(4.5) The function $G : \mathsf{R} \to C(H ; H)$ is almost-periodic, i.e.

(see e. g. [3]. p.122) it is strongly continuous with respect to t and satisfies the following property : for any $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that in any interval $[a, a+L] \subset \mathsf{R}$, one may find a number τ_ε, in such a way that $\sup_{t \in \mathsf{R}} |G(t+\tau_\varepsilon) - G(t)|_{0,R} < \varepsilon$.

Define the operator $\Gamma : X \to X$; where

$$X = \{E \in C(R ; C(\Sigma_R ; H)) ; \ E \text{ almost-periodic} ; \ E(t,.) \in \Pi, \ \forall \ t \in \mathsf{R}\}$$

and

$$(4.6) \qquad \Gamma(E(t,x)) = \int_{-\infty}^{t} e^{-(t-s)/\varepsilon} e^{-(t-s)A^*} [\varepsilon^{-1} E_\varepsilon + G](s, e^{-(t-s)A} x) ds.$$

X is a complete metric space with the metric :

$$d(E, \tilde{E}) = \sup_{t \in \mathsf{R}} |E(t) - \tilde{E}(t)|_{0,R}.$$

The function $e^{-(t-s)/\varepsilon} e^{-(t-s)A^*} [\varepsilon^{-1} E_\varepsilon + G](s, e^{-(t-s)A} x)$ belongs to $C([\theta, t] ; C(\Sigma_R ; H))$ and hence the integral

$$\int_{\theta}^{t} e^{-(t-s)/\varepsilon} e^{-(t-s)A^*} [\varepsilon^{-1} E_\varepsilon + G](s, e^{-(t-s)A}x) ds$$

is well defined. We have the estimate

$$|e^{-(t-s)/\varepsilon} e^{-(t-s)A^*} [\varepsilon^{-1} E_\varepsilon + G](s)|_{0,R}$$

$$\leqq e^{-(t-s)(\varepsilon^{-1}+\omega)} + \varepsilon^{-1} . \sup_{s\in \mathbb{R}} |E(s)|_{0,R} + e^{-(t-s)(\varepsilon^{-1}+\omega)} . \sup_{s\in \mathbb{R}} |G(s)|_{0,R},$$

and therefore the integral from the right hand of (4.6) is absolutely convergent and

$$\left| \int_{-\infty}^{t} e^{-(t-s)/\varepsilon} e^{-(t-s)A^*} [\varepsilon^{-1} E_\varepsilon + G](s) ds \right|_{0,R}$$

$$\leqq \frac{\varepsilon^{-1}}{\varepsilon^{-1}+\omega} \sup_{s\in \mathbb{R}} |E(s)|_{0,R} + \frac{1}{\varepsilon^{-1}+\omega} \sup_{s\in \mathbb{R}} |G(s)|_{0,R}.$$

(We recall that the almost-periodic functions are bounded over R.) The right hand of (4.6) is an almost-periodic function since it may be written as a sum of two almost-periodic functions (in virtue of the almost-periodicity of E and G) and the set of all almost-periodic functions is a linear space.

The strong continuity of the right hand of (4.6) results from the fact that any almost-periodic function is uniform continuous on the real line (see e.g. [3], p. 124).
Since

$$|E_\varepsilon - \tilde{E}_\varepsilon|_{0,R} \leqq |E - \tilde{E}|_{0,R},$$

the operator Γ is contractant. Indeed, for all $E, \tilde{E} \in X$, one has

$$|\Gamma(E(t)) - \Gamma(\tilde{E}(t))|_{0,R} \leqq \frac{\varepsilon^{-1}}{\varepsilon^{-1}+\omega} \sup_{t\in \mathbb{R}} |E(t) - \tilde{E}(t)|_{0,R} \leqq \rho_\varepsilon . \sup_{t\in \mathbb{R}} |E(t) - \tilde{E}(t)|_{0,R},$$

where $0 < \rho_\varepsilon < 1$. Hence

$$d(\Gamma(E), \Gamma(\tilde{E})) \leqq \rho_\varepsilon . d(E, \tilde{E}).$$

Applying the Banach fixed-point theorem and the fact that the operator $(1+\varepsilon \mathrm{E})^{-1}$ who intervenes in the definition of E_ε is nonexpansive on H, it follows that for every $\varepsilon > 0$, Equation (4.2) has a unique almost-periodic solution,

$$E \in B(\mathbb{R}; C^1_{\mathrm{Lip}}(H; H)), \quad E^\varepsilon(t, .) \in \Pi, \ \forall\, t \in \mathbb{R}.$$

Now, we shall show that this solution is bounded with respect to ε in the norms : $|\cdot|_{0,R}$, $\|\cdot\|_{0,R}$, $|\cdot|_{1,R}$ and $\|\cdot\|_{1,R}$; $\forall\, t \in \mathbb{R}$.

By (4.2), it follows that

$$(4.7) \qquad |E^\varepsilon(t)|_{0,R} \leqq \sup_{t\in \mathbb{R}} |G(t)|_{0,R} \cdot \int_{-\infty}^{t} e^{-(t-s)(\varepsilon^{-1}+\omega)} ds$$

$$+\varepsilon^{-1}\int_{-\infty}^{t} e^{-(t-s)(\varepsilon^{-1}+\omega)}|E^{\varepsilon}(s)|_{0,R}\,ds,\quad t\in\mathrm{R}.$$

We set

$$P^{\varepsilon}(t)=|E^{\varepsilon}(t)|_{0,R}\cdot e^{t(\varepsilon^{-1}+\omega)}$$

and (4.7) becomes

$$P^{\varepsilon}(t)\leqq\frac{e^{t(\varepsilon^{-1}+\omega)}}{\varepsilon^{-1}+\omega}\sup_{t\in\mathrm{R}}|G(t)|_{0,R}+\varepsilon^{-1}\int_{-\infty}^{t}P^{\varepsilon}(s)ds. \tag{4.8}$$

Applying the Gronwall lemma, (4.8) yields

$$|E^{\varepsilon}(t)|_{0,R}\leqq\frac{1}{\omega}\sup_{t\in\mathrm{R}}|G(t)|_{0,R},\quad t\in\mathrm{R}. \tag{4.9}$$

Since

$$\|E_{\varepsilon}\|_{0,R}\leqq\|E\|_{0,R},$$

one shows, identically as in the proof of (4.9), that

$$\|E^{\varepsilon}(t)\|_{0,R}\leqq\frac{1}{\omega}\sup_{t\in\mathrm{R}}\|G(t)\|_{0,R},\quad t\in\mathrm{R}. \tag{4.10}$$

By (4.2) we have

$$E^{\varepsilon}_{x}(t,x)=\int_{-\infty}^{t} e^{-(t-s)/\varepsilon}\cdot e^{-(t-s)A^{*}}\cdot[\varepsilon^{-1}(E^{\varepsilon}_{\varepsilon})_{x}+G_{x}](s,e^{-(t-s)A}x)e^{-(t-s)A}ds. \tag{4.11}$$

This yields

$$|E^{\varepsilon}(t)|_{1,R}\leqq\sup_{t\in\mathrm{R}}|G(t)|_{1,R}\cdot\int_{-\infty}^{t} e^{-(t-s)(\varepsilon^{-1}+2\omega)}ds \tag{4.12}$$

$$+\varepsilon^{-1}\int_{-\infty}^{t} e^{-(t-s)(\varepsilon^{-1}+2\omega)}|E^{\varepsilon}(s)|_{1,R}ds,\quad t\in\mathrm{R}.$$

Using the same technics (based on the Gronwall lemma) with which we have proved (4.9), it follows that

$$|E^{\varepsilon}(t)|_{1,R}\leqq\frac{1}{2\omega}\cdot\sup_{t\in\mathrm{R}}|G(t)|_{1,R},\quad t\in\mathrm{R}. \tag{4.13}$$

Coming back to (4.11) we get

$$\|E^{\varepsilon}(t)\|_{1,R}\leqq\sup_{t\in\mathrm{R}}\|G(t)\|_{1,R}\cdot\int_{-\infty}^{t} e^{-(t-s)(\varepsilon^{-1}+2\omega)}ds \tag{4.14}$$

$$+\varepsilon^{-1}\int_{-\infty}^{t} e^{-(t-s)(\varepsilon^{-1}+2\omega)}[\| E^{\varepsilon}(s)\|_{1,R}+\varepsilon|E^{\varepsilon}(s)|_{1,R}]\,ds,\quad t\in\mathrm{R}.$$

Using (4.13), this yields as above

$$(4.15)\qquad \| E^{\varepsilon}(t)\|_{1,R}\leqq\frac{1}{2\omega}(\sup_{t\in\mathrm{R}}\| G(t)\|_{1,R}+\frac{1}{2\omega}\sup_{t\in\mathrm{R}}|G(t)|_{1,R}),\quad t\in\mathrm{R}.$$

Now we define the mapping Φ from $B(\mathrm{R}\,;\,C(H\,;H))$ to $B(\mathrm{R}\,;C(H\,;H))$, by

$G\xrightarrow{\Phi}E^{\varepsilon}$, where E^{ε} is the almost-periodic solution to the approximating equation (4.2).

Using the estimate (4.9), we get

$$(4.16)\qquad |\Phi(G_1)(t)-\Phi(G_2)(t)|_{0,R}=|E_1^{\varepsilon}(t)-E_2^{\varepsilon}(t)|_{0,R}$$
$$\leqq\frac{1}{\omega}\sup_{t\in\mathrm{R}}|G_1(t)-G_2(t)|_{0,R},\quad t\in\mathrm{R}.$$

On the other hand, we have

$$(4.17)\qquad \varepsilon^{-1}(E(x)-E_{\varepsilon}(x))-E_x(x)E(x)$$
$$=\int_0^1(E'(sx+(1-s)y_{\varepsilon}(x))\,E(y_{\varepsilon}(x))-E'(x)\,E(x))\,ds,\quad x\in H,$$

where

$$y_{\varepsilon}(x)=(1+\varepsilon E)^{-1}x.$$

Using estimates (4.9), (4.10), (4.13) and (4.15), (4.17) yields

$$(4.18)\qquad |\varepsilon^{-1}(E^{\varepsilon}(t)-E_{\varepsilon}^{\varepsilon}(t))-E_x^{\varepsilon}(t)E^{\varepsilon}(t)|_{0,R}$$
$$\leqq(\| E^{\varepsilon}(t)\|_{1,R}\cdot|E^{\varepsilon}(t)|_{0,R}+|E^{\varepsilon}(t)|_{1,R}\cdot\| E^{\varepsilon}(t)\|_{0,R})\varepsilon\leqq C_R\,\varepsilon,\quad t\in\mathrm{R}.$$

Now coming back to the equation (4.2), we observe that, for each $x\in D(A)$, the almost-periodic solution $E^{\varepsilon}(t,x)$ is differentiable on R and satisfies the equation

$$(4.19)\qquad E_t^{\varepsilon}+A^*E^{\varepsilon}+E_x^{\varepsilon}A+\varepsilon^{-1}(E^{\varepsilon}-E_{\varepsilon}^{\varepsilon})=G.$$

Using (4.18), we see that the equation (4.19) is equivalent to

$$E_t^{\varepsilon}+A^*E^{\varepsilon}+E_x^{\varepsilon}A+E_x^{\varepsilon}E^{\varepsilon}=G+R^{\varepsilon},$$

where

$$|R^{\varepsilon}(t)|_{0,R}\leqq\varepsilon C_R,\qquad t\in\mathrm{R}.$$

Similarly we have

$$E_t^\nu + A^* E^\nu + E_x^\nu A + E_x^\nu E^\nu = G + R^\nu,$$

where

$$|R^\nu(t)|_{0,R} \leqq \nu C_R, \qquad t \in \mathrm{R}.$$

Then by (4.16), we get

$$|E^\varepsilon(t) - E^\nu(t)|_{0,R} \leqq \frac{C_R}{\omega}(\varepsilon + \nu)\ ;\ t \in \mathrm{R}\ ;\ \varepsilon, \nu > 0$$

and therefore

$$\lim_{\varepsilon \to 0} E^\varepsilon = E \text{ exists in } B(\mathrm{R}\ ;\ C(H\ ;H)).$$

Definition 4.1. *The almost-periodic function* $\mathrm{E}: \mathrm{R} \to C(H\ ;H)$ *is called* a weak solution *to Equation* (4.1), *if there exists a sequence* $\{E^\varepsilon\} \subset B(\mathrm{R}\ ;\ C^1_{\mathrm{Lip}}(H\ ;H))$ *with* $E^\varepsilon(t,.) \in \Pi$ *such that for* $\varepsilon \to 0$,

(j) $E^\varepsilon \to E$ in $B(\mathrm{R}\ ;\ C(H\ ;H))$.

(jj) $E^\varepsilon \in C^1(\mathrm{R}\ ;\ C(D(A)\ ;H)), E^\varepsilon : \mathrm{R} \to C(H\ ;H)$ is almost-periodic.

(jjj) $E_t^\varepsilon + A^* E^\varepsilon + E_x^\varepsilon A + E_x^\varepsilon E^\varepsilon \to G$ in $B(\mathrm{R}\ ;\ C(H\ ;H))$.

(jv) $\{E^\varepsilon(t,.),\ t \in \mathrm{R},\ \varepsilon > 0\}$ is bounded in $C^1_{\mathrm{Lip}}(H\ ;H)$.

Summarising, we have proved the following theorem

Theorem 4.1. *Suppose that assumptions* (*a*), (*b*) *and* (4.3)-(4.5) *are satisfied. Then the equation* (4.1) *has a unique weak, almost-periodic solution E. Moreover,* $E \in B(\mathrm{R}\ ;\ C_{\mathrm{Lip}}(H\ ;H))$ *and for every* $x \in D(A)$ *the function* $E(t,x)$ *is absolutely continuous on* R. *In addition, the map* $G \to E^\varepsilon$ *is Lipschitz from* $B(\mathrm{R};\ C(H; H))$ *to* $B(\mathrm{R}\ ;\ C(H\ ;H))$.

Remark 4.1. In the particular case in which G is a linear continuous self-adjoint operator on H, Equation (4.1) reduces to the Riccati equation

$$P'(t) + P(t)A + A^*P(t) + P^2(t) = Q(t). \tag{4.20}$$

In consequence, if

$$G(t,x) = \frac{1}{2}(Q(t)x, x), \text{ where } Q(t) = Q^*(t),\ Q \geqq 0,\ Q(\cdot)x \text{ is}$$

continuous on R for each $x \in H$ and Q is almost-periodic then the assumptions (a), (b) and (4.3)-(4.5) are satisfied and Theorem 4.1 gives the existence and uniqueness of a weak almost-periodic solution P to the Riccati equation (4.20).

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