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Fixed Points and a Condition Concerning Selfmappings of Ordered Sets

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Setting up some results of Kurepa Đ. 1964:4, 1975:2, we prove a general fixed point theorem relative to ordered sets $(E, \leqq)$ and involving a condition presenting a special kind of surjections on an important part of $(E, \leqq)$.

1. Let (1 : 1) $(E, \leqq)$ be an ordered set; in general (1 : 1) is not an ordered chain. For any selfmapping $s{:}E\to E$ it is natural to consider following 3 parts of E:

(1 : 2) $E^s := \{x \therefore x\in E \,\&\, x\leqq sx\}$

(1 : 3) $E_s := \{x \therefore x\in E \,\&\, sx\leqq x\}$

(1 : 4) $E(s) := \{x \therefore x\in E \,\&\, x \parallel sx\}$.

In particular, increasing (decreasing) s were studied and one remarked that decreasing selfmappings d of $(E, \leqq)$ behave otherwise than the increasing ones. On the other hand, any d has a peculiar property that E^d, E_d are respectively a left piece of $(E, \leqq)$ and a right piece of $(E, \leqq)$ and that

(1 : 5) $dE^d \subset E_d,\ dE_d \subset E^d.$

1 : 6. The sets (1 : 2)-(1 : 4) could be any ordered sets; in particular, (1 : 4) could be any ordered set. As a matter of fact, if (A, B, C) is any 3-un of pairwise disjoint sets, we could consider: their union E ordered in such a way that $A<B$ and $A \parallel C \parallel B$ and a function d on E such that dA, dB, dC be singletons in B, A, C respectively; then obviously the sets E^d, E_d, $E(d)$ are A, B, C respectively.

1 : 7. Sets R_0X, R^0X (X being ordered) are defined as the set of all initial and terminal points of X respectively.

1 : 8 **Lemma:** *For any decreasing selfmapping d* of $(E, \leqq)$ *the set* $I(E; \leqq, d)$ *of all invariant or fixed points* of E *satisfies*

$$I((E, \leqq), d) = E^d \cap E_d$$

and is an (empty or non empty) antichain in $(E, \leqq)$.

The cardinal number of I could be any given power m.

The proof of the first part of 1 : 8 is obvious. As to the power of I, let us consider for any power m any set M of power m; let $M\times R:=E$, be ordered by $\leqq$ where for (x, y), $(x', y')\in E$ one defines

$$(x, y)\leqq(x', y')\leftrightarrow x=x' \quad y=y' \text{ in } (R, \leqq),$$

$(x, y)\,\|\,(x', y')\leftrightarrow x\neq x'$, and
$d(x, y):=(x, -y)$;
then $(E, \leqq)$ is degenerate (i.e. every subcone is a chain) and $I((E, \leqq), d)=\{(x, 0)\therefore x\in M\}$ is of power m (s.n. 2.25 in Kurepa 1975:2).

1:9. What more natural than to consider a special case of (1:5) in which $\subset$ stands for $=$? Doing so, we are going to prove the following.

2. Theorem. *For any non-empty ordered set $(E, \leqq)$ and any decreasing selfmapping d in $(E, \leqq)$, such that*

$$(PS) \qquad dE^d=E_d, \quad dE_d=E^d \quad \text{and}$$
$$(2:1) \qquad E^d\neq v \ (=\textit{vacuous set})$$

the following statements are pairwise equivalent:
(F) d has a unique fixed point in E, i.e. the equality $dx=x$ has a unique solution in E: the set $I((E, \leqslant), d)$ is a singleton;
$(S=i)$ $S:=\sup E^d$ and $i:=\inf E_d$ exist in $(E, \leqq)$ and are equal;
(S) $S:=\sup E^d$ exists in $(E, \leqq)$ and satisfies $S\geqq dS$; thus $S\in E_d$;
(i) $i:=\inf E_d$ exists in $(E, \leqq)$ and satisfies $i\leqq di$, thus $i\in E^d$.

3. Proof.

3:1. $F\Rightarrow(S=i)$. As a matter of fact, by hypothesis, the set I is a singleton $\{b\}$; by 1:8 the point b is the unique point of E which belongs to E^d and E_d. Since E^d is a left piece and E_d is a right piece of $(E, \leqq)$ such that $E^d\leqq E_d$, one has necessarily $b=S$ and $b=i$, thus $(S=i)$ is holding.

3:2. $(S=i)\Rightarrow(S)$. Since $(S=i)$ says that S, i exist and since $dE^d=E_d$ one has in particular $dS=i$; this equality joint to the assumption $S=i$ implies $dS=S$, $dS\leqq S$.

3:3. $(S)\Rightarrow(i)$ because by hypothesis S exists and belongs to E_d the operation d carries E^d onto E_d, and necessarily $dS=i$ and $di=S$ because $dE_d=E^d$.

3:4. $(i)\Rightarrow F$. Since E_d is a proper right piece of $(E, \leqq)$ and by hypothesis the infimum $i:=\inf E_d$ exists and is in E^d as a maximum; consequently $S=i$. This relation jointly with $di=S$ implies $di=i$; thus I is non empty; I is the singleton because i is the maximum of E^d and because $E^d\supset I$.
The previous conclusions 3:1-3:4 finish the proof of Theorem 2.

4. An example. Let us consider a chain, say $L:=R[0, 1]$ — real numbers $\geqq 0$ and $\leqq 1$; let d be defined by $d\,[0, 1/2]=\{2/3\}$ and $d(1/2, 1]=\{1/3\}$; then $L^d=[0, 1/2]$, $L_d=(1/2, 1]$, $S=1/2=1/2$, $dL^d=\{1/3\}$, $dL_d=\{2/3\}$, $I=v$. Consequently, Theorem 2 need not hold for decreasing mappings which do not satisfy (PS).

5. Remark on the conditions (2:1). For any selfmapping s of $(E, \leqq)$ into itself the conditions

$$(PS) \qquad sE^s=E_s, \quad sE_s=E^s$$

mean that s is surjective in the set $E^s \cup E_s$ which is the complement of the set $E(s)$ defined in (1, 4); thus (PS) means that s is partially surjective in $(E, \leqq)$.

6. If $(E, \leqq)$ is a chain $(L, \leqq)$, then the conditions (PS) represent a particular surjection of L, and $L^d | L_d$ is a cut of $(L, \leqq)$. One proves readily that every surjection s of $(L, \leqq)$ carries L^s onto L_s as well as L_s onto L^s. Therefore, if we specify $(E, \leqq)=(L, \leqq)$, Theorem 2 becomes

6 : 1. **Theorem.** *For any gapless (≡conditionally complete) ordered chain* $(L, \leqq) \neq v$ *and any decreasing selfsurjection* d *on* L *the following are equivalent:*

6 : 2 *d has a unique fixed point in L;*

6 : 3 *S and i exist and are equal;*

6 : 4 *S exists and is $\geqq dS$*

6 : 5 *i exists and is $\leqq di$* (as to (6 : 2)↔(6 : 3), v. also Diaconescu 1986).

7. PS-condition (partial surjection condition) versus decreasing mapping.

7 : 1. For a selfmapping $s:(E, \leqq) \rightarrow E$ we considered in n° 2 the conditions (PS)
PS $\quad sE^s = E_s$, $sE_s = E^s$; they are called partial surjection conditions because they express the surjectivity of s in the set $E^s \cup E_s$.

7 : 2. Moreover, PS-conditions express a partial decreasing phenomenon because if $x \in E$ then

$$x \leqq sx \text{ implies } sx \geqq s^2x \text{ (i. e. if } x \in E^s \text{ then } fx \in E_s),$$
$$s \geqq sx \text{ implies } sx \leqq s^2x \text{ (i. e. if } x \in E_s \text{ then } fx \in E^s).$$

In other words, PS expresses that s is decreasing for every s-tied pair x, $sx \in E$.

7 : 3. On the other hand, one knows that for every decreasing automapping d of $(E, \leqq)$ the sets E^d, E_d are respectively a proper left and right piece of $(E, \leqslant)$. How to get this fact starting with any PS-selfmapping s on $(E, \leqq)$? In other words how to express that E^s is a left piece, i. e. that

$$a < b \in E^d \Rightarrow a \in E^d, \text{ i. e. that}$$

(7 : 4) $\quad a < b \leqq sb \Rightarrow a \leqq sa$? Where to set sa? It is sufficient to prolonge the 3-term-relation (7 : 4) by $\leqslant sa$; one gets

(7 : 5) $\quad a < b \leqq sb \leqq sa$, i. e. $a \leqq sa$ thus $a \in E^s$ (procedure of right extending of $a < b \leqq sb$ by $\leqq sa$).

Analogously, for E_s to express that E_s is a right piece of $(E, \leqq)$: If $a \in E_s$ and $a < b$ then $b \in E_s$, i. e. if

(7 : 6) $\quad fa \leqq a \leqq b$, then the simple procedure is to set down the missing fb by intercalation of fb isotonily between a and b in (7 : 5); one gets

(7 : 7) $\quad fa \leqq a \leqq fb \leqq b$, thus in particular $fb \leqq b$, i. e. $b \in E_s$. By simple extensions (7 : 5) (on setting sa) and (7 : 7) (on setting sb), s becomes decreasing in E^s and in E_s.

(7 : 8)If moreover $E^s \leqq E_s$ (such a situation occurs for every ordered chain $(E, \leqq)$), then by the above extensions (7 : 5) and

(7 : 7)the given PS-mapping s becomes a decreasing surjection on the complement of the set $E(s)$ of all points $x \in E$ such that $x \| sx$.

No rules are known on the behaviour of the set $E(s)$.

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