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Embeddings of a Poset into Cut Systems of its Directed Subsets

J. N. Stabakis

Presented by St. Negrepontis

§ 1. Introduction

It is the main purpose of this paper to show that an invariant extension system of a poset could be considered as a completion system of the space by a topological point of view, and that the whole process is analogous to this one of the linear topological spaces.

Let $(E, \leqq)$ be an order structure, $<$ the strict relation. Each subset of E can be considered as a join of directed (right or left) sets, maximal with respect to set inclusion; this principle is preserved through all the present text.

We restate some definitions.

A f^*-cut (cf. [7]) is a couple (A, B) of subsets of E satisfying the following properties:

a) The subsets A, B are directed subsets right and left respectively;

b) $(\forall\ x\in A)\ (\forall\ y\in B)\ [x<y]$;

c) The A and B are maximal with respect to set inclusion;

d) There exist linear subsets $\partial_1 \subseteq A$, $b_1 \subseteq B$ such that ∂_1 *is cofinal to* A *(for the* $\leqq$*) and* b_1 *to* B *(for the* $\geqq$*).*

The subsets A, B are called the classes of the f^*-cut (lower and upper resp.); if the classes have ends (sup A and inf B) the f^*-cut is called f^*-jump, if they have not any end it is called f^*-gap. The set of the lower (resp. upper) classes without ends is symbolized by $L^-(E)$ resp. $L^+(E)$) and the set of f^*-gaps by $L^*(E)$.

Symbolize:

$$\downarrow y:=\{x\in E : x\leqq y\},\quad \downarrow \mathring{y}:=\downarrow y\backslash\{y\}$$

$$\downarrow Y:=\cup\{\downarrow y : y\in Y\},\quad Y\subseteq E$$

$$\mathscr{M}E:=\{\downarrow y : y\in E\}\quad \text{and}\quad \mathscr{A}E:=\{\downarrow Y : Y\subseteq E\}.$$

We also define $\uparrow y:=\{x\in E : y\leqq x\}$ and in a similar way the $\uparrow \mathring{y}$ and $\uparrow Y$.

The members of $\mathscr{M}E$ are called the principal ideals, and these of $\mathscr{A}E$ the lower ends of E. Directed lower ends are often referred as ideals (see f.e. [5] and [4]).

The system of all ideals of E is symbolized by JE.

A standard extension of E is called a system Q with $\mathcal{M}E \subseteq Q \subseteq \mathcal{A}E$; if the system is closed (under arbitrary intersections), it is called a standard complétion of E. Finally an invariant extension is called a functor $\mathcal{I}$, assigning to each poset E a standard extension $\mathcal{I}E$ such that for every isomorphism $\varphi: E \to Q$, $Y \in \mathcal{I}E$ implies $\varphi[Y] \in \mathcal{I}Q$.

After a result of M. Erné ([5] Cor. 3 and the relations of p. 201) every isomorphism of E into JE is trivial, i.e. it is a composition of automorphisms of E and of the trivial isomorphism $n: E \to \mathcal{M}E$ where $y \to \downarrow y$. Thus, if we suppose that each maximal directed subset is cofinal to a linear subset, we could define the set $\mathcal{E}^- := E \cup L^-(E)$ as the system JE. On the other hand, the system $\mathcal{E}^-$ is a standard extension system (in fact it is an invariant one) and it coincides with the standard extension system l constructed by the strictly monotone chains of E (see [7], p. 82).

The meaning of the completion in the above consideration is that in a closure extension system Q, each subset has infimum. We attempt to face the completion by the aspect of a topology which is generated by the sets of the form $\downarrow \mathring{y}$ and $\uparrow \mathring{y}$ for each point y of the space (Alexandroff topology), and where the increasing sets (i.e. sets of the form $\{\uparrow Y$, Y subset of the space$\}$ are used as well as lower ends; we especially refer to the set $E^* = E \cup L^-(E) \cup L^+(E)$, the extension system of f^*-cuts of F (cf. [7], def. 1).

It is shown that E has a topological embedding into E^*, while E is embedded into another f^*-cut extension system, which has a character of completion (§ 2). This completion property permits us to generalize two results of Kowalsky. The last paragraph is devoted to three problems: a problem of the "smallest" completion, connection and order and a case where linear subsets are cofinal to directed sets.

For the first problem the solution is strict in the case of MacNeille completion, while [3] is referred to the second problem in the case of linear space.

§ 2. The embedding theorem

Let E be an ordered space and E^* its extension system of f^*-cuts; the topology we consider here is the Alexandroff one.

Proposition 1. ([7]) *The induced topology by E^* into E is the initial one and E is dense in E^*.*

In fact, the subsets of E^* of the form $\uparrow \mathring{y}$ (or $\downarrow \mathring{y}$) have as trace on E, open set. On the other hand, for each $x \in E^* \setminus E$ and for each neighbourhood O_x of x there exist a, b in E such that $a < x < b$.

The extension system of f^*-cuts has not f^*-gaps. So if $\mathcal{I}$ is a functor which to each ordered space E assigns the above system (i.e. if $\mathcal{I}E = E^*$), then $\mathcal{I}^n E = E$.

It is well-known (see f.e [1]) the next compatibility principle of ordered spaces.

Two subsets A, B are called order separated if and only if

$$(\forall x \in A)(\forall y \in B) \quad [x \lessapprox y] \text{ or } (\forall x \in A)\ (\forall y \in B)\ [y \lessapprox x],$$

where $x \lessapprox y$ means "$x < y$ or x, y non comparable".

A topology is called compatible with the order iff: for each couple a, b of points, such that there exists a neighbourhood of one, say $U(a)$ of a, which b doesn't belong in, there exists neighbourhood $V(a) \subseteq U(a)$ of a such that b and $V(a)$ to be order separated. A space is called O_0 iff for each couple of points there exists a neighbourhood of one, order separated from the other. In the same way are defined the O_1, O_2 spaces.

Proposition 2. *The Alexandroff topology is compatible with the order.*

Proof. Let V_y be an open neighbourhood of y, $x \notin V_y$. There exist finite numbers e_i, $e_i < y$ as well as finite numbers e_j, $e_j > y$ such that:

$$V_y = (\bigcap_I \uparrow \mathring{e}_i) \bigcap (\bigcap_J \downarrow \mathring{e}_j), \quad I, J \text{ finite.}$$

Because $x \notin V_y$, there exists, say e_{j^*}, such that $x \notin \downarrow \mathring{e}_{j^*}$. The set $\uparrow \mathring{e}_{i^*} \cap \downarrow \mathring{e}_{j^*}$, where $i^* \in I$, is neighbourhood of y and order separated from x.

We say that the space fulfils the f^*-cut condition of completion (f^*.c.c.) iff for every f^*-cut (A, B) $\mathrm{Cl}\,A \cap \mathrm{Cl}\,B \neq \emptyset$, where $\mathrm{Cl}\,A$ is the closure of A (cf. [7], def. 2, where the notion is slightly different). This property is an expression of completion and in the case of linear spaces it holds if for every subset there exists an end, but it generally fails in a poset.

From now on we suppose that every directed set maximal with respect to set inclusion is cofinal to a linear subset.

Proposition 3. *If the ordered space E endowed by Alexandroff topology is without f^*-jumps, then it can be embedded into an ordered space $\tilde{E}$, which has not f^*-gaps and (for the same topology) fulfils the (f^*.c.c.).*

Proof. Construction of $\tilde{E}$.

To each $e \in E$ assigns a set $K(e)$ of couples (A_i, B_j), subsets of E formed as follows: let $(A_i)_{i \in I}$ (resp. $(B_j)_{j \in J}$) be the analysis of $\downarrow \mathring{e}$ (resp. $\uparrow \mathring{e}$) in maximal directed right (resp. left) subsets.

If card $I \leqq 1$ and card $J \leqq 1$, define $K(e) = \emptyset$.

In the other cases, the $K(e)$ is the set of all the couples (A_i, B_j) for which there does not exist $x \in E \setminus \{e\}$ between the elements of A_i and B_j.

$$\text{Put } K(E) = \bigcup_{e \in E} K(e), \quad E' = E \cup K(E).$$

Define a relation R on E' as follows (x, y in E, (A_i, B_i) in $K(E)$, $i \in \{1, 2\}$).

$$xRy \leftrightarrow x \leqq y, \quad xR(A_1, B_1) \leftrightarrow x \in A_1, \quad (A_1, B_1)Rx \leftrightarrow x \in B_1$$

$$(A_1, B_1)R(A_2, B_2) \leftrightarrow (A_1, B_1) = (A_2, B_2) \text{ or } A_2 \cap B_1 \neq \emptyset.$$

The relation R is an order, extension of the initial one.

The demanded set $\tilde{E}$ is the extension system of f^*-cuts of E'.

The set $\tilde{E}$ is without f^*-gaps.

If an element of the form $(A_i, B_j) \in K(e)$ has an end, then between this end and the e does not exist any other element, i.e. it is a f^*-jump here, absurd. So the E' is without f^*-jumps too and the $\tilde{E}$ has not f^*-gaps as an extension system of f^*-cuts.

The density of E.

First: between two comparable elements of $\tilde{E}$, there exists an element of E. In fact, say $z<z'$, $z=(A, B)\in L^*(E')$ and $z'\in E$, then $z'\in B$, next there exists $z''\in B$, $z''<z'$ and finally there exists $z_0\in E$ such that $z<z''<z_0<z'$.

Next consider $e=(A, B)\in \tilde{E}\setminus E$ and open V_e, $e\in V_e$. The V_e is the finite intersection of sets of the form $\uparrow \mathring{e}_i$ and $\downarrow \mathring{e}_j$, let $\downarrow \mathring{e}_j$ be one of them, then $e_j\in B$, and being B without end, there exists $e_0\in B$, $e_0<e_j$ and finally $x\in E$, $e<e_0<x<e_j$, hence $x\in V_e$.

The E fulfils the $(f^*.c.c)$.

Let (A^*, B^*) be a f^*-cut of $\tilde{E}$ and (A', B'), (A, B) its trace on E' and E respect. Because of non-existence of f^*-gaps in $\tilde{E}$, there exists an end, say $e=\min B^*$.

The A is a directed set. In fact, it is $\neq\emptyset$ and if x, y in A, there exists $z\in A^*$, $x\leqq z$, $y\leqq z$, the A^* is without end hence there exists $z_0\in A$, with $x\leqq z_0$, $y\leqq z_0$, $z\leqq z_0$.

Distinguish now two cases.

a) The end $e\in E$. The couple (A, B) is a f^*-cut, because A is right directed and $e\in B$. Analyse the subset $\uparrow \mathring{e}$ into the family $(B_j)_{j\in J}$ of maximal left directed subsets and remark that a couple of the form (A, B_j) defines a $s\in E'$ such that $\mathrm{Cl}\,A\cap\mathrm{Cl}\,B\neq\emptyset$, hence $\mathrm{Cl}\,A\cap\mathrm{Cl}\,B\neq\emptyset$ and finally $\mathrm{Cl}\,A^*\cap\mathrm{Cl}\,B^*\neq\emptyset$.

b) The element $e\in\tilde{E}\setminus E$. Then $e=(A_1, B_1)\in E'\setminus E$, $(A_1, B_1$ subsets of $E)$ or $e=(A_1, B_1)\in\tilde{E}\setminus E'$ $(A_1, B_1$ subsets of $E')$. In the former case it is $A=A_1$, $B=B_1$ and $e\in\mathrm{Cl}\,A\cap\mathrm{Cl}\,B$, in the last one it is $A'=A_1$, $B'=B_1$, hence $e\in\mathrm{Cl}\,A'\cap\mathrm{Cl}\,B'$, hence $\mathrm{Cl}\,A^*\cap\mathrm{Cl}\,B^*\neq\emptyset$.

Application. If F is a real function defined on E, (A, B) a f^*-cut and $(A_i^e)_i$, $(B_j^e)_j$ the analysis of the sets $\downarrow\mathring{e}$, $\uparrow\mathring{e}$ respectively for an $e\in E$, it is natural to say that F has as. limit the value l with respect to the direction A_i if and only if:

$$(\forall\varepsilon>0)\ (\exists x_0\in A_i)\ (\forall x\in A_i)\ [x_0<x\rightarrow|F(x)-l|<\varepsilon]\ (\lim_{A_i}F(x)=l),$$

and in a same way to get the $\lim_B F(x)$. Similarly to say that F has limit from the left (resp. right) of l if $\lim_{A_i^e}F(x)=l$ (respectively $\lim_{B_j^e}F(x)=l$) for all i (resp. j) and for the same l.

Proposition 4. *If F is a real valued function defined on an ordered space E without f^*-jumps and it has limit with respect to all the directions and from the left and right of all the elements, then F can be extended to a continuous function over an extension system of E.*

Proof. The extended set SE is constructed as follows: consider the extension system $\tilde{E}$ of E constructed in the Proposition 3 and $SE=\tilde{E}\cup L^-(\tilde{E})\cup L^+(\tilde{E})$, where $L^-(\tilde{E})$, $L^+(\tilde{E})$ are the lower and upper classes of the f^*-cuts of $\tilde{E}$ without ends.

Define a function $\tilde{F}$ as follows:

If $x\in E$, put $\tilde{F}(x)=f(x)$.

If $x\in E'\setminus E$ (E', as it has been defined in Prop. 3), i.e. if $x\in K(e)$, put $F(x)=F(e)$.

If $x\in\tilde{E}\setminus E'$, i.e. if $x=(A', B')\in L^*(E')$, then the traces A, B of A', B' on E consist the classes of a cut in E and (by hypothesis) there exist $\lim_A F(x)$ and $\lim_B F(x)$, say l^-, l^+ resp. and $l^-\leqq l^+$. Put $\tilde{F}(x)=l$, such that $l^-\leqq l\leqq l^+$.

If $x \in LE \setminus \tilde{E}$, x is the end of a class, say a lower class A^* of a f*-cut of $\tilde{E}$, the trace A of it in E is the lower class of a f^* – cut of E, so we put $F(x)=l$, where $l=\lim_A F(x)$.

The function $\tilde{F}$ is an extension of F and it is continuous over SE. In fact, let $e \in \tilde{E}$ and A_i, B_j the traces on E of two maximal directed subsets of $\tilde{E}$, which the $\downarrow \overset{\circ}{e}$, $\uparrow \overset{\circ}{e}$ are analysed in. We suppose that $F(e)=l$, $\lim_{A_i} F(x)=l^-$, $\lim_{B_j} F(x)=l^+$ and $l^- < l < l^+$. There exist the elements e_i, e_j of $LE \setminus \tilde{E}$ which are ends in LE of A_i, B_j, hence $\tilde{F}(e_j)=l^-$, $\tilde{F}(e_j)=l$. If V_l a neighbourhood of l in R, then $\tilde{F}(\uparrow \overset{\circ}{e}_i \cap \downarrow \overset{\circ}{e}_j)=\{l\}$.

If we apply the last results in the case where $E=Q$ or $E=$ R, we find as $\tilde{E}$ the set R (in both cases) but the assigned LE is a set larger than R. A real non-continuous valued function defined on R (or on Q) having limits (from right and left) in each point, can be extended in this way. In this case we can study the extended function as continuous one and then pull back to R to deduce results for the initial function. This is exactly the point in [2] (p. 101).

§3. Some consequences of (f^*. c. c.) property

In this paragraph we generalize two results referring by [6] (p. 50) to the case of a linear space. We again suppose that in the ordered space E each maximal directed set is cofinal to a linear subset.

For each $x \in E$, put $K_x = \uparrow x \cup \downarrow x$. We say that a subset E° of E is a **branch** of E if and only if:

(1) $$(\forall x \in E^\circ)\ [K_x \subset E^\circ] \text{ and}$$

(2) $$(\forall x \in E^\circ)\ (\forall y \in E^\circ)\ [K_x \cap K_y \neq \emptyset].$$

Proposition 5. *If an ordered space endowed by Alexandroff topology fulfils the* (f^*. *c. c.*), *all its branches are connect subsets.*

Proof. Let $\overset{\circ}{E}$ be a branch of the space, non singleton; it is evident that $\overset{\circ}{E}$ is a clopen subset. Let $S \neq \emptyset$ be also a clopen proper subset of $\overset{\circ}{E}$. There exist $p \in S$ and $q \in \overset{\circ}{E} \setminus S$ comparable elements, say $p<q$. Consider the set $\mathcal{M}$ of $x \in E$ which satisfy the following:

"$p<x<q$ and there exists a chain, subset of S, which has p as the first element and x as the last one".

It is $\mathcal{M} \neq \emptyset$ ($p \in \mathcal{M}$).

Consider, now, a chain $a_1 \subset \mathcal{M}$, maximal with respect to set inclusion, which contains p. Then the set $A_1 = \downarrow a_1$ is a directed subset cofinal to a_1 and the lower class of a f^*-cut (A_1, B_1) of $\overset{\circ}{E}$; such a f^*-cut exists because q is greater than all the elements of A_1.

There exists $m \in \mathrm{Cl}\, A_1 \cap \mathrm{Cl}\, B_1$, each of the neighbourhoods of m meets the subsets S and $\overset{\circ}{E} \setminus S$, hence $\mathrm{Cl}\, S \cap \mathrm{Cl}(\overset{\circ}{E} \setminus S) \neq \emptyset$, while $\mathrm{Cl}\, S \cap \mathrm{Cl}(\overset{\circ}{E} \setminus S) = S \cap (\overset{\circ}{E} \setminus S) = \emptyset$, which is absurd.

Remark. If the initial space is itself a branch, it doesn't need the space to be endowed by its Alexandroff topology.

We say **the filter** $\mathcal{U}$ **is bounded by the elements** p and q (where $p<q$) if $(\uparrow p \cap \downarrow q) \in \mathcal{U}$.

Proposition 6. *In an ordered space without f^*-jumps, endowed by Alexandroff topology and which fulfils the ($f^*.c.c.$), each ultrafilter, which has as an element a chain, maximal with respect to set inclusion and which is bounded by two elements, converges.*

Proof. Let p, q be the two elements of the ultrafilter $\mathcal{U}$ and a the chain of the supposition. Consider the set

$$\mathcal{M}=\{x\in a:\uparrow x\in\mathcal{U}\}.$$

It is $\mathcal{M}\neq\emptyset$ ($p\in\mathcal{M}$), while if $\uparrow q\in\mathcal{M}$, $\lim\mathcal{U}=\{q\}$. Let $\uparrow q\notin\mathcal{M}$.
Consider the cut (A, B) of a, where:

$$B_1=\{y\in a:(\forall x\in\mathcal{M})\,[x<y]\} \text{ and } A_1=\{x\in a:(\forall y\in B_1)\,[x<y]\}$$

and the f^*-cut of the space (A, B), where $A=\downarrow A_1$ cofinal to $\mathcal{M}$ and $B=\uparrow B_1$ cofinal to $aC\mathcal{M}$. There exists $s\in\mathrm{Cl}\,A\cap\mathrm{Cl}\,B$; for each neighbourhood V_s of s there exist $a'\in A_1$ and $b\in B_1$ such that $\uparrow\mathring{a}'\cap\downarrow\mathring{b}\subseteq V_s$ and because the space has not f^*-jumps and a' is not an end of A, there exists $a\in A_1$ such that $\uparrow a\cap\downarrow\mathring{b}\subset V_s$. On the other hand, it is $\uparrow b\notin\mathcal{U}$ and $\uparrow a\in\mathcal{U}$, hence $\uparrow a\cap\downarrow b\in\mathcal{U}$.

Proposition 7. *If each ultrafilter which has a maximal chain as one of its elements and is bounded by two elements, converges, the space fulfils the ($f^*.c.c$).*

Proof. Let (A, B) be a f^*-cut, a_1 (resp. b_1) maximal chain cofinal to A (resp. B), $p\in a_1$ (resp. $q\in b_1$). Consider the class of subsets:

$$V=\{\uparrow x\cap a_1:x\in a_1\}.$$

This class constitutes a base of a filter and let $\mathcal{U}$ be ultrafilter containing this filter. It is $\uparrow p\cap\downarrow q\in\mathcal{U}$ and the maximal chain $a_1\cup b_1\in\mathcal{U}$. Then the ultrafilter $\mathcal{U}$ converges to a point s whose each neighbourhood intersects the A and B.

§4. Three problems on compatibility and directed-cut systems

In this paragraph we give certain answers to three problems relative to the theory which has been developed.

Problem 1. The problem is whether there exists the "smallest" extension system of a poset which fulfils the ($f^*.c.c$).

It is evident that if $(E_1, \leqq)$, $(E_2, \leqq)$ are linear order structures, completions of another linear structure $(E, \leqq)$ fulfiling the properties: they are lattices, the E is dense into them (by the aspect of the order as well as the interval topology) and the elements of $E_1\backslash E$ and $E_2\backslash E$ are not ends of jumps into E_1 and E_2 respectively, then E_1 and E_2 are similar.

On the other hand, if E_0 is the MacNeille's completion of E and E' is another lattice completion of E, then there exists an isotone function $f:E_0\to E'$ such that the sets E_0 and $f(E_0)$ to be similar. It is sufficient to define f as follows:
if $x\in E$, then $f(x)=x$,
if $x=(A, B)\in E_0\backslash E$, then $f(x)=\sup A$ in E'.
Now we have the following:

Proposition 8. *Let E be an ordered space without f^*-jumps and O_1 for Alexandroff topology, and E^* be its extension system of f^*-cuts. If E_1 is an* *extension system of E endowed by the same topology and moreover* (1) *it is without f^*-gaps,* (2) *it is O_1,* (3) *fulfils the* ($f^*.c.c$) *and* (4) $\mathrm{Cl}E=E_1$, *then there exists an isotone function $f:E^*\to E_1$ such that the sets $f(E^*)$ and E^* to be similar.*

Proof. *Definition of f*: for $x\in E$, put $f(x)=x$. Let $e=(A, B)\in E^*\setminus E$. Consider the subsets of E_1:

$$B'=\{y:(\forall x\in A)[x<y]\} \text{ and } A'=\{x:(\forall y\in B')[x<y]\}.$$

Analyse A', B' in maximal directed subsets and take as A_1, B_1 these of them which contain A, B respectively. Because of (1) there exists an end of A_1 or B_1, say $e_1=\min B_1$.

We will show that the trace of the $\downarrow e_1$ on E is A and of the $\uparrow e_1$ is B, hence in the f^*-cut (A, B) assigns e_1 uniquely.

First, each neighbourhood of e_1 intersects A_1 and B_1. In fact, if not, there exists $x\in E_1$ (because of (3)) such that $x\in\mathrm{Cl}A_1\cap\mathrm{Cl}B_1$. Thus for each neighbourhood V_x of x there exist $x_1\in A_1$, $y_1\in B$ such that $x_1<e_1y_1$ and $\uparrow x_1^\circ\cap\downarrow y_1^\circ\subset V_x$, which is absurd, because of (2). On the other hand, if there exists another maximal directed right subset (except of A_1), the set $\uparrow e_1$ is open without intersecting A_1 absurd.

Thus $\downarrow e_1=A_1$ and because of (4) the trace of A_1 in E is A. Analogously it is shown that $\uparrow e_1=B_1$ and the trace of it in E is B and our assertion has been proved.

Put $f(e)=e_1$.

The function f is one-to-one.

The restriction of f in E is the identity map.

Let $e_1=(A_1, B_1)$, $e_2=(A_2, B_2)$ be f^*-gaps in E, $e_1\neq e_2$. These f^*-cuts have one, at least, of their classes different, say $B_1\neq B_2$. Construct (as in the first part of the demonstration) f^*-cuts in $E_1(\tilde{A}_1, \tilde{B}_1)$, $(\tilde{A}_2, \tilde{B}_2)$ with $\tilde{A}_i$, $\tilde{B}_i$, $i\in\{1, 2\}$, containing the A_1, B_{1_2} respectively. Then there exist $\tilde{e}_i\in E_1\setminus E$, ends of one class and such that $\tilde{e}_i\in\mathrm{Cl}\tilde{A}_i\cap\mathrm{Cl}\tilde{B}_i$. If $\tilde{e}_1=\tilde{e}_2$, then each of them has on its right two maximal directed subsets, thus the set $\downarrow e_i$ will be open and non intersecting $\tilde{B}_i$, which is absurd.

Hence $\tilde{e}_1\neq\tilde{e}_2$.

The function f preserves the order.

If $e=(A, B)$ is a f^*-gap in E, then $f(x)=x<f(e)<f(y)=y$, for each $x\in A$ and $y\in B$. Thus if one of the two elements of E^* is an element of E, the proof is obvious.

Let, now, $e_i=(A_1, B_1)$, $e_2=(A_2, B_2)$ be f^*-gaps in E. If $e_1<e_2$, then there exists $x\in E$, $x\in B_1\cap A_2$ and $f(e_1)<f(x)=x<f(e_2)$. On the other hand, if e_1, e_2 are non comparable and $f(e_1)<f(e_2)$, then it must exist an element $x\in E$ between $f(e_1)$ and $f(e_2)$, hence $x\in B_1\cap A_2$, which is absurd.

Problem 2. We refer to a connected space and to conditions which define an order relation in this, compatible with the topology of the space.

Let E be a connected Hausdorff space E; we can prove that if the set $(E\times E)\setminus D$, where D is the diagonal, ceases to be connect, then the sets $E\setminus\{x\}$ for all x, but at most two of them, are non connected.

In fact, let (L, M) an open partition of $E\times E\setminus D$ and a, b elements such that the sets $E\setminus\{a\}$, $E\setminus\{b\}$ to be connected. For each k put:

$$L_k=\{x\in E:(x, k)\in L\}, \quad M_k=\{x\in E:(x, k)\in M\}$$

as well as:

$$L_k^* = \{y \in E : (k, y) \in L\}, \quad M_k^* = \{y \in E : (k, y) \in M\}.$$

The join of the sets $\{a\} \times (E \setminus \{a\})$ and $(E \setminus \{a\}) \times \{a\})$ is nonconnected in $E \times E \setminus D$, so each of them belongs to a different component. On the other hand, if $M_a^* = M_b^* = \emptyset$, then $(a, b) \in L$ and $(b, a) \in L$ which is absurd. Thus, if $M_a^* = \emptyset$, it is $L_b^* = \emptyset$ and for any other element $k \in E$, it is $M_k^* \neq \emptyset$ (because $(k, a) \in M$) and $L_k^* \neq \emptyset$ (because $(k, b) \in L$).

It has been proved ([8], p. 43) that if in a connected Hausdorff space there exist two points such that each third one divides them, then the space is linearly orderable. Let now E a connected Hausdorff space and a, b, c three elements such that the subspaces $E \setminus \{a\}$, $E \setminus \{b\}$, $E \setminus \{c\}$ are connected while, each of the other elements divides c and one of a, b, i.e. for each other x the set $E \setminus \{x\}$ is divided into an open partition A_x, C_x with $c \in C_x$ and one of a or b (or both of them) belongs to A_x.

It is easy to see that the sets $\{x\} \cup A_x$, $\{x\} \cup C_x$ are closed as well as A_x and C_x are open. Define an (partial) order in E as follows: put $x < y$ if and only if a belongs to the sets A_x, A_y and $A_x \subseteq A_y$ or b belongs to A_x, A_y and $A_x \subseteq A_y$. In the case where both of a and b belong to the sets A_x, A_y, it is proved that $A_x \subseteq A_y$ or $A_y \subseteq A_x$. In fact, put $A = A_x \cap A_y$, $B = C_y \cap A_x$, $C = A_y \cap C_x$ and $D = C_x \cap C_y$. Let $y \in C_x$; we must prove that $B = \emptyset$. If not, the subset B is open (by definition) and on the other hand $B = [\{y\} \cup C_y] \cap [\{x\} \cup A_x]$, hence B is closed, which is absurd.

The above relation is an order; the sets of the form $\downarrow \mathring{x}$ are the open sets A_x, while the sets of the form $\downarrow x$ are the closed $A_x \cup \{x\}$.

Let now be V_a an open neighbourhood of an element a, $b \notin V_a$ and c, d two elements belonging to V_a such that $c < b < d$. If $a < b$, the set $V_a \cap \downarrow \mathring{b}$ is open, contains the element a and it is order separated by b. In the other cases the set $V_a \setminus (\downarrow b)$ is a neighbourhood of a order separated by b. Thus the topology is compatible to the defined order.

Problem 3. Is there a cofinal linear subset of any directed set? We give a relative proposition and the sketch of the proof.

Proposition 9. *If the width of each antichain of an order structure is smaller than a concrete natural number, then there exists a cofinal chain subset for each directed set.*

Proof. Let E be an ordered set whose all the antichains have width smaller than $m \in N$ and which has not a cofinal chain as subset. Consider a linear subset I_0 of A, maximal with respect to set inclusion. There exist $p \in A \setminus I_0$ such that p is not smaller than any element of I_0 and p is not greater than all the elements of I_0. Hence there exists $a_0^0 \in I_0$ such that p to be non comparable with all the elements of I_0 which are greater than a_0^0. (In the next we call parallel the non comparable elements.)

The generality of the demonstration is preserved if we consider the chain I_0 as a sequence $(a_i^0)_{i \in N}$ with the above a_0^0 as its first element.

Consider now $p_1 \in A \setminus I_0$, which is parallel to all the elements of a final segment of I_0, hence there exists $a_0^1 \in E$ such that $a_0^1 > p_1$, $a_0^1 > a_0^0$ and a_0^1 is parallel to all the elements of a final segment of I_0. Inductively we construct a sequence

$(a_i^1)_{i\in N} := I_1$, which is a chain and in the same way we construct the chains $(I_n)_{n\in N}$ which fulfil the following:

(1) The chains I_n, $n\in N$ are sequences and cofinal subsets to maximal chains of the set A.

(2) They have not common elements.

(3) If $\tau<\sigma$, it is $a_i^\tau < a_i^\sigma$ for all $i\in N\cup\{0\}$.

(4) For any $\tau\in N$, each of a_i^τ is parallel to all the elements of a final segment of $I_0, I_1, \ldots, I_{\tau-1}$.

(5) For any $\tau\in N$, there are τ parallel elements.

The above construction leads to the existence of an antichain of width larger than m.

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