# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 2, 1988, Fasc. 4

# On the Tangent Bundle and on the Bundle of the Linear Frames over an Affinely Connected Manifold

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*Presented by M. Putinar*

## Introduction

Let $M$ be a non compact and connected manifold. A soul of $M$ is any connected submanifold $N$ of $M$, such that $\dim N < \dim M$ and the inclusion $i: N \to M$ is a homotopy equivalence.

If $M$ is a complete Riemannian manifold, with non negative sectional curvature, a well-known theorem of J. Cheeger and D. Gromoll states that $M$ has a compact soul.

In a paper of 1984, ([5]), T. Higa considers non compact, connected manifolds with a complete connection $\nabla$, which is supposed symmetric, for simplicity. He proves that the existence of non constant affine functions implies the existence of souls. Furthermore, he studies the compactness of such souls, he constructs a special vector space $V(M, \nabla)$ and proves that, if $\dim V(M, \nabla) = m > 0$, then $M$ is diffeomorphic to $\mathbb{R}^m \times M'$, where $M'$ is a totally geodesic submanifold of $M$.

In this paper, given a connected manifold $M$ with a symmetric and complete connection $\nabla$, we study the manifolds $T(M)$, $L(M)$, with respect to the complete lifts of $\nabla$.

In § 1, we give the links between the complete and the vertical lifts of several geometric objects to $L(M)$ and to $T(M)$.

In § 2, 3, we determine the spaces of the parallel 1-forms and of the affine functions on $(L(M), \nabla^C)$ and on $(T(M), \nabla^c)$. We get that the existence of parallel 1-forms on $M$ is a sufficient condition to obtain souls for $(T(M), \nabla^c)$. Then (see § 4), when it is possible, we find the links between the above souls and the corresponding souls of $(M, \nabla)$. We point out that such souls are totally geodesic submanifolds. The aim of Section 5 is to determine the dimension of $V(T(M), \nabla^c)$, when $\nabla$ is the Levi-Civita connection on $M$. In this case, the existence of parallel 1-forms on $M$ allows us to state reducibility theorems for $T(M)$, (see §6).

Many of these results are an application of the theorems given by T. Higa in [5], [6]. Such an application is possible, since $\nabla^c$ is complete on $T(M)$. On the other hand, the connection $\nabla^C$ on $L(M)$ is not complete; so, the same methods fall for $L(M)$.

(*) This paper has been partially supported by M. P. I.

The authors will deal with $L(M)$ and $T^*(M)$ in another paper.

In the following, $M$ will denote an $n$-dimensional connected manifold with a symmetric and complete connection $\nabla$; $\mathscr{J}(M)$ will be the module of the real valued $C^\infty$-differentiable functions on $M$.

Furthermore, $\mathscr{X}(M)$, $\Lambda^1(M)$, $\mathscr{J}^r_s(M)$ will be, respectively, the $\mathscr{J}(M)$-modules of the vector fields, of the 1-forms and of the tensor fields of type $(r, s)$ on $M$. $P^1(M, \nabla)$ will denote the real vector space consisting of the parallel 1-forms on $M$, and $A(M, \nabla)$ the real vector space consisting of the affine functions on $M$.

Finally, we put $a(M, \nabla)=\dim A(M, \nabla)-1$.

We recall that a function $f\in\mathscr{J}(M)$ is said to be affine if, for any geodesic $c: I\to M$, there exist $\alpha, \beta\in\mathrm{R}$ such that $f(c(t))=\alpha t+\beta$, $t\in I$. The symmetry of $\nabla$ implies the equivalence:

$$f\in A(M, \nabla)\leftrightarrow df\in P^1(M, \nabla).$$

Others properties of the affine functions can be found in [5], § 1, 2.

## § 1. Complete and vertical lifts to $L(M)$ and links with the analogous lifts to $T(M)$

We denote with $\pi: T(M)\to M$ and $\pi_L: L(M)\to M$, respectively, the tangent bundle and the linear frames bundle over $M$.

As regards to the complete and the vertical lifts to $T(M)$, see [13]. In [1], [2], [3], [10] the lifts of tensor fields and of connections to $L(M)$ have been defined and studied. We report here some basic definitions and we state other properties. To avoid confusion, we use the capital letters $C$, $V$ to indicate the lifts to $L(M)$, and the small letters $c$, $v$ referred to $T(M)$.

We recall that, if $f\in\mathscr{J}(M)$, the function $f^V=f\circ\pi_L$ is the vertical lift of $f$ to $L(M)$.

Furthermore, for any $\alpha\in\{1,\dots, n\}$, the complete lift of $f$, of index $\alpha$, is the function $f^C_\alpha$ on $L(M)$ such that

$$(1.1)\qquad f^C_\alpha(p, X_1,\dots, X_n)=X_\alpha(f)\qquad (p, X_1,\dots, X_n)\in L(M).$$

Given $X\in\mathscr{X}(M)$, $\alpha\in\{1,\dots, n)$, the vertical lift of $X$, of index $\alpha$, is the vector field $X^{\alpha V}$ on $L(M)$ such that

$$(1.2)\qquad X^{\alpha V}(f^C_\beta)=X(f)^V\delta^\alpha_\beta\qquad f\in\mathscr{J}(M),\ \beta\in\{1,\dots, n\}.$$

The complete lift of $X$, $X^C$, is defined by

$$(1.3)\qquad X^C(f^C_\alpha)=X(f)^C_\alpha\qquad f\in\mathscr{J}(M),\ \alpha\in\{1,\dots,n).$$

Fixed $T\in\mathscr{J}^0_s(M)$, $s\geqq 1$, $\alpha\in\{1,\dots, n\}$, the complete lift $T^C_\alpha$ of $T$, with index $\alpha$, is defined by

$$(1.4)\qquad T^C_\alpha(X^C_1,\dots, X^C_s)=T(X_1,\dots, X_s)^C_\alpha\qquad X_1,\dots, X_s\in\mathscr{X}(M),$$

whereas the vertical lift $T^V$ is defined by

$$(1.5)\qquad T^V=\pi^*_L(T).$$

Note that the complete lifts of $f \in \mathscr{I}(M)$ and of $T \in \mathscr{I}^{o}_{s}(M)$ to $L(M)$ defined in [1] can be obtained as follows:

$$f^C = \sum_{\alpha=1}^{n} f^C_\alpha \qquad T^C = \sum_{\alpha=1}^{n} T^C_\alpha .$$

Given $\omega \in \Lambda^1(M)$, $\alpha \in \{1, \ldots, n\}$, $\gamma_\alpha(\omega)$ is the function on $L(M)$ defined by

$$(1.6) \quad \gamma_\alpha(\omega)(p, X_1, \ldots, X_n) = \omega_p(X_\alpha) \qquad (p, X_1, \ldots, X_n) \in L(M),$$

and $\gamma(\omega)$ is the function on $T(M)$ such that

$$(1.7) \quad \gamma(\omega)(p, X) = \omega_p(X) \qquad (p, X) \in T(M).$$

Furthermore, the complete lift $\nabla^C$ to $L(M)$ of a connection $\nabla$ on $M$ is obtained by means of

$$(1.8) \quad \nabla^C_{X^C}(Y^C) = (\nabla_X Y)^C \qquad X, Y \in \mathscr{X}(M).$$

In [10], K. Mok proved that

$$(1.9) \quad \nabla^C(H^C) = (\nabla H)^C \qquad H \in \mathscr{I}^1_s(M).$$

The following result holds.

**Proposition 1.1. For each $T \in \mathscr{I}^{o}_{s}(M)$, we have:**

$$(1.10) \quad \nabla^C(T^V) = (\nabla T)^V$$

$$(1.11) \quad \nabla^C(T^C_\alpha) = (\nabla T)^C_\alpha \qquad \alpha \in \{1, \ldots, n\}.$$

As regards the geodesics of $\nabla^C$, we have this result, (see [10]): a curve $\bar{c} : I \to L(M)$, $\bar{c}(t) = (c(t), X_1(t), \ldots, X_n(t))$ is a $\nabla^C$-geodesic iff $c$ is a $\nabla$-geodesic and each $X_i$ is a Jacobi vector field along $c$. It follows that $\nabla^C$ is not complete, while the same is not true for $\nabla^c$ on $T(M)$. In fact, the completeness of $\nabla^C$ would imply the non existence of zeros for any Jacobi vector field along any geodesic on $M$.

Furthermore, the following relations hold for the exterior differentiation operators:

$$(1.12) \quad d(f^C_\alpha) = (df)^C_\alpha, \quad d(f^V) = (df)^V \qquad f \in \mathscr{I}(M)$$

$$(1.13) \quad d(\omega^C_\alpha) = (d\omega)^C_\alpha, \quad d(\omega^V) = (d\omega)^V \qquad \omega \in \Lambda^1(M)$$

$$(1.14) \quad \gamma_\alpha(df) = f^C_\alpha \qquad f \in \mathscr{I}(M)$$

$$(1.15) \quad d(\gamma_\alpha(\omega)) = \omega^C_\alpha \qquad \omega \in \Lambda^1(M).$$

We establish now the links between the lifts to $L(M)$ and to $T(M)$. Let $(U, \Phi)$ be a local chart on $M$, with coordinates $\{x^i\}$; the induced coordinates on $\pi_L^{-1}(U)$ are denoted by $\{x^i, y^i_\alpha\}$, (see [8]). If $\{e_i\}$ is the natural base in $U$, then the natural base in $\pi_L^{-1}(U)$ is given by $\{e_i^C, e_i^{\alpha V}\}$. Analogously, the natural base in $\pi^{-1}(U)$ is given by $\{e_i^c, e_i^v\}$. Finally, see [10], [13] for the transformation laws of such bases. Now, for a fixed $\xi=(\xi^1,\dots,\xi^n)\in \mathsf{R}^n$, we put $\tau_\xi : L(M)\to T(M)$ the map such that:

$$(1.16)\quad \tau_\xi(p, X_1,\dots, X_n)=(p, \xi^\alpha X_\alpha) \qquad (p, X_1,\dots, X_n)\in L(M).$$

It follows:

$$(1.17)\qquad \pi\circ\tau_\xi=\pi_L,$$

and, if we denote with $\{\varepsilon_1,\dots,\varepsilon_n\}$ the canonical base of $\mathsf{R}^n$, and put $\tau_\alpha=\tau_{\varepsilon_\alpha}$, we have:

$$(1.18)\qquad \tau_\xi=\xi^\alpha\tau_\alpha$$

$$(1.19)\qquad \tau_\alpha(L(M))=T'(M)=T(M)-\{O\},\quad \alpha\in\{1,\dots,n\}.$$

In fact, (1.18) is a consequence of (1.16). To obtain (1.19), fix $(p, X)$ in $T'(M)$ and complete $\{X\}$ to a base of $T_p(M)$.

**Proposition 1.2.** *Fixed a connection $\nabla$ on $M$ and $\alpha\in\{1,\dots,n\}$, the map $\tau_\alpha:(L(M), \nabla^C)\to(T(M), \nabla^c)$ is totally geodesic.*

Let be $\bar c: I\to L(M)$, with $\bar c(t)=(c(t), X_1(t),\dots, X_n(t))$ a $\nabla^C$-geodesic. Then, $\tau_\alpha(\bar c(t))=(c(t), X_\alpha(t))$, so $\tau_\alpha\circ\bar c$ is a $\nabla^c$-geodesic (see [13]).

**Proposition 1.3.** *Given $0\neq\xi\in\mathsf{R}^n$, the map $\tau_\xi^*:\mathscr{J}(T(M))\to\mathscr{J}(L(M))$ satisfies the following relations:*

i) $\tau_\xi^*(f^v)=f^V\quad f\in\mathscr{J}(M)$,

ii) $\tau_\xi^*(f^c)=\xi^\alpha f_\alpha^C\quad f\in\mathscr{J}(M)$

iii) $\tau_\alpha^*(f^c)=f_\alpha^C$

iv) $\tau_\alpha^*(\gamma(\omega))=\gamma_\alpha(\omega)\quad \omega\in\Lambda^1(M)$

v) $\tau_\alpha^*$ *is a monomorphism, for each* $\alpha\in\{1,\dots,n\}$.

From the definitions of the vertical lifts of a function and by means of (1.17) we have i). Given $u=(p, X_1,\dots, X_n)\in L(M)$, we have $f^c\circ\tau_\xi(u)=f^c(p, \xi^\alpha X_\alpha)$ $=\xi^\alpha X_\alpha(f)=\xi^\alpha f_\alpha^C(u)$, and so ii). Obviously, ii) implies iii). Furthermore, if $u\in L(M)$, we have $\tau_\alpha^*(\gamma(\omega))(u)=\gamma(\omega)(p, X_\alpha)=\omega_p(X_\alpha)=\gamma_\alpha(\omega)(u)$, and so iv).

Finally, given $\alpha\in\{1,\dots,n\}$, consider $\bar f\,\bar g\in\mathscr{J}(T(M))$ such that $\tau_\alpha^*(\bar f)=\tau_\alpha^*(\bar g)$. For any $(p, Y)\in T'(M)$, there exists $u=(p, X_1,\dots, X_n)\in L(M)$ with $X_\alpha=Y$; so $\tau_\alpha^*(\bar f)(u)=\tau_\alpha^*(\bar g)(u)\Rightarrow\bar f(p, Y)=\bar g(p, Y)$, i.e. $\bar f_{|T'(M)}=\bar g_{|T'(M)}$. Since $T'(M)$ is dense in $T(M)$, we get $\bar f=\bar g$.

**Proposition 1.4.** *For any $0\neq\xi\in\mathsf{R}^n$, $u\in L(M)$, the map $(\tau_\xi)_{*u}:T_u(L(M))$ $\to T_{\tau_\xi(u)}(T(M))$ satisfies the following relations:*

i) $(\tau_\xi)_{*u}(X_u^{\alpha V})=\xi^\alpha X_{\tau_\xi(u)}\quad \alpha\in\{1,\dots,n\},\quad X\in\mathscr{X}(M)$

ii) $(\tau_\xi)_{*u}(X_u^C)=X^C_{\tau_\xi(u)}$

iii) $(\tau_\xi)_{*u}$ *is an epimorphism.*

*In particular, we have*

iv) $(\tau_\beta)_{*_u}(X^{\beta V}_u)=X^v_{\tau_\beta(u)}$ $\quad\beta\in\{1,\dots,n\},\quad X\in\mathscr{X}(M)$

v) $(\tau_\beta)_{*_u}(X^{\alpha V}_u)=O$ $\quad\alpha\neq\beta$

vi) $(\tau_\beta)_{*_u}(X^C_u)=X^c_{\tau_\beta(u)}$

A direct computation gives i), ii), and then iv), v), vi). For iii), we have, for any $i,\ \alpha\in\{1,\dots,n\}$:

$$(\tau_\xi)_{*_u}((e_i)^C_u)=(e_i)^c_{\tau_\xi(u)} \quad\text{and}\quad (\tau_\xi)_{*_u}((e_i)^{\alpha V}_u)=\xi^\alpha(e_i)^v_{\tau_\xi(u)}$$

Therefore, fixed $\alpha\in\{1,\dots,n\}$, such that $\xi^\alpha\neq 0$, it is easy to show that

$$\{(e_i)^c_{\tau_\xi(u)},\ \xi^\alpha(e_i)^v_{\tau_\xi(u)}\}\ \text{is a base of}\ T_{\tau_\xi(u)}(T(M)).$$

**Proposition 1.5.** *For any* $0\neq\xi\in\mathbb{R}^n$, *the map* $\tau^*_\xi:\Lambda^1(T(M))\to\Lambda^1(L(M))$ *satisfies the following relations:*

i) $\tau^*_\xi(\omega^v)=\omega^V\quad \omega\in\Lambda^1(M)$

ii) $\tau^*_\xi(\omega^c)=\xi^\alpha\omega^C_\alpha$

iii) $\tau^*_\beta$ *is a monomorphism, for any* $\beta\in\{1,\dots,n\}$.

A direct computation and the Proposition 1.4 imply i) and ii). Suppose, now, $\tau^*_\beta(\bar\omega)=0$, $\bar\omega\in\Lambda^1(T(M))$. For any $\bar u\in T'(M)$, by (1.19), we have $\bar u=\tau_\beta(u)$, $u\in L(M)$. Then, for each $X\in\mathscr{X}(M)$,

$$\bar\omega_{\bar u}(X^c_{\bar u})=\bar\omega_{\tau_\beta(u)}((\tau_\beta)_{*_u}(X^C_u))=(\tau^*_\beta\bar\omega)_u(X^C_u)=0,$$

and so $\bar\omega_{|T'(M)}=0$; hence, by continuity, $\bar\omega=0$.
Finally, the following result holds.

**Proposition 1.6.** *Let* $\nabla$ *be a connection on* $M$. *For* $0\neq\xi\in\mathbb{R}^n$ *and* $\omega\in\Lambda^1(T(M))$, *we have*: $\tau^*_\xi(\nabla^c\omega)=\nabla^C(\tau^*_\xi(\omega))$.

## §2. The space $A(L(M),\ \nabla^C)$.

At first, we study the space $P^1(L(M),\ \nabla^C)$.

**Proposition 2.1.** *Fixed* $\omega\in\Lambda^1(M)$, *the following conditions are equivalent*:

i) $\omega\in P^1(M,\ \nabla)$

ii) $\omega^V\in P^1(L(M),\ \nabla^C)$

iii) $\omega^C_\alpha\in P^1(L(M),\ \nabla^C)$, *for any* $\alpha\in\{1,\dots,n\}$

iv) *there exists* $\alpha\in\{1,\dots,n\}$ *such that* $\omega^C_\alpha\in P^1(L(M),\ \nabla^C)$.

i) $\leftrightarrow$ ii) and i) $\Rightarrow$ iii) follow from the Proposition 1.1;

iii) $\Rightarrow$ iv) is obvious. Finally, suppose $\nabla^C(\omega^C_\alpha)=0$, for some $\alpha\in\{1,\dots,n\}$. Then, for any $X,\ Y\in\mathscr{X}(M)$, we have:

$$0=(\nabla\omega)^C_\alpha(X^C,\ Y^{\alpha V})=((\nabla\omega)(X,\ Y))^V \quad\text{so}\quad (\nabla\omega)(X,\ Y)=0 \quad\text{and}\quad \omega\in P^1(M,\ \nabla).$$

**Proposition 2.2.** *For any* $\omega\in P^1(L(M),\ \nabla^C)$ *there exist, uniquely determined, 1-forms* $\pi,\ \pi_1,\dots,\ \pi_n\in P^1(M,\ \nabla)$ *such that* $\omega=\pi^V+\sum_{\beta=1}^{n}(\pi_\beta)^C_\beta$.

Let (U, Φ) be a local chart, with coordinates $\{x^i\}$, and put $\omega_{|\pi^{-1}(U)} = \omega_i dx^i + \omega_j^\beta dy_\beta^j$. From $\nabla^C \omega = 0$ and (6.2) [10], we have:

$$\frac{\partial \omega_i}{\partial x^j} - \Gamma_{ji}^p \omega_p - y_\alpha^h \frac{\partial \Gamma_{ji}^p}{\partial x^h} \omega_p^\alpha = 0 \quad i,\ j \in \{1, \dots, n\} \tag{2.1}$$

$$\frac{\partial \omega_i}{\partial y_\beta^j} - \Gamma_{ji}^p \omega_p^\beta = 0 \qquad i,\ j,\ \beta \in \{1, \dots, n\} \tag{2.2}$$

$$\frac{\partial \omega_j^\beta}{\partial x^i} - \Gamma_{ji}^p \omega_p^\beta = 0 \qquad i,\ j,\ \beta \in \{1, \dots, n\} \tag{2.3}$$

$$\frac{\partial \omega_j^\beta}{\partial y_\alpha^i} = 0 \qquad i,\ j,\ \alpha,\ \beta \in \{1, \dots, n\}. \tag{2.4}$$

The relation (2.4) implies that the components $\omega_j^\beta$ are independent on $\{y_\alpha^i\}$, and so they are the vertical lift of functions $h_j^\beta$ defined on $U$. For any $\beta \in \{1, \dots, n\}$, put $(\pi_\beta)_U = h_j^\beta dx^j$.

Let $(V, \Phi')$ be another chart on $M$, with coordinates $\{x'^i\}$, and $U \cap V \neq \emptyset$. The same construction gives rise to 1-forms $(\pi'_\beta)_V = h'^\beta_j dx'^j$. From the transformation laws of the component of $\omega$ on $\pi_L^{-1}(U) \cap \pi_L^{-1}(V)$, it follows $(\pi'_\beta)_{U \cap V} = (\pi_\beta)_{U \cap V}$. Therefore, for any $\beta \in \{1, \dots, n\}$, we obtain a global 1-form on $M$, denoted by $\pi_\beta$. The relation (2.3) implies $\nabla(\pi_\beta) = 0$. Now, if we put $\theta = \omega - \sum_{\beta=1}^{n} (\pi_\beta)_\beta^C$, using (2.1), (2.2), we have that $\theta$ is the vertical lift of a parallel 1-form $\pi$ on $M$.

Finally, let us suppose

$$\omega = \pi^V + \sum_{\beta=1}^{n} (\pi_\beta)_\beta^C = \rho^V + \sum_{\beta=1}^{n} (\rho_\beta)_\beta^C. \tag{2.5}$$

For any $\alpha \in \{1, \dots, n\}$, $X \in \mathscr{X}(M)$, since $\pi^V(X^{\alpha V}) = \rho^V(X^{\alpha V}) = 0$, we have:

$$\sum_{\beta=1}^{n} (\pi_\beta)_\beta^C (X^{\alpha V}) = \sum_{\beta=1}^{n} (\rho_\beta)_\beta^C (X^{\alpha V}),$$

$$\text{and so} \quad \pi_\beta(X) = \sum_{\alpha=1}^{n} \pi_\alpha(X) \delta_\alpha^\beta = \sum_{\alpha=1}^{n} \rho_\alpha(X) \delta_\alpha^\beta = \rho_\beta(X), \quad \beta \in \{1, \dots, n\}.$$

It follows $\pi_\beta = \rho_\beta$, and, by (2.5), $\pi = \rho$.

**Proposition 2.3.** *The map* $\Phi : (P^1(M, \nabla))^{n+1} \to P^1(L(M), \nabla^C)$ *defined by:*

$$\Phi(\pi, \pi_1, \dots, \pi_n) = \pi^V + \sum_{\beta=1}^{n} (\pi_\beta)_\beta^C, \quad \pi,\ \pi_i \in P^1(M, \nabla), \tag{2.6}$$

*is an isomorphism of vector spaces.*

*In particular, we have:*

i) $\dim P^1(L(M), \nabla^C) = (n+1) \dim P^1(M, \nabla)$,

ii) $P^1(L(M), \nabla^C) = V(P^1(M, \nabla)) \oplus \bigoplus_{\beta=1}^{n} C_\beta(P^1(M, \nabla))$,

*where $V$, $C_\beta$ denote the operators of vertical and complete lifts on 1-forms.*

The proof follows from the Proposition 2.2.

**Proposition 2.4.** *Given $\omega \in \Lambda^1(M)$, we have:*

i) *if $\omega \in P^1(M, \nabla)$, then, for any $\alpha \in \{1, \dots, n\}$, $\gamma_\alpha(\omega) \in A(L(M), \nabla^C)$;*

ii) if $\gamma_\alpha(\omega) \in A(L(M), \nabla^C)$ *for some $\alpha \in \{1, \dots, n\}$, then $\omega \in P^1(M, \nabla)$.*

Note that, if $\omega \in P^1(M, \nabla)$, then $d\omega = 0$, and by (1.15), $d(\gamma_\alpha(\omega)) = \omega_\alpha^C$. From the Proposition 2.1, it follows $d(\gamma_\alpha(\omega)) \in P^1(L(M), \nabla^C)$, and so $\gamma_\alpha(\omega)$ is $\nabla^C$-affine. Let now suppose $\gamma_\alpha(\omega) \in A(L(M), \nabla^C)$. Locally, we have: $\omega_{|U} = \omega_i dx^i$ and

$$d(\gamma_\alpha(\omega))_{|\pi_L^{-1}(U)} = y_\alpha^r \frac{\partial \omega_r}{\partial x^i} dx^i + \omega_r dy_\alpha^r.$$

Applying (2.3) to $d(\gamma_\alpha(\omega)) \in P^1(L(M), \nabla^C)$, we get:

$$\frac{\partial \omega_j}{\partial x^i} - \Gamma_{ji}^p \omega_p = 0; \text{ i.e. } \omega \in P^1(M, \nabla).$$

**Proposition 2.5.** *Given $f \in \mathcal{F}(M)$, we have:*

i) $f \in A(M, \nabla) \leftrightarrow f^V \in A(L(M), \nabla^C)$;

ii) $f \in A(M, \nabla) \leftrightarrow f_\alpha^C \in A(L(M), \nabla^C)$, *for some $\alpha \in \{1, \dots, n\}$ (and hence for any $\alpha$).*

The Proposition 1.1 and the relation (1.12) imply:

$$\nabla^C(d(f^V)) = \nabla^C((df)^V) = (\nabla(df))^V; \quad \nabla^C(d(f_\alpha^C)) = \nabla^C((df)_\alpha^C) = (\nabla(df))_\alpha^C.$$

From the first relation we obtain i).
Furthermore, if $f$ is affine, from the second relation we have $f_\alpha^C \in A(L(M), \nabla^C)$. Vice versa, if $f_\alpha^C$ is affine for $\nabla^C$, we have $(\nabla(df))_\alpha^C = 0$. As in the proof of the Proposition 2.1, we obtain $df \in P^1(M, \nabla)$, so $f$ is affine.

**Proposition 2.6.** *Let $f$ be an affine function on $(L(M), \nabla^C)$. Then there exist, uniquely determined, parallel 1-forms $\pi_1, \dots, \pi_n$ on $M$, and an affine function $h$ on $M$ such that:*

$$f = h^V + \sum_{\beta=1}^{n} \gamma_\beta(\pi_\beta). \tag{2.7}$$

If $f$ is affine for $\nabla^C$, then $df \in P^1(L(M), \nabla^C)$, and so, applying the Proposition 2.2, we get:

$$df = \pi^V + \sum_{\beta=1}^{n} (\pi_\beta)_\beta^C = \pi^V + \sum_{\beta=1}^{n} d(\gamma_\beta(\pi_\beta)),$$

with $\pi, \pi_1, \dots, \pi_n \in P^1(M, \nabla)$ uniquely determined. Now, put

(2.8) $$g=f-\sum_{\beta=1}^{n}\gamma_\beta(\pi_\beta),$$

and obtain $dg=\pi^V$. It follows that $g$ is constant along the fibres of $L(M)$, and so $g=h^V$, $h\in\mathscr{J}(M)$, and (2.8) becomes:

$$f=h^V+\sum_{\beta=1}^{n}\gamma_\beta(\pi_\beta).$$

Finally, using the above relation, we get $h^V\in A(L(M), \nabla^C)$, and then $h\in A(M, \nabla)$. The uniqueness in (2.7) is trivial.

**Proposition 2.7.** *The map* $\psi: A(M, \nabla)\times(P^1(M, \nabla))^n\to A(L(M), \nabla^C)$ *defined by*

$$\psi(h, \pi_1,\dots, \pi_n)=h^V+\sum_{\beta=1}^{n}\gamma_\beta(\pi_\beta)$$

*is an isomorphism of vector spaces. In particular, we have:*

(2.9) $$\dim A(L(M), \nabla^C)=n\dim P^1(M, \nabla)+\dim A(M, \nabla).$$

**Proposition 2.8.** *Let us put* $r=a(M, \nabla)$, $r+s=\dim P^1(M, \nabla)$. *Then, we have*:

(I) $r>0$, $s>0$. *Fixed a base* $\{1, f_1,\dots, f_r\}$ of $A(M, \nabla)$, *given* $\{\omega_1,\dots, \omega_s\}$ *with* $\{df_i, \omega_j\}$ *base of* $P^1(M, \nabla)$, *then* $\{1, f_i^V, (f_i)_\alpha^C, \gamma_\alpha(\omega_j)\}$ *with* $i\in\{1,\dots, r\}$, $j\in\{1,\dots, s\}$, $\alpha\in\{1,\dots, n\}$, *is a base of* $A(L(M), \nabla^C)$.

(II) $r>0$, $s=0$. *Fixed a base* $\{1, f_1,\dots, f_r\}$ of $A(M, \nabla)$, *then* $\{1, f_i^V, (f_i)_\alpha^C\}$, with $i\in\{1,\dots, r\}$, $\alpha\in\{1,\dots, n\}$ *is a base of* $A(L(M), \nabla^C)$.

(III) $r=0$, $s>0$. *Fixed a base* $\{\omega_j\}$, $1\leq j\leq s$ *of* $P^1(M, \nabla)$, *then* $\{1, \gamma_\alpha(\omega_j)\}$ *with* $j\in\{1,\dots, s\}$, $\alpha\in\{1,\dots, n\}$, *is a base of* $A(L(M), \nabla^C)$.

Let $\{df_i, \omega_j\}$ be a base of $P^1\{M, \nabla)$. The Proposition 2.6 and (1.14) imply that $\{1, f_i^V, (f_i)_\alpha^C, \gamma_\alpha(\omega_j)\}$ spans $A(L(M), \nabla^C)$. From (2.9), we get that this family is a base of $A(L(M), \nabla^C)$. Analogous arguments hold in the cases (II), (III).

## § 3. The space $A(T(M), \nabla^c)$

**Proposition 3.1.** *Given* $\omega\in\Lambda^1(M)$, *the following statements are equivalent:*

i) $\omega\in P^1(M, \nabla)$;

ii) $\omega^v\in P^1(T(M), \nabla^c)$;

iii) $\omega^c\in P^1(T(M), \nabla^c)$.

Our assertion is a consequence of the relations: $\nabla^c(\omega^c)=(\nabla\omega)^c$, $\nabla^c(\omega^v)=(\nabla\omega)^v$, (see [13]).

**Proposition 3.2.** *For any* $\omega\in P^1(T(M), \nabla^c)$ *there exist, uniquely determined,* $\omega_1$, $\omega_2\in P^1(M, \nabla)$ *such that* $\omega=\omega_1^v+\omega_2^c$.

For $\omega\in P^1(T(M), \nabla^c)$, applying the Proposition 1.6 and (2.2), we have $\tau_1^*(\omega)\in P^1(L(M), \nabla^C)$, and so:

(3.1) $$\tau_1^*(\omega)=\pi^V+\sum_{\beta=1}^{n}(\pi_\beta)_\beta^C$$

with $\pi, \pi_1, \ldots, \pi_n \in P^1(M, \nabla)$, uniquely determined. Locally, we have:

(3.2) $$\pi_{\beta|U}=h_j^\beta dx^j, \ (h_j^\beta)^V=(\tau_1^*\omega)_j^\beta, \ \beta, j\in\{1,\ldots, n\},$$

(see the proof of the Proposition 2.2).
On the other hand, if $\omega_{|\pi^{-1}(U)}=\omega_i dx^i+\omega_J dy^j$, we have:

$$(\tau_1^*\omega)_j^\beta=(\tau_1^*\omega)\left(\frac{\partial}{\partial y_\beta^j}\right)=(\tau_1^*\omega)((e_j)^{\beta V})=\delta_1^\beta\omega_J.$$

By means of (3.2), we obtain $\pi_{\beta|U}=0$, for any $\beta\in\{2,\ldots, n\}$.
Using (3.1) and the Proposition 1.5, we get:

$$\tau_1^*(\omega)=\pi^V+(\pi_1)_1^C=\tau_1^*(\pi^v)+\tau_1^*(\pi_1^c).$$

Since $\tau_1^*$ is injective, we obtain: $\omega=\pi^v+\pi_1^c$.
The uniqueness in the representation of $\omega$ is trivial.

**Proposition 3.3.** *The map $\Phi: P^1(M, \nabla)\times P^1(M, \nabla)\to P^1(T(M), \nabla^c)$ defined by: $\Phi(\omega_1, \omega_2)=\omega_1^v+\omega_2^c$ is an isomorphism of vector spaces. In particular, we have:*
*i) $\dim P^1(T(M), \nabla^c)=2\dim P^1(M, \nabla)$;*
*ii) $P^1(T(M), \nabla^c)=v(P^1(M, \nabla))\oplus c(P^1(M, \nabla))$,*
*where $v$, $c$ denote the operators of vertical and complete lift acting, on the 1-forms.*

**Proposition 3.4.** *For any 1-form $\omega$ on $M$, we have:*

$$\omega\in P^1(M, \nabla)\leftrightarrow\gamma(\omega)\in A(T(M), \nabla^c).$$

The condition $\nabla\omega=0$ implies $d\omega=0$, then we have:

(3.3) $$d(\gamma(\omega))=\omega^c.$$

By the Proposition 3.1, we get $\gamma(\omega)\in A(T(M), \nabla^c)$.
Conversely, if $\gamma(\omega)$ is affine, then $d(\gamma(\omega))\in P^1(T(M), \nabla^c)$, and so $d(\gamma(\omega))=\pi^v+\sigma^c$, where $\pi, \sigma\in P^1(M, \nabla)$ are uniquely determined by the Proposition 3.2. Using (3.3), we obtain $d(\gamma(\omega-\sigma))=\pi^v$, and so $\gamma(\omega-\sigma)$ is constant along the fibres of $T(M)$, that is $\omega-\sigma=0$. We deduce $\omega\in P^1(M, \nabla)$.

**Proposition 3.5.** *For a fixed $f\in\mathcal{J}(M)$ we have:*

$$f\in A(M, \nabla)\leftrightarrow f^v\in A(T(M), \nabla^c)\leftrightarrow f^c\in A(T(M), \nabla^c).$$

The statement is a consequence of the Proposition 3.1 and of the relations $d(f^v)=(df)^v$, $d(f^c)=(df)^c$; (see [13]).

**Proposition 3.6.** *Let $f$ be an affine function on $(T(M), \nabla^c)$. Then there exist, uniquely determined, $h\in A(M, \nabla)$, $\pi\in P^1(M, \nabla)$ such that*

(3.4) $$f = h^v + \gamma(\pi).$$

Since $f \in A(T(M), \nabla^c)$, we have $df \in P^1(T(M), \nabla^c)$, and so $df = \omega^v + \pi^c$, where $\omega$, $\pi \in P^1(M, \nabla)$ are uniquely determined by the Proposition 3.2. Then, $\gamma(\pi) \in A(T(M), \nabla^c)$ and, using (3.3), $df = \omega^v + d(\gamma(\pi))$, that is $d(f - \gamma(\pi)) = \omega^v$. It follows that $f - \gamma(\pi)$ is constant along the fibres of $T(M)$, and so $f - \gamma(\pi) = h^v$, with $h^v \in A(T(M), \nabla^c)$. This implies $h \in A(M, \nabla)$ and (3.4). The uniqueness in (3.4) is trivial.

**Proposition 3.7.** *The map $\Psi : A(M, \nabla) \times P^1(M, \nabla) \to A(T(M), \nabla^c)$ defined by $\Psi(h, \pi) = h^v + \gamma(\pi)$ is an isomorphism of vector spaces. In particular, we have:*

(3.5) $$\dim A(T(M), \nabla^c) = \dim A(M, \nabla) + \dim P^1(M, \nabla).$$

**Proposition 3.8.** *Put $r = a(M, \nabla)$, and $\dim P^1(M, \nabla) = r + s$. We have:*

(I) $r > 0$, $s > 0$. *Fixed a base $\{1, f_1, \dots, f_r\}$ of $A(M, \nabla)$, and given $\{\omega_1, \dots, \omega_s\}$ such that $\{df_i, \omega_j\}$ is a base of $P^1(M, \nabla)$, then $\{1, f_i^v, f_i^c, \gamma(\omega_j)\}$ with $i \in \{1, \dots, r\}$, $j \in \{1, \dots, s\}$ is a base of $A(T(M), \nabla^c)$.*

(II) $r > 0$, $s = 0$. *Fixed a base $\{1, f_1, \dots, f_r\}$ of $A(M, \nabla)$, then $\{1, f_i^v, f_i^c\}$ with $i \in \{1, \dots, r\}$, is a base of $A(T(M), \nabla^c)$.*

(III) $r = 0$, $s > 0$. *Fixed a base $\{\omega_j\}$, $1 \leqq j \leqq s$, of $P^1(M, \nabla)$, then $\{1, \gamma(\omega_j)\}$, with $1 \leqq j \leqq s$, is a base of $A(T(M)\ \nabla^c)$.*

Let $\{df_j, \omega_j\}$ be a base of $P^1(M, \nabla)$. By means of the Proposition 3.6 and of the relation $\gamma(df) = f^c$, we deduce that $\{1, f_i^v, f_i^c, \gamma(\omega_j)\}$ spans $A(T(M), \nabla^c)$. This family is a base of $A(T(M), \nabla^c)$, taking account of (3.5).
The cases (II), (III) can be discussed analogously.

Remark. The relation (3.5) implies that the existence of non constant affine functions on $(T(M), \nabla^c)$ is equivalent to the condition $\dim P^1(M, \nabla) > 0$. For instance, if $M$ is compact and simply connected, then $T(M)$ is not compact and has constant $\nabla^c$-affine functions only.

## §4. The souls of $T(M)$

The existence of non constant affine functions on $M$ leads to the construction of souls of $M$ (see theorem 3.5, [5]).

In this section we use the same method as in [5] to determine souls of $T(M)$ and, when it is possible, we establish the links with the souls of $M$.

We recall that a distribution $D$ on $(M, \nabla)$, $\nabla$ symmetric, is said to be flat if, for each $X$, $Y \in D$, $\nabla_X Y$ belongs to $D$ (see [11]).

Furthermore $D$ is said to be geodesic if for each $x \in M$, $X \in D_x$, the $\nabla$-geodesic $\gamma : I \to M$ with initial conditions $(x, X)$ satisfies: $\dot{\gamma}(t) \in D_{\gamma(t)}$, $t \in I$.

It is well known that if $D$ is flat, then $D$ is integrable and geodesic.

**Proposition 4.1.** *Assume that $\dim P^1(M, \nabla) = h > 0$. Then, there exists a (differentiable), $(n-h)$-dimensional distribution $D$ on $M$, which is flat. Furthermore, the maximal integral manifolds of $D$ are totally geodesic.*

Let $\{\omega_i\}_{1 \leqslant i \leqslant h}$ be a base of $P^1(M, \nabla)$. Since each $\omega_i$ is parallel, $\omega_i$ are linearly independent at each point of $M$. Therefore, they define the distribution $D$ on $M$ such that

$$D_x = \bigcap_{i=1}^{h} \operatorname{Ker}(\omega_i)_x, \quad x \in M.$$

For $X, Y \in D$, and $i \in \{1, \ldots, h\}$, we have $\omega_i(\nabla_X Y) = \nabla_X(\omega_i(Y)) - (\nabla\omega_i)(Y, X) = 0$, and so $\nabla_X Y \in D$. Since $D$ is geodesic and the maximal integral manifolds of $D$ are connected, the last assertion holds.

Finally, note that $D$ is independent on the base chosen in $P^1(M, \nabla)$.

Now, we examine the three cases as in the Proposition 3.8.

The first case: $\dim P^1(M, \nabla) = h = r + s$, $r = a(M, \nabla) > 0$, $s > 0$.

Fixed $\{1, f_1, \ldots, f_r\}$ as a base of $A(M, \nabla)$, and $\{df_i, \omega_j\}$ as a base of $P^1(M, \nabla)$, let $\{1, f_i^v, f_i^c, \gamma(\omega_j)\}$ be the corresponding base of $A(T(M), \nabla^c)$. It is known, (see [5]), that the map $p: M \to \mathrm{R}^r$ such that

$$(4.1) \qquad p(x) = (f_1(x), \ldots, f_r(x)), \quad x \in M,$$

is a totally geodesic, surjective submersion; $p$ is uniquely determined up to affine transformations of $\mathrm{R}^r$. Furthermore, for each $a \in \mathrm{R}^r$, $N_a = p^{-1}\{a\}$ is a soul of $M$, with $\dim N_a = n - r$.

Since $\pi: T(M) \to M$ is a totally geodesic submersion, also the map $p \circ \pi : T(M) \to \mathrm{R}^r$ is a surjective, totally geodesic submersion. Furthermore, for any $a \in \mathrm{R}^r$, $(p \circ \pi)^{-1}\{a\}$ is the total space of the bundle $T(M)_{|N_a}$.

Remark. Given $a \in \mathrm{R}^r$ and $x \in N_a$, the maximal integral manifold of $D$ through $x$ is a submanifold of $N_a$.

In fact, if $x \in N_a$, let $V_x$ be the maximal integral manifold of $D$ through $x$, and fix $y \in V_x$. Let us consider a curve $\gamma: [0, 1] \to V_x$ from $x$ to $y$. Then, we have:

$$\frac{d}{dt}(f_i(\gamma(t))) = (df_i)_{\gamma(t)}(\dot\gamma(t)) = 0, \quad t \in [0, 1];$$

therefore, $f_i \circ \gamma$ is constant. It follows $f_i(y) = f_i(x) = a_i$, and so $y \in N_a$.

Using the theorem 3.5 [5], we obtain that the map $p': T(M) \to \mathrm{R}^{2r+s}$ such that, for any $(x; X) \in T(M)$

$$(4.2) \quad p'(x) = (f_1(x), \ldots, f_r(x), (df_1)_x(X), \ldots, (df_r)_x(X), (\omega_1)_x(X), \ldots, (\omega_s)_x(X))$$

is a surjective, totally geodesic submersion.

Furthermore, for any $\alpha \in \mathrm{R}^{2r+s}$, $N'_\alpha = p'^{-1}\{\alpha\}$ is a $(2(n-r)-s)$-dimensional soul of $T(M)$. Obviously, the following diagram:

$$\begin{array}{ccc} T(M) & \xrightarrow{p'} & \mathrm{R}^r \times \mathrm{R}^r \times \mathrm{R}^s \\ \pi\downarrow & & \downarrow \quad p_1 \\ M & \xrightarrow{p} & \mathrm{R}^r \end{array}$$

commutes, where $p_1(a, b, c) = a$. It follows:

$$(4.3) \qquad \pi(N'_\alpha) = N_{(\alpha_1, \ldots, \alpha_r)}.$$

**Proposition 4.2.** *Fixed $a \in \mathbb{R}^r$, $N'_{(a,0)}$ is the total space of the bundle $D_{|N_a}$. Furthermore, $D_{|N_a}$ is a subbundle of $T(N_a)$.*

Put $a=(a_1, \dots, a_r) \in \mathbb{R}^r$; then

$$N'_{(a,0)}=\{(x, X) \in T(M) | f_i(x)=a_i, (df_i)_x(X)=0=(\omega_j)_x(X)\}$$

$$=\{(x, X) \in T(M) | x \in N_a, X \in D_x\}.$$

This proves the first assertion.

Given $(x, X) \in N'_{(a,0)}$, the conditions $(df_i)_x(X)=0$ imply that $X \in T_x(N_a)$. In fact, for $c$, geodesic on $M$ with initial conditions $(x, X)$, we have: $f_i(c(t))=\alpha_i t+\beta_i$; then $\alpha_i=(df_i)_x(\dot{c}(0))=0$, $a_i=f_i(x)=f_i(c(0))=\beta_i$. Therefore, the curve $c$ belongs to $N_a$, and so $X=\dot{c}(0)$ is tangent to $N_a$.

**Corollary 4.1.** *The total space $D(M)$ of the bundle $D$ coincides with the $\bigcup_{a \in \mathbb{R}^r} N'_{(a,0)}$.*

In fact, given $(x, X) \in D(M)$, we put $p(x)=a \in \mathbb{R}^r$. Since $X \in D_x$, we have $p'(x, X)=(a, 0)$, and so $(x, X) \in N'_{(a,0)}$. Vice versa, given $(x, X) \in N'_{(a,0)}$, we have $p(x)=a$, $X \in D_x$, and so $(x, X) \in D(M)$.

We establish now the relations between the homotopies on $T(M)$ and on $M$, which realize the homotopy equivalence for $N'_\alpha$ and $N_{p_1(\alpha)}$ (see [5]).

**Lemma 4.1.** *Let $g$ be an arbitrary Riemannian metric on $M$, and $g^D$ the diagonal lift of $g$ to $T(M)$ with respect to $\nabla$. Then the following relations hold:*

(4.4) $\operatorname{grad}^D(f^v)=(\operatorname{grad} f)^H=(\operatorname{grad} f)^c-\gamma(\nabla(\operatorname{grad} f))$ $\quad f \in A(M, \nabla)$

(4.5) $\operatorname{grad}^D(\gamma(\omega))=Y^v_\omega$ $\quad \omega \in P^1(M, \nabla)$

(4.6) $\operatorname{grad}^D(f^c)=(\operatorname{grad} f)^v$ $\quad f \in A(M, \nabla)$,

*where $Y_\omega$ is the vector field associated with $\omega$ with respect to $g$.*

The relations (4.4), (4.5) are obtained by a direct computation; (4.6) follows from (4.5), using the formulas: $\gamma(df)=f^c$, $Y_{df}=\operatorname{grad} f$. The above lemma implies the following construction.

Let us consider the symmetric matrix $\bar{A} \in GL(2r+s, \mathbb{R})$ defined by $\bar{A}=(\bar{a}_{\alpha\beta})=(g^D(\operatorname{grad}^D(h_\alpha), \operatorname{grad}^D(h_\beta)))$, where $h_i=f_i^v$, $h_{r+i}=f_i^c$ for $i \in \{1, \dots, r\}$; $h_{2r+j}=\gamma(\omega_j)$ for $j \in \{1, \dots, s\}$. Then, we have:

$$\bar{A}=\begin{matrix} A^v & 0 & 0 \\ 0 & A^v & C^v \\ 0 & ({}^tC)^v & D^v \end{matrix},$$

where $A=(a_{ij})=(g(\operatorname{grad} f_i, \operatorname{grad} f_j))$; $C=(c_{ij})=(g(\operatorname{grad} f_i, Y_{\omega_j}))=(Y_{\omega_j}(f_i))$; $D=(d_{ij})=(g(Y_{\omega_i}, Y_{\omega_j}))$.

The inverse matrix of $\bar{A}$ is of this kind:

$$\bar{B} = \begin{matrix} B & 0 & 0 \\ 0 & Z & T \\ 0 & {}^tT & R \end{matrix}$$

where $B=(A^{-1})^v$. By means of $\bar{B}$, fixed $(a, b, c) \in \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^s$, we define the following vector field on $T(M)$:

$$(4.7) \qquad \bar{X}(a, b, c) = \sum_{i,j=1}^{r} a_i B_{ij} \operatorname{grad}^D(f_j^v) + \sum_{i,j=1}^{r} b_i Z_{ij} \operatorname{grad}^D(f_j^c)$$

$$+ \sum_{i=1}^{r} \sum_{j=1}^{s} b_i T_{ij} \operatorname{grad}^D(\gamma(\omega_j)) + \sum_{i=1}^{s} \sum_{j=1}^{r} c_i T_{ji} \operatorname{grad}^D(f_j^c)$$

$$+ \sum_{i,j=1}^{s} c_i R_{ij} \operatorname{grad}^D(\gamma(\omega_j)) = X(a)^H + W(b, c)^v,$$

where $X(a) = \sum_{i,j=1}^{r} a_i B_{ij} \operatorname{grad} f_j$, (see [5]) and $W(b, c)$ is a suitable vector field on $M$, depending on $b, c$.

**Proposition 4.3.** *Let us denote by* $\exp^c$ *the exponential map with respect to* $\nabla^c$. *Then the following diagram:*

$$\begin{array}{ccc} T(T(M)) & \xrightarrow{\exp^c} & T(M) \\ T\pi \downarrow & & \downarrow \pi \\ T(M) & \xrightarrow{\exp} & M \end{array}$$

*commutes.*

Let us fix $\xi = (u, \tilde{X}) \in T(T(M))$, with $u = (x, X)$. Then, we have: $\exp^c(\xi) = \exp^c_u(\tilde{X}) = \bar{\gamma}(1)$, where $\bar{\gamma}$ is the $\nabla^c$-geodesic with initial conditions $(u, \tilde{X})$. Then, $\gamma = \pi \circ \bar{\gamma}$ is the $\nabla$-geodesic with initial conditions $\gamma(0) = \pi(u) = x$, $\dot{\gamma}(0) = \pi_{*_u}(\tilde{X})$. It follows that:

$$\pi(\exp^c(\xi)) = \pi(\exp^c_u(\tilde{X})) = \gamma(1) = \exp_x(\pi_{*_u}(\tilde{X})) = (\exp \circ T_\pi)(\xi).$$

**Corollary 4.2.** *For any* $X \in \mathscr{X}(M)$, $u \in T(M)$, *we have:*

$$(4.8) \qquad \pi(\exp^c_u(X^c_u)) = \exp_{\pi(u)}(X_{\pi(u)}).$$

*For any* $u \in T(M)$, *and any vertical vector* $X$ *tangent at* $u$, *we have:*

$$(4.9) \qquad \pi(\exp^c_u(X)) = \pi(u).$$

Applying the construction of T. Higa ([5], 52-53) to this case, we consider the map $\bar{G} : \mathbb{R} \times \mathbb{R}^{2r+s} \times T(M) \to T(M)$ such that

$$\bar{G}(t, \alpha, u) = \exp^c_u(t\bar{X}(\alpha)_u) \quad t \in \mathbb{R}, \ \alpha \in \mathbb{R}^{2r+s}, \ u \in T(M).$$

Fixed $\alpha \in \mathbb{R}^{2r+s}$, let us consider the maps $\bar{H}_\alpha : \mathbb{R} \times T(M) \to T(M)$ and $\bar{r}_\alpha : T(M) \to N'_\alpha$ defined by $\bar{H}_\alpha(t, u) = \bar{G}(t, \alpha - p'(u), u)$ and $\bar{r}_\alpha(u) = \bar{G}(1, \alpha - p'(u), u)$.

Then, the inclusion $i_\alpha : N'_\alpha \to T(M)$ is an homotopy equivalence, since, for any $u \in T(M)$, we have $\bar{H}_\alpha(0, u) = u$, $\bar{H}_\alpha(1, u) = I_\alpha \circ \bar{r}_\alpha(u)$.
Furthermore, if we consider the soul $N_{p_1(\alpha)}$ of $M$ corresponding to $p_1(\alpha)$, we denote with $H_{p_1(\alpha)} : \mathsf{R} \times M \to M$ the map defined by:

$$H_{p_1(\alpha)}(t, x) = G(t, p_1(\alpha) - p(x), x) = \exp_x(tX(p_1(\alpha) - p(x))_x),$$

and we consider the retraction $r_{p_1(\alpha)} : M \to N_{p_1(\alpha)}$ such that:

$$r_{p_1(\alpha)}(x) = H_{p_1(\alpha)}(1, x).$$

The homotopy and the retraction corresponding to the soul $N'_\alpha$ of $T(M)$ project onto those which correspond to the soul $N_{p_1(\alpha)}$ of $M$. In fact, the following result is valid.

**Proposition 4.4.** *The diagrams*

$$(1)\quad \begin{array}{ccc} \mathsf{R} \times \mathsf{R}^{2r+s} \times T(M) & \xrightarrow{\bar{G}} & T(M) \\ id \times p_1 \times \pi \downarrow & & \downarrow \pi \\ \mathsf{R} \times \mathsf{R}^r \times M & \xrightarrow[G]{} & M \end{array} \qquad (2)\quad \begin{array}{ccc} \mathsf{R} \times T(M) & \xrightarrow{\bar{H}_\alpha} & T(M) \\ id \times \pi \downarrow & & \downarrow \pi \\ \mathsf{R} \times M & \xrightarrow[H_{p_1(\alpha)}]{} & M \end{array}$$

*commute. In particular, we have:* $\pi \circ \bar{r}_\alpha = r_{p_1(\alpha)} \circ \pi$

As regards the diagram (1), using (4.7), the corollary (4.2) and the relation $X^H = X^c - \gamma(\nabla X)$, we get:

$$\pi(\bar{G}(t, (a, b, c), u)) = \pi(\exp^c_u(t\bar{X}(a, b, c)_u)) = \pi(\exp^c_u(tX(a)^H_u + tW(b, c)^v_u))$$

$$= \exp_{\pi(u)}(tX(a)_{\pi(u)}) = G(t, a, \pi(u)).$$

As regards the diagram (2), we have:

$$\pi(\bar{H}_\alpha(t, u)) = \pi(\bar{G}(t, \alpha - p'(u), u)) = G(t, p_1(\alpha) - p(\pi(u)), \pi(u)) = H_{p_1(\alpha)}(t, \pi(u)).$$

Finally, we obtain:

$$\pi(\bar{r}_\alpha(u)) = \pi(\bar{G}(1, \alpha - p'(u), u)) = G(1, p_1(\alpha) - p(\pi(u)), \pi(u)) = r_{p_1(\alpha)}(\pi(u)).$$

The second case: $\dim P^1(M, \nabla) = r = a(M, \nabla)$, $r > 0$.

Fix $\{1, f_1, \ldots, f_r\}$ as a base of $A(M, \nabla)$, and $\{1, f^v_i, f^c_i\}$ as a base of $A(T(M), \nabla^c)$. Let $p : M \to \mathsf{R}^r$, $p' : T(M) \to \mathsf{R}^{2r}$ be the submersions corresponding to the given bases. The relations analogous to (4.2), (4.3) hold, with $s = 0$. As in the Proposition 4.2, we obtain the following result.

**Proposition 4.5.** *For any $a \in \mathsf{R}^r$, the soul $N'_{(a,0)}$ of $T(M)$ is the total space of the bundle $T(N_a)$. Furthermore, the total space of the bundle* D *coincides with the* $\bigcup_{a \in \mathsf{R}^r} N'_{(a,0)}$.

As in the first case, we fix a riemannian metric $g$ on $M$, and we consider the square matrix $\bar{A}$ of order $2r$, constructed in an analogous way. This matrix has the form

$$\bar{A}=\begin{pmatrix} A^v & 0 \\ 0 & A^v \end{pmatrix} \quad \text{with inverse matrix } \bar{B}=\begin{pmatrix} (A^{-1})^v & 0 \\ 0 & (A^{-1})^v \end{pmatrix}.$$

It follows that, for any $(a, b) \in R^r \times R^r$, the corresponding vector field on $T(M)$ is $\bar{X}(a, b)=X(a)^H+X(b)^v$.
Also in this case we consider the maps $\bar{G}$, $\bar{H}_\alpha$, $\bar{r}_\alpha$, $\alpha \in R^{2r}$, and we obtain a proposition which is analogous to the Proposition 4.4.

The third case: $\dim P^1(M, \nabla)=s>0$, $a(M, \nabla)=0$.

Let $\{1, \gamma(\omega_j)\}$ be the base of $A(T(M), \nabla^c)$, corresponding to a given base $\{\omega_j)$ of $P^1(M, \nabla)$. We consider the submersion $p': T(M) \to R^s$ such that

$$p'(x, X)=((\omega_1)_x(X), \ldots, (\omega_s)_x(X)). \tag{4.10}$$

**Proposition 4.6.** *The soul $N'_0=p'^{-1}\{0\}$ of $T(M)$ is the total space of the bundle $D$.*

The proof is trivial.
Note that, in this case, $N'_0$ is independent on the choice of the base of $P^1(M, \nabla)$. For a fixed riemannian metric $g$ on $M$, the square matrix, of order $s$, $\bar{A}=(g^D(\mathrm{grad}^D(\gamma(\omega_i)), \mathrm{grad}^D(\gamma(\omega_j))))$ coincides with $D^v$, (see the first case). Therefore, for each $\alpha \in R^s$, we have:

$$\bar{X}(\alpha)=\Big(\sum_{i,j=1}^{s} \alpha_i (D^{-1})_{ij} Y_{\omega_j}\Big)^v = W(\alpha)^v. \tag{4.11}$$

**Lemma 4.2.** *For any $u=(x, X) \in T(M)$, $Y \in \mathcal{X}(M)$, we have:*

$$\exp^c_u(Y^v_u)=(x, X+Y_x).$$

In fact, the $\nabla^c$-geodesic $\bar{\gamma}$ with initial condition $(u, Y^v_u)$ is defined by $\bar{\gamma}(t)=(x, X+tY_x)$.
Using (4.11) and the lemma 4.2, we obtain that the map $\bar{G}$ acts as follows, for any $u=(x, X) \in T(M)$, $t \in R$, $\alpha \in R^s$:

$$\bar{G}(t, \alpha, u)=exp^c_u(t\bar{X}(\alpha)_u)=(x, X+tW(\alpha)_x). \tag{4.12}$$

Finally, we have:

**Proposition 4.7** *Given $\alpha \in R^s$, the homotopy $\bar{H}_\alpha$ and the retraction $\bar{r}_\alpha$ act along the fibres of $T(M)$.*

In fact, using (4.12), *for each* $t \in R$ and $(x, X) \in T(M)$, we get:

$$\pi(\bar{H}_\alpha(t, (x, X)))=\pi(x, X+tW(\alpha-p'(x, X))_x)=x,$$

$$\pi(\bar{r}_\alpha(x, X))=\pi(x, X+W(\alpha-p'(x, X))_x)=x.$$

## §5. The vector space $V(T(M), \nabla^c)$

We recall that, for a given connected manifold $M$, with a (symmetric) connection $\nabla$, it is possible to define the real spaces:

$$\tilde{W}(M,\nabla)=\{X\in\mathscr{X}(M)\,|\,\forall f\in A(M,\nabla)\ \ X(f)=\text{const.},\ \nabla_X X=0\},$$

$$W(M,\ \nabla)=\{X\in\tilde{W}(M,\ \nabla)\,|\,\forall\, Y\in\tilde{W}(M,\ \nabla)\ \ \nabla_Y X=0\},$$

$$W_0(M,\ \nabla)=\{X\in W(M,\ \nabla)\,|\,\forall f\in A(M,\ \nabla)\ \ X(f)=0\},$$

$$V(M,\ \nabla)=W(M,\ \nabla)/_{W_0(M,\nabla)};$$

(see [5]). Furthermore, we have:

(5.1) $$\nabla X=0\Rightarrow X\in W(M,\ \nabla)$$

(5.2) $$\dim V(M,\nabla)\leqq a(M,\ \nabla)$$

(5.3) If $\nabla$ is the Levi-Civita connection on $(M,\ g)$ then

$$\dim\ V(M,\ \nabla)=a(M,\ \nabla).$$

**Lemma 5.1.** If $X\in\mathscr{X}(M)$ is $\nabla$-parallel, then we have $X^c,\ X^v\in W(T(M),\ \nabla^c)$.
The statement follows from the relations:

$$\nabla^c(X^v)=(\nabla X)^v,\ \nabla^c(X^c)=(\nabla X)^c,\ \text{using (5.1).}$$

We denote by $L^1(M,\nabla)$ the real vector space consisting of the $\nabla$-parallel vector fields on M.

**Proposition 5.1.** For any $\tilde{X}\in L^1(T(M),\ \nabla^c)$ there exist, uniquely determined, vector fields $T,Z\in L^1(M,\ \nabla)$, such that

(5.4) $$\tilde{X}=T^v+Z^c=T^v+Z^H.$$

Let us put $\tilde{X}=\tilde{T}+\tilde{Z}$, with $\tilde{T}$ vertical vector field, $\tilde{Z}$ horizontal vector field. The condition $\nabla^c\tilde{X}=0$ implies, for any $Y\in\mathscr{X}(M)$:

(5.5) $$\nabla^c_{Y^v}\tilde{T}+\nabla^c_{Y^v}\tilde{Z}=0,$$

(5.6) $$\nabla^c_{Y^c}\tilde{T}+\nabla^c_{Y^c}\tilde{Z}=0.$$

Let us put $\tilde{T}_{|\pi^{-1}(U)}=\tilde{T}^{\bar{\imath}}\dfrac{\partial}{\partial y^i},\ \tilde{Z}_{|\pi^{-1}(U)}=\tilde{Z}^i\{\dfrac{\partial}{\partial x^i}-y^r\Gamma^h_{ri}\dfrac{\partial}{\partial y^h}\}$, where $U$ is a chart of $M$, with local coordinates $\{x^i\}$. The relation (5.5) implies $Y^j\dfrac{\partial\tilde{Z}^i}{\partial y^j}=0=Y^j\dfrac{\partial\tilde{T}^{\bar{\imath}}}{\partial y^j}$. Since $Y$ is arbitrary, we deduce that the components $\tilde{Z}^i,\tilde{T}^{\bar{\imath}}$ are constant along the fibres of $T(M)$ and so $\tilde{T}=T^v,\ \tilde{Z}=Z^H$, with $T,\ Z\in\mathscr{X}(M)$. Substituting in (5.6), we have $(\nabla_Y T)^v+(\nabla_Y Z)^c-\nabla^c_{Y^c}(\gamma(\nabla z))=0$. Since the first and the last terms are vertical, it follows $\nabla_Y Z=0$, for each $Y\in\mathscr{X}(M)$, and so $Z\in L^1(M,\ \nabla)$. Finally, we obtain $T\in L^1(M,\ \nabla)$. The uniqueness in the representation (5.4) is trivial.

**Corollary 5.1.** *The map* $\lambda : L^1(M, \nabla) \times L^1(M, \nabla) \to L^1(T(M), \nabla^c)$ *such that* $\lambda(X, Y) = X^c + Y^v$ *is an isomorphism of vector spaces. In particular, we have:* $\dim L^1(T(M), \nabla^c) = 2 \dim L^1(M, \nabla)$.

**Proposition 5.2.** *Given* $\omega \in \Lambda^1(M)$, *the following conditions are equivalent:*

a) $\omega \in P^1(M, \nabla)$

b) $\gamma(\omega) \in A(T(M), \nabla^c)$

c) *for any* $Z \in \mathscr{X}(T(M))$, $Z$ *horizontal, we have* $Z(\gamma(\omega)) = 0$.

For a)$\leftrightarrow$b) see the Proposition 3.4. Note that, if $Z$ is an horizontal vector field, locally we have:

$$Z_{|\pi^{-1}(U)} = Z^i\{\frac{\partial}{\partial x^i} - y^r\Gamma^h_{ri}\frac{\partial}{\partial y^h}\}. \text{ It follows that}$$

$$Z(\gamma(\omega))_{|\pi^{-1}(U)} = Z^i y^r\{\frac{\partial \omega_r}{\partial x^i} - \Gamma^j_{ri}\omega_j\} = Z^i y^r (\nabla\omega)_{ri},$$

thus proving the equivalence a)$\leftrightarrow$c).
To determine the dimension of $V(T(M), \nabla^c)$, we suppose that $(M, g)$ is a riemannian manifold and $\nabla$ is the Levi-Civita connection. We are going to prove that $\dim V(T(M), \nabla^c) = a(T(M), \nabla^c)$. This result is a direct consequence of (5.3), when $g$ is flat; in fact, $\nabla^c$ is the Levi-Civita connection on $(T(M), g^D)$ iff $g$ is flat.

**Proposition 5.3.** *Let* (M, g) *be a riemannian manifold, and* $\nabla$ *be the Levi-Civita connection. Given* $0 \neq \omega \in P^1(M, \nabla)$, *we have:*

(5.7) $$Y^v_\omega \in W(T(M), \nabla^c),\ Y^v_\omega \notin W_0(T(M), \nabla^c);$$

(5.8) $$Y^c_\omega \in W(T(M), \nabla^c);$$

(5.9) $$Y^c_\omega \in W_0(T(M), \nabla^c) \leftrightarrow Y_\omega \in W_0(M, \nabla).$$

Since $\omega \in P^1(M, \nabla)$, we have $Y_\omega \in L^1(M, \nabla)$ and then, using the

Lemma 5.1, $Y^v_\omega$, $Y^c_\omega \in W(T(M), \nabla^c)$. Suppose now $Y^v_\omega \in W_0(T(M), \nabla^c)$. Since $\gamma(\omega) \in A(T(M), \nabla^c)$, we have $0 = Y^v_\omega(\gamma(\omega)) = \omega(Y_\omega)^v = g(Y_\omega, Y_\omega)^v$, and so $\omega = 0$. Furthermore, using the Proposition 3.6, the condition $Y^c_\omega \in W_0(T(M), \nabla^c)$ is equivalent to the conditions:

(5.10) $$0 = Y^c_\omega(f^v) = Y_\omega(f)^v, \qquad f \in A(M, \nabla)$$

(5.11) $$0 = Y^c_\omega(\gamma(\pi)), \qquad \pi \in P^1(M, \nabla).$$

Since $Y_\omega \in L^1(M, \nabla)$, we have $Y^c_\omega = Y^H_\omega$; by means of the Proposition 5.2, we obtain that (5.11) is always valid. Therefore, the equivalence in (5.9) is proved.

**Proposition 5.4.** *Put* dim $P^1(M,\nabla)=r+s,\ r=a(M,\nabla)$. *Then, we have:*

(I) $r>0, s>0$. *Fixed* $\{df_i,\ \omega_j\}$ *as a base of* $P^1(M,\ \nabla)$, *then*

$$\{[(\text{grad } f_i)^v],\ [(\text{grad } f_i)^c],\ [Y^v_{\omega_j}]\} \text{ is a base of } V(T(M),\ \nabla^c).$$

(II) $r>0, s=0$. *Fixed a base* $\{1, f_1, \ldots f_r\}$ *of* $A(M,\ \nabla)$, *then*

$$\{[(\text{grad } f_i)^v],\ [(\text{grad } f_i)^c]\} \text{ is a base of } V(T(M),\ \nabla^c).$$

(III) $r=0, s>0$. *Fixed a base* $\{\omega_j\},\ 1\leqq j\leqq s$, *of* $P^1(M,\nabla)$, *then*

$$\{[Y^v_{\omega_j}]\} \text{ is a base of } V(T(M),\ \nabla^c).$$

*In any case, we have* dim $V(T(M),\ \nabla^c)=a(T(M),\ \nabla^c)$.

Let us consider $\{df_i, \omega_j\}$ as a base of $P^1(M,\nabla)$. It is known ([5]), that $\{[\text{grad } f_i]\},\ 1\leqq i\leqq r$, is a base of $V(M,\nabla)$. Therefore, grad $f_i \notin W_0(M,\ \nabla)$, and, since grad $f_i = Y_{df_i}$, the Proposition 5.3 implies: $(\text{grad} f_i)^c \in W(T(M),\ \nabla^c)\backslash W_0(T(M),\nabla^c)$. Now, we show that $\{[(\text{grad } f_i)^v],\ [(\text{grad } f_i)^c],\ [Y^v_{\omega_j}]\}$ is free. Suppose that $a^i[(\text{grad } f_i)^v]+b^i[(\text{grad } f_i)^c]+c^j[Y^v_{\omega_j}]=0$, and put $Z=a^i$ grad $f_i + c^j Y_{\omega_j}$, $T=b^i \text{grad } f_i$. Then, we have $Z^v+T^c \in W_O(T(M),\nabla^c)$, and so, for any $f\in A(M,\nabla)$, $(Z^v+T^c)(f^v)=T(f)^v=0$. Therefore, $T\in W_O(M,\ \nabla)$, and $0=[T]=b^i[\text{grad } f_i]$ implies $b^i=0,\ i=1,\ldots r$.

Since $T=0$, we have $Z^v \in W_O(T(M),\ \nabla^c)$, and so:

$$0=Z^v(f_i^c)=Z(f_i)^v=g(\text{grad } f_i,\ Z)^v,\ i=1,\ldots r$$

$$0=Z^v(\gamma(\omega_j))=\omega_j(Z)^v=g(Y_{\omega_j},\ Z)^v,\ j=1,\ldots s.$$

It follows: $\|Z\|^2=a^i g(\text{grad} f_i,\ Z)+c^j g(Y_{\omega_j}, Z)=0$, that is $Z=0$. Since $\{\text{grad} f_i, Y_{\omega_j}\}$ is a base of $L^1(M,\ \nabla)$, we have $a^i=0,\ c^j=0$, for any $i$ and $j$. Therefore, $2r+s\leqq \dim V(T(M),\ \nabla^c)$, and, using (5.2), we obtain the equality. The cases (II), (III) can be proved in an analogous way.

## §6. Theorems of reducibility for $T(M)$ and existence of compact souls

Let $(M,\ g)$ be a connected, complete riemannian manifold, and $\nabla$ the corresponding Levi-Civita connection. It is known that $a(M,\nabla)=\dim V(M,\nabla)$. If $a(M,\ \nabla)=r>0$, then, applying the theorem 3.2 [6], we obtain that, up to diffeomorphisms, $M$ is the riemannian product $\mathrm{R}^r\times \bar{M}$, where $\bar{M}$ is a connected riemannian manifold, without non constant affine functions. Furthermore, $\bar{M}=N_0$ is the soul of $M$ corresponding to the zero vector of $\mathrm{R}^r$, determined by means of a base $\{1, f_1, \ldots, f_r\}$ of $A(M,\ \nabla)$ such that:

$$df_i(\text{grad } f_j)=\delta_{ij} \qquad i,j\in\{1,\ldots r\}. \tag{6.1}$$

We consider now the manifold $(T(M), \nabla^c)$. Since, generally, $\nabla^c$ is not the riemannian connection on $(T(M), g^D)$, we cannot apply the above theorem. On the other hand, if dim $P^1(M, \nabla)>0$, then dim $V(T(M), \nabla^c)=a(T(M), \nabla^c)>0$, and so we can apply the theorem 3.8 [5]. We get the following result.

**Proposition 6.1.** *If* dim $P^1(M, \nabla)=r+s>0$, *then there exists a connected, totally geodesic submanifold* $\bar{M}'$ *of* $T(M)$ *such that* $T(M)$ *is diffeomorphic to the product* $R^{2r+s}\times \bar{M}'$.

At first, suppose $r=a(M, \nabla)>0, s>0$, and fix an orthonormal base $\{df_i, \omega_j\}$ of $P^1(M, \nabla, g)$. Let us consider the free family

$$\{(\text{grad } f_i)^c, (\text{grad } f_i)^v, Y^v_{\omega_j}\}, \tag{1}$$

and note that

$$\{1, f_i^v, f_i^c, \gamma(\omega_j)\} \tag{2}$$

is a base of $A(T(M), \nabla^c)$. It is easy to show that (1), (2) satisfy a condition which is analogous to (6.1). Therefore, applying the theorem 3.8 [5], the manifold $T(M)$ is diffeomorphic to $R^{2r+s}\times \bar{M}'$, where $\bar{M}'=N'_0$ is the soul of $(T(M), \nabla^c)$ obtained by means of the submersion $p'$ defined in (4.2).
Furthermore, considering $\{1, f_1, .. f_r\}$ as a base of $A(M, \nabla)$ and the corresponding submersion $p: M \to R^r$, $M$ is diffeomorphic to the product $R^r\times N_0$, where $N_0$ is the soul of $M$ corresponding to $0\in R^r$. From (4.3), we have $\pi(N'_0)=N_0$, and $N'_0$ is the total space of the bundle $D_{|N_0}$ (see Proposition 4.2). We have up to diffeomorphisms of bundles

$$T(M)=T(R^r)\times T(N_0)=T(R^r)\times((N_0\times R^s)\oplus D_{|N_0})=(M\times R^{r+s})\oplus D\cdot$$

Let now suppose $s=0, r>0$. With the same procedure, using the theorem 3.8 [5], we obtain that $T(M)$ is diffeomorphic to $R^{2r}\times N'_0$, $M$ is diffeomorphic to $R^r\times N_0$ and $N'_0$ is the total space of the bundle $T(N_0)$, (see Proposition 4.5). Furthermore, up to diffeomorphisms of bundles, we have $T(M)=T(R^r)\times T(N_0)$.
Finally, suppose that $r=0, s>0$. Then, applying the theorem 3.8, [5], the manifold $T(M)$ is diffeomorphic to $R^s\times N'_0$, with $N'_0=D(M)$. So, up to diffeomorphisms of bundles, we have $T(M)=(M\times R^s)\oplus D$, (see Proposition 4.6). As regards the existence of compact souls of a non compact manifold $M$, we note that, if there is a compact soul $N_a$ then each soul $N_b$ is compact (see theorem 3.5 [5]). Furthermore, if $M$ is the product $R^k\times \bar{M}$, with $k=dim\ V(M, \nabla)$,then $\bar{M}$ is compact iff $k(M)=n-k$, where $k(M)$ is a positive integer defined by the conditions:

$$H_i(M, Z_2)=0 \qquad i>k(M)$$

$$H_i(M, Z_2)\neq 0 \qquad i=k(M).$$

For a connected manifold $M$, we have $k(M)\leq$ dim $M$, and the equality holds iff $M$ is compact.

**Proposition 6.2.** *Let $(M, g)$ be a complete, non compact, riemannian manifold, with $r=a(M, \nabla)>0$. Then $T(M)$ has a compact soul iff $M$ has a compact soul and $g$ is flat. In this case, $T(M)$ is the trivial bundle over $M$.*
We distinguish two cases. At first, let be dim $P^1(M, \nabla)=r+s, r>0, s>0$. If $T(M)$ has a compact soul, then $N'_O$ is compact. It follows that the soul of $M$, $N_0=\pi(N'_O)$ is compact, (see Proposition 6.1). Furthermore, we have $k(T(M))=2(n-r)-s$, $k(M)=n-r$. The equality $k(T(M))=k(M)$ implies $n=r+s=dim\ P^1(M, \nabla)$, and so $g$ is flat. Conversely, if $g$ is flat and $M$ has a compact soul, we get dim $P^1(M, \nabla)=n$ and $N_O$ is compact. Therefore, $k(T(M))=k(M)=n-r=s$. Since dim $N'_o=2(n-r)-s=s$, we have dim $N'_O=k(N'_O)$, and so $N'_o$ is compact. Let us suppose that $N'_O$ is compact. We have $n=r+s$. Since $N'_O$ is the total space of the bundle $D_{|N_0}$, we get dim $N'_O=n-r=dimN_0$, and so $N'_O=N_O\times\{0\}$. Up to diffeomorphisms, for the total space of the tangent bundle we have $T(M)=\mathrm{R}^{2r+s}\times(N_0\times\{0\})=\mathrm{R}^n\times(\mathrm{R}^r\times N_O)=\mathrm{R}^n\times M$. In the second case, that is dim $P^1(\mathrm{M}, \nabla)=r>0$, we suppose that $T(M)$ has a compact soul. Therefore, $N'_0$ is compact, which implies that $N_o=\pi(N'_O)$ is a compact soul of $M$. Furthermore, we have $n-r=k(M)=k(T(M))=2(n-r)$, and so $n=r$. Also in this case g is flat. Vice versa, if $g$ is flat and $M$ has a compact soul, we have $n=$dim $P^1(M, \nabla)=r$, and $N_O$ is compact. The relations dim $N'_O=2(n-r)=0$ and $k(N'_o)=k(T(M))=k(M)=k(N_O)=0$, imply the compactness of $N'_O$. Note that, if $N'_0$ is compact, $N'_O$ and $N_0$ both reduce at a point; $(M, g)$ is isometric to $(\mathrm{R}^n, g_0)$ (see theorem 3.1 [6]), and $T(M)=\mathrm{R}^{2n}$, up to diffeomorphisms.

**Proposition 6.3.** *Let $(M, g)$ be a complete and riemannian manifold with $a(M, \nabla)=0$ and dim $P^1(M, \nabla)=s>0$. Then $T(M)$ has a compact soul iff $M$ is compact and $g$ is flat. In this case, $T(M)$ is the trivial bundle over $M$.*

If $T(M)$ has a compact soul, then $N'_O=D(M)$ is compact, and so $M$ is compact and $n=k(M)=k(T(M))=2n-s$, that is $n=s$ and $g$ is flat. Vice versa, if $g$ is flat and $M$ is compact, we have:

$$k(N'_O)=k(T(M))=k(M)=n, \qquad \dim N'_O=2n-s=n,$$

and so $N'_O$ is compact.
If $N'_o$ is compact, then $N'_o=M\times\{0\}$ and, up to diffeomorphisms, for the manifold $T(M)$, we get $T(M)=\mathrm{R}^s\times N'_o=\mathrm{R}^n\times M$.

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*Received 17. 02. 1988*