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On Nanson's Inequality and on Some Inequalities Related to It

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1. The following result is due to E. J. Nanson:
"If a real sequence $a=(a_1, a_2, \ldots, a_{2n+1})$ is convex, then

$$\frac{a_1+a_3+\ldots+a_{2n+1}}{n+1} \geqq \frac{a_2+a_4+\ldots+a_{2n}}{n}, \tag{1}$$

with equality if and only if a is an arithmetic progression.

One can find this result in [1, p. 205], where its proof is also given. This proof is correct, but it appeared to the authors rather artificially constructed. That incited them to try to find a different proof, which would be, without regard to the length, essentially more natural. The form of the proof of Nanson's inequality that they — tending to this purpose — obtained, conducted them to a more general result. Besides that general proposition and its corollaries this paper contains several other results, related in a different way to Nanson's inequality.

Before passing to our main results, we shall give some auxiliarly assertions.

Lema 1. *If a function $f(x)$ is convex (strictly convex) on the intervals $[a, b]$ and $[b, c]$ $(a<b<c)$, and $f'_-(b) \leqq f'_+(b)$, then this function is also convex (strictly convex) on the interval $[a, c]$.*

Proof. By hypothesis, we have, with $a \leqq y_1 < b < y_2 \leqq c$,

$$\frac{f(b)-f(y_1)}{b-y_1} \underset{(<)}{\leqq} f'_-(b) \leqq f'_+(b) \underset{(<)}{\leqq} \frac{f(y_2)-f(b)}{y_2-b}.$$

Hence we get for $a \leqq x_1 < x_2 < b < x_3 \leqq c$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \underset{(<)}{\leqq} \frac{f(b)-f(x_2)}{b-x_2} \underset{(<)}{\leqq} \frac{f(x_3)-f(b)}{x_3-b}$$

and further, using the fact that the second inequality implies

$$\frac{f(b)-f(x_2)}{b-x_2} \underset{(<)}{\leqq} \frac{f(x_3)-f(x_2)}{x_3-x_2},$$

we obtain the inequality

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \underset{(<)}{\leqq} \frac{f(x_3)-f(x_2)}{x_3-x_2}. \tag{2}$$

If $a \leqq x_1 < b < x_2 < x_3 \leqq c$, we have

$$\frac{f(x_3)-f(x_2)}{x_3-x_2} \underset{(>)}{\geqq} \frac{f(x_2)-f(b)}{x_2-b} \underset{(>)}{\geqq} \frac{f(b)-f(x_1)}{b-x_1}$$

and further, since the second inequality implies

$$\frac{f(x_2)-f(b)}{x_2-b} \underset{(>)}{\geqq} \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

we conclude that (2) holds in this case, too. Therefore (2) is valid for all x_1, x_2, $x_3 \in [a, c]$ such that $x_1 < x_2 < x_3$, which means that the function f is convex (strictly convex) on $[a, c]$.

Lemma 2. *If a sequence $a_n (n \in \mathrm{N})$ is convex, then the function f whose graph is the polygonal line with corner points $(n, a_n)(n \in \mathrm{N})$ is also convex on $[1, \infty)$.*

Therefore, a convex sequence $a_n (n \in \mathrm{N})$ has the property

$$a_q \leqq \frac{(r-q)a_p+(q-r)a_r}{r-p} \quad (1 \leqq p < q < r)$$

with strict inequality for fixed p, q and r if and only if the sequence (a_n) is not arithmetic for $p \leqq n \leqq r$.

Similar assertions concerning finite sequence are valid.

Proof. The supposition implies the convexity of the function f on each interval $[k, k+1]$ $(k \in \mathrm{N})$ and its property $f'_-(k) \leqq f'_+(k)$ $(k \in \mathrm{N},\ k \geqq 2)$. Hence, using Lemma 1 and applying mathematical induction, one can easily prove the convexity of f on each interval $[1, k]$ $(k \in \mathrm{N},\ k \geqq 2)$, and that obviously implies its convexity on $[1, +\infty)$.

It follows from the established, that

$$a_q = f(q) \leqq \frac{(r-q)f(p)+(q-r)f(r)}{r-p} = \frac{(r-q)a_p+(q-r)a_r}{r-p} \quad (1 \leqq p < q < r).$$

According to a known property of convex function, this inequality is equality if and only if f is linear on $[p, r]$, that is if and only if (a_n) is arithmetic for $p \leqq n \leqq r$.

It is clear that all we previously established is also valid, in a corresponding form, in the case of a finite sequence.

Lemma 3. *If a sequence (a_n) is convex, then the sequences (a_{2n-1}), (a_{2n}), and generally all sequences (a_{mn-p}) with fixed $m \in \mathrm{N}$ and $p \in \{0, \ldots, m-1\}$, are convex.*

Lemma 4. *Let A and B be real, and α and β — positive numbers. Then any of the inequalities*

$$\frac{A}{\alpha} \underset{(<)}{\leqq} \frac{B}{\beta}, \quad \frac{A}{\alpha} \underset{(<)}{\leqq} \frac{A+B}{\alpha+\beta}, \quad \frac{A+B}{\alpha+\beta} \underset{(<)}{\leqq} \frac{B}{\beta}$$

implies remaining two of them, i.e. the inequality

$$\frac{A}{\alpha} \underset{(<)}{\leqq} \frac{A+B}{\alpha+\beta} \underset{(<)}{\leqq} \frac{B}{\beta}.$$

Lemma 3 follows immediately from Lemma 2. The assertion of Lemma 4 is evident and well-known.

Let us remark that we have, according to Lemma 4, the following corollary of Nanson's inequality

$$\frac{\sum_{k=0}^{n} a_{2k+1}}{n+1} \geqq \frac{\sum_{k=1}^{2n+1} a_k}{2n+1} \geqq \frac{\sum_{k=1}^{n} a_{2k}}{n}$$

holding for convex sequence $(a_1, \ldots, a_{2n+1})$ (a similar result is given in [5, pp. 59-60]).

2. We pass to a result of a general kind.

Theorem 1. *Let $a=(a_1, \ldots, a_n)$ be a convex sequence and let $I_n=\{1, \ldots, n\}$. Let further I, J and M be nonempty subsets of I_n, the sets I and J being non-overlapping. Let these three sets have cardinal numbers α, β and γ, respectively, and let*

$$u=\sum_{i\in I} i, \quad v=\sum_{i\in J} i, \quad w=\sum_{i\in M} i.$$

Then, under the conditions

$$\frac{u}{\alpha}<\frac{v}{\beta} \wedge (\forall\, m\in M)\, \frac{u}{\alpha} \leqq m \leqq \frac{v}{\beta}, \tag{4}$$

the inequality

$$\sum_{i\in M} a_i \leqq \frac{\gamma v-\beta w}{\alpha v-\beta u} \sum_{i\in I} a_i + \frac{\alpha w-\gamma u}{\alpha v-\beta u} \sum_{i\in J} a_i \tag{5}$$

holds. If besides (4) *the condition*

$$\frac{w}{\gamma}=\frac{u+v}{\alpha+\beta} \tag{6}$$

is satisfied, then

$$\frac{\sum_{i\in M} a_i}{\gamma} \leqq \frac{\sum_{i\in I\cup J} a_i}{\alpha+\beta} \tag{7}$$

holds. If, in addition, M and $I \cup J$ are non-overlapping sets, the inequality

$$\frac{\sum_{i\in M} a_i}{\gamma} \leqq \frac{\sum_{i\in I\cup J\cup M} a_i}{\alpha+\beta+\gamma} \leqq \frac{\sum_{i\in I\cup J} a_i}{\alpha+\beta} \tag{7'}$$

is also true.

Each of the inequalities (5), (7) *and* (7′) *becomes an equality if and only if a is an arithmetic progression on the set*

$$S = \{\min(I\cup J),\ \min(I\cup J)+1, \dots,\ \max(I\cup J)\}.$$

Without the supposition

$$(\forall m \in M) \frac{u}{\alpha} \leqq m \leqq \frac{v}{\beta}, \tag{4'}$$

the conditions $\frac{u}{\alpha} < \frac{v}{\beta}$ *and* (6) *do not imply the inequality* (7).

Proof. For $m \in M$ arbitrarily chosen, we can find nonnegative numbers P and Q such that $P+Q>0$ and

$$m = \frac{Pu+Qv}{P\alpha+Q\beta},$$

i. e.

$$(\alpha m - u)P = (v - \beta m)Q.$$

This is satisfied for

$$P = v - \beta m, \quad Q = am - u,$$

these numbers P and Q being, according to (4), nonnegaive and with positive sum. Using Lemma 2 and Jensen's inequality applied to the function f mentioned in that Lemma, we obtain

$$a_m = f(m) \leqq \frac{P\alpha f(\frac{u}{\alpha}) + Q\beta f(\frac{v}{\beta})}{P\alpha+Q\beta} = \frac{P\alpha f(\frac{1}{\alpha}\sum_{i\in I} i) + Q\beta f(\frac{1}{\beta}\sum_{i\in J} i)}{P\alpha+Q\beta}$$

$$\leqq \frac{P\sum_{i\in I} f(i) + Q\sum_{i\in J} f(i)}{P\alpha+Q\beta},$$

i. e.

(8) $$a_m \leqq \frac{(v-\beta m)A+(\alpha m-u)B}{\alpha v-\beta u},$$

where $A=\sum_{i\in I} a_i$, $B=\sum_{i\in J} a_i$. It follows further from (8)

$$\sum_{m\in I} a_m \leqq \frac{1}{\alpha v-\beta u}\{A \sum_{m\in M} (v-\beta m)+B \sum_{m\in M} (\alpha m-u)\}$$

$$=\frac{\gamma v-\beta w}{\alpha v-\beta u}A+\frac{\alpha w-\gamma u}{\alpha v-\beta u}B,$$

that is the inequality (5).

If the condition (6) is satisfied, then we have

$$\frac{\gamma v-\beta w}{\alpha v-\beta u}=\frac{\alpha w-\gamma u}{\alpha v-\beta u}=\frac{\gamma}{\alpha+\beta},$$

so that (5) becomes, I and J being non-overlapping sets,

$$\frac{\sum_{i\in M} a_i}{\gamma} \leqq \frac{\sum_{i\in I\cup J} a_i}{\alpha+\beta},$$

i.e. (7). If, in addition, $M\cap(I\cup J)=\emptyset$, this inequality implies (7'), according to Lemma 4.

Further, one can easily see that if the sequence a is not an arithmetic progression (linear sequence) on the set S, then any of the inequalities (8) is strict, and so the inequality (7) is strict, too.

Finally, the last assertion is confirmed by the example of the sequence $a_n=n^2$ ($n=1,\ldots, 8$), with $I=\{1, 3, 4\}$, $J=\{5, 6, 8\}$, $M=\{2, 7\}$. In this case the condition (6) is satisfied, but

$$\frac{u}{\alpha}=\frac{8}{3}>2\in M,$$

and so (4') do not hold. Since

$$\frac{\sum_{i\in M} a_i}{\gamma}=\frac{2^2+7^2}{2}=\frac{53}{2}, \quad \frac{\sum_{i\in I\cup J} a_i}{\alpha+\beta}=\frac{151}{6}<\frac{159}{2.3}=\frac{53}{2},$$

the inequality (7) is not satisfied.

This completes the proof of Theorem 1.

Corollary 1. *Let $a=(a_1,\ldots, a_n)$ be a convex sequence. Then for $m\in\mathbb{N}$ and $2m\leqq n-1$ the inequalities*

$$\frac{1}{n-2m}\sum_{i=m+1}^{n-m} a_i \leqq \frac{1}{n}\sum_{i=1}^{n} a_i \leqq \frac{1}{2m}\Big(\sum_{i=1}^{m} a_i + \sum_{i=n-m+1}^{n} a_i\Big)$$

hold. Both inequalities become equalities if and only if a is an arithmetic progression.

3. Using Theorem 1, we shall prove the following generalization of Nanson's inequality.

Theorem 2. *Let the sequence* $a=(a_1, a_2, \ldots, a_{2n+1})$ *be convex. Then for* $m \in \mathbf{N}$ *and* $2m \leqq n+1$ *the following inequality*

$$\frac{1}{n-2m+2}\sum_{k=m}^{n+1-m} a_{2k} \leqq \frac{1}{n+1}\sum_{k=1}^{n+1} a_{2k-1} \tag{9}$$

holds, with equality if and only if a is an arithmetic progression.

Proof. We start from the inequality defining a convex sequence

$$2a_{2k} \leqq a_{2k-1} + a_{2k+1}.$$

Adding from $k=m$ to $k=n+1-m$ one gets

$$2\sum_{k=m}^{n+1-m} a_{2k} \leqq a_{2m-1} + 2\sum_{k=m+1}^{n+1-m} a_{2k-1} + a_{2(n+1-m)+1}$$

$$= 2\sum_{k=m}^{n-m+2} a_{2k-1} - (a_{2m-1} + a_{2(n-m)+3})$$

$$\leqq 2\sum_{k=m}^{n-m+2} a_{2k-1} - \frac{2}{n-2m+3}\sum_{k=m}^{n-m+2} a_{2k-1},$$

i.e.

$$\sum_{k=m}^{n+1-m} a_{2k} \leqq \frac{n-2m+2}{n-2m+3}\sum_{k=m}^{n-m+2} a_{2k-1}. \tag{10}$$

Here we use the inequality

$$\frac{1}{n-2m+3}\sum_{k=m}^{n-m+2} a_{2k-1} \leqq \frac{a_{2m-1} + a_{2(n-m)+3}}{2}$$

which follows immediately from Theorem 1. It follows also from Theorem 1 (first inequality (7′)) the inequality

$$\frac{1}{n-2m+3}\sum_{k=m}^{n-m+2} a_{2k-1} \leqq \frac{1}{n+1}\sum_{k=1}^{n+1} a_{2k-1}. \tag{11}$$

Combining (10) and (11) one gets (9). The statement concerning equality in (9) is clear, taking into consideration the corresponding assertion of Theorem 1. This completes the proof.

Remark 1. Using the substitution $a_i = b_{i+1}$ one gets from (9), with $n \geqq 2$ instead of $n+1$,

$$\frac{1}{n-2m+1} \sum_{k=m}^{n-m} a_{2k+1} \leqq \frac{1}{n} \sum_{k=1}^{n} a_{2k} \quad (m \in \mathrm{N},\ 2m \leqq n),$$

and especially, for $m=1$,

$$\frac{1}{n-1} \sum_{k=1}^{n-1} a_{2k+1} \leqq \frac{1}{n} \sum_{k=1}^{n} a_{2k},$$

or, according to Lemma 4,

$$\frac{1}{n-1} \sum_{k=1}^{n-1} a_{2k+1} \leqq \frac{1}{2n-1} \sum_{k=2}^{2n} a_k \leqq \frac{1}{n} \sum_{k=1}^{n} a_{2k}.$$

4. As it has been established in [1, pp 205-206, 3.2.27], Nanson's inequality with $a_k = x^{k-1}$ becomes the inequality of J. M. Wilson

$$\frac{1+x^2+x^4+\ldots+x^{2n}}{x+x^3+\ldots+x^{2n-1}} \geqq \frac{n+1}{n} \quad (x>0).$$

Using Theorem 2 one can, clearly, obtain a generalization of Wilson's inequality. On the other hand, starting from Corollary 1 one can also deduce the results given in [1, p. 198, 3.2.4] (or in [5, p. 139, 2.3.1.4]), in [3, p. 142, 2.3.1.5] and in [3, p. 144, 2.3.1.6].

The following statement generalizes many results of this kind.

Theorem 3. *Let p and q be real numbers and let $p>q$.*

1° *If $q>0 \vee (p>0>q \wedge p+q<0)$, then*

$$\frac{a^{p+q}-1}{a^p-a^q} > \frac{p+q}{p-q} \quad (0<a \neq 1). \tag{12}$$

2° *If $(p>0>q \wedge p+q>0) \vee p<0$, then* (12) *with reverse inequality is valid.*

Proof. As the substitutions $a \to 1/a$ do not change the left side of (12), it suffices to consider this inequality for $a>1$. For the function

$$f(a) = a^{p+q} - 1 - \frac{p+q}{p-q}(a^p - a^q) \quad (a>1)$$

we have

$$f'(a) = \frac{p+q}{p-q} a^{q-1}((p-q)a^p - pa^{p-q} + q) \quad (a>1).$$

If we put

$$g(a) = (p-q)a^p - pa^{p-q} + q,$$

then

$$g'(a) = (p-q)pa^{p-q-1}(a^q-1).$$

For $p>q>0$, we have $g'(a)>0$ $(a>1)$ and consequently $g(a)>g(1)=0$. That implies $f'(a)>0$ $(a>1)$ and further $f(a)>f(1)=0$, and so (12) holds in this case. If $p>0>q$ and $p+q<0$, we have $g'(a)<0$ and consequently $g(a)<g(1)=0$ $(a>1)$, which implies $f'(a)>0$ $(a>1)$ and further $f(a)>f(1)=0$ $(a>1)$, so that (12) holds again. For $p>0>q$ and $p+q>0$ we have, as in the preceding case, $g(a)<0$ $(a>1)$, which now implies $f'(a)<0 (a>1)$ and hence $f(a)<f(1)=0 (a>1)$ and so the reverse inequality of (12) holds. Finally, under the conditions $0>p>q$ we have $g'(a)>0$ $(a>1)$ and so $g(a)>g(1)=0$ $(a>1)$, wherefrom it follows $f'(a)<0$ $(a>1)$, so that $f(a)<\;\;(1)=0$, i.e. the reverse inequality in (12) holds again. The proof is finished.

Remark 2. Besides the generalizations of Wilson's inequality and other mentioned results, Theorem 3 gives improvement of a result of D. Ž. Djoković [3, pp 162-163, 2.3.2.8] (also see [1, p. 276, 3.6.26]), and also improves the inequality in [1, p. 279, 3.6.31] (or in [3, 2.5.18]).

Corollary 2. *The function* $h(a)=\dfrac{x^a-x^{-a}}{a}$, *with fixed* $x>1$, *is strictly increasing for* $a>0$.

First proof. Let us put in Theorem 3: $a=x>1$, $p=u+v$, $q=u-v$ with $u>v>0$. Then we get

$$\frac{x^{2u}-1}{x^{u+v}-x^{u-v}}>\frac{u}{v},$$

or

$$\frac{x^u-x^{-u}}{u}>\frac{x^v-x^{-v}}{v}\quad(x>1,\ u>v>0)$$

which was to be proved.

Second proof. As we have

$$(13)\quad h(a)=\frac{x^a-x^{-a}}{a}=2\frac{\operatorname{sh}(a\ln x)}{a}=2\sum_{n=0}^{\infty}\frac{\ln^{2n+1}x}{(2n+1)!}a^{2n}\quad(a\neq0,\ x>0),$$

the function h with fixed $x>1$ increases strictly for $a>0$.

Remark 3. From Corollary 2 immediately follows the mentioned improvement of 3.6.31 in [1, p. 279]. On the other hand, it follows from (13) that, with a fixed $x>1$, the function $h(a)$ strictly increases together with $|a|$. By using this fact and writing preliminarily (12) in the form

$$\frac{x^{\frac{p+q}{2}}-x^{-\frac{p+q}{2}}}{x^{\frac{p-q}{2}}-x^{-\frac{p-q}{2}}}>\frac{p+q}{p-q}\quad(0<x\neq1),$$

one can simply prove in another way all assertions of Theorem 3.

Corollary 3. *Let* $x>y>0$. *Then*

$$\frac{x^s-y^s}{s\sqrt{x^sy^s}} > \frac{x^r-y^r}{r\sqrt{x^ry^r}} > \ln x - \ln y \quad (0<r<s); \tag{14}$$

$$\frac{x-y}{\ln x - \ln ny} > \frac{r(x-y)}{x^r-y^r}\sqrt{x^ry^r} > \sqrt{xy} \quad (0<r<1); \tag{15}$$

for $r=1/2$ *the inequality* (15) *reduces to the following result from* [4]

$$\frac{x-y}{\ln x - \ln y} > \frac{1}{2}(x^{3/4}y^{1/4}+x^{1/4}y^{3/4}) > \sqrt{xy}.$$

Proof. According to Corollary 2, we have

$$h(\frac{s}{2}) > h(\frac{r}{2}) > \lim_{a\to+0} h(a) = 2\ln x,$$

wherefrom one deduces (14) using the substitution $x\to x/y$. (15) is a simple consequence of (14).

Remark 4. For $y=1$ and $r=1$ the second inequality (14) gives a Karamata's inequality [1, p. 272, 3.6.15]; more precisely, this Karamata's result in the form of a strict inequality. One really first obtains, directly, Karamata's inequality for $x>1$, and then, by substitution $x\to 1/x$, also for $0<x<1$.

Remark 5. It follows from (13) that $h'(a)>0$ $(a>0)$. Using this inequality we can easily obtain the inequality 3.6.17 in [1, p. 273]:

$$\frac{x-y}{\ln x - \ln y} < \frac{x+y}{2} \quad (0<y<x).$$

This result enables us to prolong the sequence (14) of inequalities by

$$\ln x - \ln y > \frac{2}{r}\frac{x^r-y^r}{x^r+y^r} > \frac{2}{s}\frac{x^s-y^s}{x^s+y^s} \quad (0<y<x,\ 0<r<s),$$

where the last inequality follows from the fact that the function $\frac{\operatorname{th} t}{t}$ decreases strictly for $t>0$.

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