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Functions of Bounded Boundary Rotation of Complex Order

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A function $f(z)$ analytic in the unit disk E, normalized by the conditions $f(0)=0, f'(0)=1$ and $f'(z)\neq 0$ in E is said to belong to $V_k(b)$, where b is a nonzero complex number if

$$\int_0^{2\pi} |\operatorname{Re}(1+\frac{1}{b}\cdot\frac{zf''(z)}{f'(z)})| d\theta \leqq k\pi$$

for all $z=re^{i\theta}\in$ E.

In this paper we obtain various representation theorems and using an invariant transformation we obtain the sharp radius of convexity for the class $V_k(b)$. Using Ahlfor's criterion for univalence we determine a condition on k and b for which a function belonging to $V_k(b)$ is univalent. Also using Goluzin's method of variation of parameters we have solved an extremal problem in the class $V_k(b)$ and used it to obtain the sharp distortion and rotation bounds for the functions in the class. Further by employing a technique due to Krzyż, we determine the sharp radius of close-to-convexity of the class $V_k(b)$.

1. Introduction

Let A be the class of functions $f(z)$ regular in the unit disk $\mathrm{E}=\{Z:|z|<1\}$ and normalized by the conditions $f(0)$ and $f'(0)=1$.

Definition. *A function $f(z)\in A, f'(z)\neq 0$ in* E *is said to belong to the class $V_k(b)$, where b is a non-zero complex number if*

$$\int_0^{2\pi} |\operatorname{Re}(1+\frac{1}{b}\cdot\frac{zf''(z)}{f'(z)})| d\theta \leqq k\pi \tag{1.1}$$

for all $z=re^{i\theta}\in$ E.

The functions of the class $V_k(b)$ are said to be functions of bounded boundary rotation of complex order b. For different values of k and b, $V_k(b)$ reduces to important subclasses studied by various authors.

(i) $V_k(1)$ is the well-known class V_k of functions of bounded boundary rotation at most $k\pi$ introduced by V. Paatero [12].

(ii) $V_k(1-\rho)$ is the class introduced by K. S. Padmanabhan and R. Parvatham [13].
(iii) $V_k(\cos\alpha e^{-i\alpha})$ is the class introduced by E. J. Moulis, Jr. [8].
(iv) $V_k[(1-\beta)\cos\alpha e^{-i\alpha}]$ is studied by the author [19].
(v) $V_2(b)$ is the class of convex functions of complex order b introduced by P. Wiatrowski [21] and also studied by M. A. Nasr and M. A. Aouf [10]. Also $V_2(\cos\alpha e^{-i\alpha})$ has been studied by R. J. Libera and M. R. Ziegler [7] and $V_2[(1-\beta)\cos\alpha e^{-i\alpha}]$ has been studied by P. N. Chichra [2], P. I. Sizuk [17] and the author [18].

Hence the results obtained in this paper generalize the results due to [7], [8], [10], [18], [19].

2. Preliminary Results

Let M_k denote the class of functions $\mu(t)$ of bounded variation with

$$\int_0^{2\pi} d\,\mu(t)=2 \quad \text{and} \quad \int_0^{2\pi} 1\,d\,\mu(t)|\leqq k.$$

Lemma 1. *If $f(z)\in V_k(b)$ there exists a function $\mu(t)\in M_k$ such that*

$$f'(z)=\exp\{-\mathrm{b}\int_0^{2\pi}\log(1-ze^{it})\,d\mu(t)\}. \tag{2.1}$$

Proof. If $\mathrm{f(z)}\in V_k(b)$ then (1.1) is satisfied for all $z\in \mathrm{E}$. Hence by a theorem due to V. Paatero [12] there exists a function $\mu(t)\in M_k$ such that

$$1+\frac{1}{b}\cdot\frac{zf''(z)}{f'(z)}=\frac{1}{2}\int_0^{2\pi}\frac{1+ze^{it}}{1-ze^{it}}\cdot \mathrm{d}\mu(t).$$

A little simplification and an integration gives the desired result.

Lemma 2. *$f(z)\in V_k(b)$ if and only if there exists a function $g(z)\in V_k(1)=V_k$ such that*

$$f'(z)=[g'(z)]^b. \tag{2.2}$$

Proof. Differentiating (2.2) logarithmically we get

$$1+\frac{1}{\mathrm{b}}\cdot\frac{zf''(z)}{f'(z)}=1+\frac{zg''(z)}{g'(z)}$$

$$\therefore \int_0^{2\pi}|\,\mathrm{Re}(1+\frac{1}{b}\cdot\frac{zf''(z)}{f'(z)})|\,\mathrm{d}\theta=\int_0^{2\pi}|\,\mathrm{Re}(1+\frac{zg''(z)}{g'(z)})|\,\mathrm{d}\theta,$$

where $z=re^{i\theta}\in \mathrm{E}$.

Hence the result follows.

Lemma 3. *$f(z)\in V_k(b)$ if and only if there exist two functions $g(z)$ and $h(z)$ in $V_2(b)=C(b)$ such that*

$$f'(z)=\frac{[g'(z)]^{\frac{k}{4}+\frac{1}{2}}}{[h'(z)]^{\frac{k}{4}-\frac{1}{2}}}. \tag{2.3}$$

Proof. For every function $\mu(t)\in M_k$ there exist two non-decreasing functions $u(t)$ and $v(t)$ such that $\mu(t)=u(t)-v(t)$

and $$\int_0^{2\pi} du(t)\leqq\frac{k}{2}+1, \qquad \int_0^{2\pi} dv(t)\leqq\frac{k}{2}-1.$$

Now using the representation (2.1), (2.3) follows.

Theorem 1. *If $f(z)=z+a_2z^2+a_3z^3+\ldots\in V_k(b)$ then*

$$|a_2|\leqq\frac{k}{2}\cdot|b|. \tag{2.4}$$

The result is sharp.

Proof. If $f(z)\in V_k(b)$ then by Lemma 2 there exists a function $g(z)=z+b_2z^2+\ldots\in V_k(1)$ such that

$$f'(z)=[g'(z)]^b.$$

$$\therefore 1+2a_2z+3a_3z^2+\ldots=[1+2b_2z+3b_3z^2+\ldots]^b.$$

Comparing the coefficients we get $a_2=b_2$. But $|b_2|\leqq\frac{k}{2}$ [11]. Hence $|a_2|\leqq\frac{k}{2}\cdot|b|$.

Equality in (2.4) can be obtained for a function $f(z)$ of the form

$$f'(z)=\left[\frac{(1-\varepsilon_1z)^{\frac{k}{2}-1}}{(1-e_1z)^{\frac{k}{2}+1}}\right]^b, \quad |\varepsilon_1|=|e_1|=1 \tag{2.5}$$

for $\varepsilon_1=-1$ and $e_1=1$.

Theorem 2. *Let $f(z)\in V_k(b)$ and $x\in E$. Then the function $F(z)$ defined by*

$$F'(z)=\frac{f'\left(\frac{z+x}{1+\bar{x}z}\right)}{f'(x)(1+\bar{x}z)^{2b}} \quad F(0)=0 \tag{2.6}$$

also belongs to $V_k(b)$.

Proof. Now $f(z)\in V_k(b)$, hence by Lemma 2 there exists a function $g(z)\in V_k(1)$ such that $f'(z)=[g'(z)]^b$. M. S. Robertson [15] has shown that $G(z)$ defined by

$$G(z)=\frac{g(\frac{z+x}{1+\bar{x}z})-g(x)}{g'(x).(1-|x|^2)} \quad \text{is also in } V_k(1),$$

whenever $g(z)$ is in $V_k(1)$. Hence there exists a function $F(z)\in V_k(b)$ such that $F'(z)=[G'(z)]^b$.

$$\therefore \quad F'(z)=\frac{g'(\frac{z+x}{1+\bar{x}z})}{[g'(x)]^b.(1+\bar{x}z)^{2b}}=\frac{f'(\frac{z+x}{1+xz})}{f'(x).(1+\bar{x}z)^{2b}},$$

which completes the proof of the theorem.

Theorem 3. *If $f(z)\in V_k(b)$ then $f(z)$ is convex for $|z|<r_0$, where r_0 is the least positive root of the equation*

$$(2.7) \qquad 1-k.|b|.r+(2\,\mathrm{Re}b-1)r^2=0.$$

The result is sharp.

Proof. If $f(z)\in V_k(b)$ then $F(z)$ defined by (2.6) also belongs to $V_k(b)$. If $F(z)=z+A_2z^2+\ldots$ then from (2.4), $|A_2|\leqq\frac{k}{2}\cdot|b|$.

But $$A_2=\frac{F''(0)}{2!}=\frac{1}{2}\{(1-|x|^2)\frac{f''(x)}{f'(x)}-2b.\bar{x}\}.$$

Replacing x by z and using the above inequality, we get

$$\left|\frac{zf''(z)}{f'(z)}-\frac{2b|z|^2}{1-|z|^2}\right|\leqq\frac{k.|b|.|z|}{1-|z|^2}.$$

Therefore,

$$\mathrm{Re}(1+\frac{zf''(z)}{f'(z)})\geqq\frac{1-k.|b|.r+(2\,\mathrm{Re}\,b-1)r^2}{1-r^2},$$

where $r=|z|$. Hence $\mathrm{Re}(1+\frac{zf''(z)}{f'(z)})>0$, for $|z|<r_0$, where r_0 is the least positive root of the equation (2.7).

The result is sharp for a function $f(z)$ defined by (2.5), where

$$\varepsilon_1=\frac{r-e^{-i\delta}}{1-re^{-i\delta}},\ e_1=\frac{r+e^{-i\delta}}{1+re^{-i\delta}},\ \delta=\arg b \text{ at } z=r.$$

If $b=(1-\beta)\cos\alpha e^{-i\alpha}$, we get a result obtained by the author [19].

Theorem 4. *A function $f(z)\in V_k(b)$ is univalent in* E *if* $|b|<\frac{1}{k}$.

Proof. If $f(z)\in V_k(b)$ then

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2b.|z|^2\right|\leqq k|b|.|z|<k|b|.$$

Then by using L. V. Ahlfor's [1] criterion for univalence where $c=2b$, $f(z)$ is univalent if $k|b|<1$, i.e. $|b|<\frac{1}{k}$.

If $b=(1-\rho)\cos\alpha e^{-i\alpha}$, Theorem 4 reduces to a result obtained by E. J. Moulis Jr. [9].

3. Distortion and Rotation Bounds

By using G. M. Goluzin's [4] method of variation of parameters we have solved an extremal problem for the class $V_k(b)$. This method has been previously used by B. Pinchuk [14], E. M. Silvia [16] author [18] to study some important subclasses. We omit the details, for they are similar to those of B. Pinchuk [14], E. M. Silvia [16] and the author [18].

Theorem 5. *Let $w\neq 0$ be a point of* E *and let $F(x_1, x_2, \dots x_{n+1})$ be analytic in a neighbourhood of each point*

$$F(f'(w), f''(w), \dots, f^{(n)}(w)\,;\ w),\quad f(z)\in V_k(b).$$

Then the functional $J(f')=\operatorname{Re}F(f'(w), f''(w), \dots f^{(n)}(w), w)$ attains its maximum or minimum in $V_k(b)$ only for a function of the form

$$f'(z)=\prod_{j=1}^{M}(1-\varepsilon_j z)^{bv_j}.\prod_{j=1}^{N}(1-e_j z)^{-bu_j}, \tag{3.1}$$

where

$$\sum_{j=1}^{N}u_j\leqq\frac{k}{2}+1,\quad \sum_{j=1}^{M}v_j\leqq\frac{k}{2}-1,$$

and $1\varepsilon_j=|e_j|=1$.

Thus choosing $J(f')=\operatorname{Re}\log f'(z)$ $(n=1)$ or $J(f')=\operatorname{Im}\log f'(z)$ in the above theorem the maximum or minimum of $J(f')$ can be attained only for a function of the form (2.5). Using elementary calculus we can prove the following theorems. Throughout the rest of the paper we assume that $b=c+id$.

Theorem 6. (*Distortion*) *If $f(z)\in V_k(b)$ then for $|z|=r<1$*

$$L(r)\leqq\log|f'(z)|\leqq M(r), \tag{3.2}$$

where

$$L(r)=(\frac{k}{2}-1).c.\log\frac{\sqrt{|b|^2-d^2r^2-cr}}{|b|}$$

$$-(\frac{k}{2}+1).c.\log\frac{\sqrt{|b|^2-d^2r^2+cr}}{|b|}-kd.\sin^{-1}(\frac{dr}{|b|})$$

and

$$M(r)=(\frac{k}{2}-1).c.\log\frac{\sqrt{|b|^2-d^2r^2+cr}}{|b|}$$

$$-(\frac{k}{2}+1).c.\log\frac{\sqrt{|b|^2-d^2r^2-cr}}{|b|}+kd\sin^{-1}(\frac{dr}{|b|}).$$

The result is sharp.

Theorem 7. (*Rotation*) *If* $f(z)\in V_k(b)$ *then for* $|z|=r<1$,

$$P(r)\leqq \arg f'(z)\leqq Q(r), \tag{3.3}$$

where

$$P(r)=(\frac{k}{2}-1).d.\log\frac{\sqrt{|b|^2-c^2r^2-dr}}{|b|}$$

$$-(\frac{k}{2}+1).d.\log\frac{\sqrt{|b|^2-c^2r^2+dr}}{|b|}-ck.\sin^{-1}(\frac{cr}{|b|})$$

and

$$Q(r)=(\frac{k}{2}-1).d.\log\frac{\sqrt{|b|^2-c^2r^2+dr}}{|b|}$$

$$-(\frac{k}{2}+1).d.\log\frac{\sqrt{|b|^2-c^2r^2-dr}}{|b|}+ck.\sin^{-1}(\frac{cr}{|b|}).$$

The result is sharp.

If $b=(1-\beta)\cos\alpha e^{-i\alpha}$ Theorems 6 and 7 reduce to Theorems 11 and 12 of the author [19]. If $k=2$, we get the results obtained by M. A. Nasr and M. A. Aouf [10]. For $b=1$, we get the well-known [14] results for the class V_k.

4. Radius of close-to-convexity

In [5], W. Kaplan has shown that a function $f(z)\in A$ and $f'(z)\neq 0$ in E maps $|z|=r<1$ onto a close-to-convex curve if and only if

$$\arg[z_2f'(z_2)]-\arg[z_1f'(z_1)]\geqq -\pi$$

for all z_1 and z_2 satisfying $z_2=z_1 e^{i\theta}$, $0<\theta<2\pi$ and $|z_2|=|z_1|=r$. By using the techniques introduced by J. Krzyz [6] we find the sharp radius of close-to-convexity of the class $V_k(b)$.

Theorem 8. *Let r_0 be the radius of convexity of $V_k(b)$ and let $x_0=\cos\frac{\theta_0}{2}$ be the unique root in* (0, 1] *of the equation*

$$Ax^4-Bx^2+C=0, \tag{4.1}$$

where

$$A=16(1-c)^2.r^4+4k^2d^2r^4,$$

$$B=4k^2d^2r^4+k^2r^2|b|^2\ (1-r^2)^2$$

$$+8(1-c)r^2(1+r^2)[1+(1-2c)r^2],$$

$$C=(1+r^2)^2[1+(1-2c)r^2]^2$$

for $r\in[r_0,\ 1)$, $x=\cos\frac{\theta}{2}$, $0\leqq\theta<\pi$

and

$$\Delta(r)=\theta_0+2c.\tan^{-1}(\frac{r^2\sin\theta_0}{1-r^2\cos\theta_0})$$

$$\begin{aligned}&+(\frac{k}{2}-1)d.\log\{[(1-r^2)^2+\frac{4d^2r^2\sin^2\theta_0/2}{|b|^2}]^{1/2}+\frac{2dr}{|b|}.\sin\frac{\theta_0}{2}\}\\ &-(\frac{k}{2}+1)d.\log\{[(1-r^2)^2+\frac{4d^2r^2\sin^2\theta_0/2}{|b|^2}]^{1/2}+\frac{2dr}{|b|}.\sin\frac{\theta_0}{2}\}\end{aligned} \tag{4.2}$$

$$-kc.\sin^{-1}\{\frac{2cr\sin\theta_0/2}{|b|(1-2r^2\cos\theta_0+r^4)^{1/2}}\}+2d.\log(1-r^2).$$

The radius of close-to-convexity of $V_k(b)$ is the unique root r_1 of the equation $\Delta(r)=-\pi$ in the interval $(r_0,\ 1)$.

Proof. Let $\Delta(r,\ \theta)=\inf\arg\{\frac{z_2.f'(z_2)}{z_1.f'(z_1)}\}$

$$f(z)\in V_k(b),$$

where $z_1,\ z_2$ are any two points satisfying $|z_1|=r<1$, $z_2=z_1e^{i\theta}$, $0<\theta<2\pi$ and the argument is so chosen as to vary continuously from an initial value of zero.

Let $f(z)\in V_k(b)$. Then from Theorem 2, $F(z)$ defined by

$$F'(z)=\frac{f'\left(\dfrac{z+x}{1+\bar{x}z}\right)}{f'(x).(1+\bar{x}z)^{2b}}, \quad F(0)=0$$

is also in $V_k(b)$. Let $w_0=\dfrac{z_2-z_1}{1-\bar{z}_1 . z_2}$.

Hence

$$F'(w_0)=\frac{f'(z_2)}{f'(z_1)}.\{\frac{1-\bar{z}_1 . z_2}{1-|z_1|^2}\}^{2b}.$$

$$\Delta(r,\ \theta)=\arg\frac{z_2}{z_1}\{\frac{1-|z_1|^2}{1-\bar{z}_1 . z_2}\}^{2b}+\inf_{F\in V_k(b)}\arg\ F'(w_0). \tag{4.3}$$

Also we have

$$|w_0|=r\{\frac{2(1-\cos\theta)}{1-2r^2\cos\theta+r^4}\}^{1/2}$$

and

$$\arg\frac{z_2}{z_1}\ \{\frac{1-|z_1|^2}{1-\bar{z}_1 . z_2}\}^{2b}=\theta+2c.\tan^{-1}(\frac{r^2\sin\theta}{1-r^2\cos\theta})$$

$$+2d.\log\{\frac{1-r^2}{(1-2r^2\cos\theta+r^4)^{1/2}}\}.$$

Using these results and Theorem 7 in (4.3), we get

$$\Delta(r,\ \theta)=\theta+2c.\tan^{-1}(\frac{r^2\sin\theta}{1-r^2\cos\theta})$$

$$+(\frac{k}{2}-1).d.\log\{[(1-r^2)^2+\frac{4d^2r^2}{|b|^2}\ \sin^2\frac{\theta}{2}]^{\frac{1}{2}}-\frac{2dr}{|b|}.\sin\frac{\theta}{2}\}$$

$$-(\frac{k}{2}+1).d.\log\{[(1-r^2)^2+\frac{4d^2r^2}{|b|^2}.\sin^2\frac{\theta}{2}]^{1/2}+\frac{2dr}{|b|}.\sin\frac{\theta}{2}\} \tag{4.4}$$

$$-ck\sin^{-1}\{\frac{2\,cr\sin\theta/2}{|b|.(1-2r^2\cos\theta+r^4)^{1/2}}\}+2d\log(1-r^2).$$

Further there exists a function $f(z)\in V_k(b)$ for which equality holds in (4.4), for fixed z_1 and z_2 letting $g(z)$ in $V_k(b)$ be defined by

$$f'(z)=\frac{g'(\dfrac{z-z_1}{1-\bar{z}_1 . z})}{g'(-z_1).(1-\bar{z}_1 . z)^{2b}}, \quad f(0)=0$$

it follows that

$$\Delta(r,\ \theta)=\arg\{\frac{z_2 g'(z_2)}{z_1 g'(z_1)}\}.$$

Let $\Delta(r)=\inf\limits_{0\leqq\theta<2\pi}\Delta(r,\theta)$. Then $\Delta(r)$ is a decreasing function of r and the radius of close-to-convexity is the root r_1 of the equation $\Delta(r)=-\pi$, provided such a root exists.

We now show that $\Delta(r)$ is given by (4.2) and establish the existence of the root r_1.

Since r_0 is the radius of convexity of $V_k(b)$, $\Delta(r)\geqq 0$ for $r\leqq r_0$, $\Delta(r_0)=0$ and so $r_1>r_0$. Thus, we assume that $r\in[r_0,\ 1)$.

Differentiating (4.4) w.r.t. θ, we get

$$\frac{\partial\Delta(r,\ \theta)}{\partial\theta}=\frac{P(x)-Q(x)}{R(x)},\ \text{where}$$

$$P(x)=(1+r^2)[1+(1-2c)r^2]-4(1-c).r^2x^2$$

$$Q(x)=kr|b|x[(1-r^2)^2+\frac{4d^2r^2}{|b|^2}(1-x^2)]^{1/2}$$

$$R(x)=(1+r^2)^2-4r^2x^2,$$

where

$$x=\cos\frac{\theta}{2}.$$

Now for $\theta\in[\pi,\ 2\pi)$, $\frac{\partial\Delta(r,\ \theta)}{\partial\theta}$ has no zeros and for $\theta\in(0,\ \pi)$, the zeros of $\frac{\partial\Delta(r,\ \theta)}{\partial\theta}$ correspond to the roots of $H(x)=(P(x))^2-(Q(x))^2=0$,
i.e., $H(x)=A.x^4-Bx^2+C=0$, where A, B, C are as given in the statement of the theorem.

The roots of $H(x)=0$ are

$$x_0=[\frac{B-\sqrt{B^2-4AC}}{2A}]^{1/2};$$

$$x_1=[\frac{B+\sqrt{B^2-4AC}}{2A}]^{1/2}.$$

Now $H(0)=(1+r^2)^2\ [1+(1-2c)r^2]^2>0$ and

$$H(1)=A-B+C=(1-r^2)^2\ \{1+kr|b|+(2c-1)r^2\}\ \{1-kr|b|+(2c-1)r^2\}<0$$

if and only if $r\in(r_0,\ 1)$.

Hence there exists a root in (0, 1) of the equation $H(x)=0$ for $r\in(r_0,\ 1)$. Since $B-A-C>0$ for $r\in(r_0,\ 1)$, $x_1>1$.

Hence x_0 is the only root of $H(x)=0$ in (0, 1).

It can be easily verified that $\frac{\partial^2 \Delta(r, \theta)}{\partial \theta^2} > 0$ for $\theta \in (0, \pi)$. Therefore minimum of $\Delta(r, \theta)$ is attained at $x = x_0 = \cos\frac{\theta_0}{2}$ which yields (4.2).

We have $\Delta(r_0)=0$ and $\Delta(r) \to -\infty$ as $r \to 1^-$. Hence $\Delta(r)$ is a continuous decreasing function of r, therefore there exists a unique root r_1 of the equation $\Delta(r) = -\pi$ in the interval $(r_0, 1)$ and this root r_1 is the radius of close-to-convexity of $V_k(b)$. This completes the proof of the theorem.

The above theorem generalizes the results of the author [18] and [20] when $b = \cos\alpha e^{-i\alpha}$ and $b = 1-\rho$, M. A. Nasr and M. A. Aouf [10] when $k=2$, H. B. Coonce and M. R. Ziegler [3] when $b=1$ and that of R. J. Libera and M. R. Ziegler [7] when $b = \cos\alpha e^{-i\alpha}$ and $k=2$.

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