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Some Special Iterative Procedures for Solving Stochastic Differential Equations of Ito Type

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In this paper some iterative methods for solving stochastic differential equation of Ito type, analogous to classical iterative methods for solving deterministic ordinary differential equations, are considered. Sequences of processes converging almost surely to the solution of the equation are constructed.

1. Preliminaries

In the paper [6] one general iterative method for solving deterministic ordinary differential equations of first order is described. The generality of this method means that it includes, as its special cases, some well-known historical important analitic methods, for instance the Picard method of successive approximations, Chaplygin methods of chords and tangents, etc. It was very logical to suggest analogous iterative procedure for stochastic differential equation of Ito type (shortly SDE) (see [3]). In the present paper we give some special algorithms for solving the SDE by appropriate linear SDE-s in the sense of almost sure convergence. The method used here is similar to the classical Chaplygin methods for ordinary differential equations. Some results of the paper [3] and some comparison theorems are used.

W shall give definitions of some notions which are needed in our discussion. Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which all random variables and random processes are defined. Let $W=(W_t, \mathcal{F}_t)$, $t\in[0, T]$, $T=\text{const}>0$, be a standard Wiener process adapted to the increasing family $(\mathcal{F}_t)$, $t\in[0, T]$, of sub-σ-algebras of $\mathcal{F}$. We suppose that the real functions $a(t, x)$, $b(t, x)$, $a_n(t, x)$ and $b_n(t, x)$, $n\in N$, defined on $[0, T]\times R$ with values on R, are measurable with respect to a pair of variables (t, x). Following the tradition of the classical theory of SDE-s (see [2]), they are assumed to satisfy the global Lipschitz condition and the condition of the restriction on growth, i.e. for a positive constant L, we have

$$|a(t, x)-a(t, y)|\leqq L|x-y| \tag{1.1}$$

and

$$a^2(t, x)\leqq L^2(1+x^2), \tag{1.2}$$

and analogously for the other functions b, a_n and b_n, $n\in N$.

The pair of functions (a, b) and the Wiener process W define the stochastic process $X=(X(t),\ t\in[0, T])$ as a unique strong solution of the following SDE

$$(1.3) \qquad dX(t)=a(t,\ X(t))dt+b(t,\ X(t))dW(t), \quad X(0)=X_0.$$

The sequence of stochastic processes $\{X_n,\ n\in N\}=\{(X_n(t),\ t\in[0, T]),\ n\in N\}$ will be formed as follows:
– X_1 is an arbitrary stochastic process with $X_1(0)=X_0$,

$$E\{|X_0|^2\}<\infty \text{ and } E\{\sup_t |X_1(t)|^2\}<\infty;$$

– $X_{n+1},\ n\in N$, is the unique strong solution of the SDE

$$(1.4) \qquad dX_{n+1}(t)=a_n(t,\ X_{n+1}(t))dt+b_n(t,\ X_{n+1}(t))dW(t),$$
$$X_{n+1}(0)=X_0.$$

Let the functions $a,\ b,\ a_n$ and $b_n,\ n\in N$, satisfy the condition

$$(1.5) \qquad \sum_{n=1}^{\infty} \sup_{t,x}\{|a(t,\ x)-a_n(t,\ x)|+|b(t,\ x)-b_n(t,\ x)|\}<\infty.$$

Note that this condition is used essentially in order to prove the following theorem.

Theorem 1.1. *Let the functions $a,\ b,\ a_n$ and $b_n,\ n\in N$, and the random variable X_0 be defined as above and the condition* (1.5) *be satisfied. Then the sequence of stochastic processes $\{X_n,\ n\in N\}$ converges almost surely, uniformly in $t,\ t\in[0, T]$, to the process X as $n\to\infty$.*

The proof can be found in the paper [3].

Let us note that the conditions of this theorem could be weakened in some sense. For example, all functions $a,\ b,\ a_n$ and $b_n,\ n\in N$, can be random functions. In this case, the conditions (1.1), (1.2) and (1.5) can be replaced by

$$(1.6) \qquad \sum_{n=1}^{\infty} E\{\sup_t[|a(t,\ X_n(t))-a_n(t,\ X_n(t))|^2$$
$$+|b(t,\ X_n(t))-b_n(t,\ X_n(t))|^2]\}<\infty.$$

Thus we obtain a special iterative procedure for which we use the notion a Z-algorithm, analogously to the consideration in [6]. The sequence of functions $\{(a_n(t,\ x),\ b_n(t,\ x)),\ n\in N\}$ is called determined sequence for the Z-algorithm. It is clear that the rate of convergence and the effective solving of the Z-algorithm depend on the choice of the starting process and of the determined sequence. Because of the fact that we can solve explicitly any linear SDE (see [2] or [7]), we should try to make linearizations of the functions $a(t,\ x)$ and $b(t,\ x)$ in the argument x.

It is well-known that the Chaplygin methods of chords and tangents introduce some forms of linearizations of the ordinary differential equation

$y' = f(x, y)$, $y(x_0) = y_0$, where $f(x, y)$ is continuous, twice differentiable in y and satisfying the Lipschitz condition in y on a domain containing the point (x_0, y_0). It is proved that there exist two sequences of functions $\{u_n(x), n \in N\}$ and $\{v_n(x), n \in N\}$, such that for each $x \in [x_0, x_0 + h]$, $h = \text{const} > 0$, it holds $u_n(x) \leqq u_{n+1}(x) \leqq y(x) \leqq v_{n+1}(x) \leqq v_n(x)$ and $v_n(x) - u_n(x) \leqq c \cdot 2^{-2^n}$, $c = \text{const} > 0$, $n \in N$. The initial idea about it is based on a Chaplygin differential inequality theorem and on a geometrical interpretation of the chord and tangent of the curve $z = f(x, y)$ at fixed x (see [5]). As we have already mentioned, it was shown in the paper [6] these methods are special cases of the Z-algorithm.

In order to describe an analogous Z-algorithm for solving the SDE (1.3), we should have to use some comparison theorems. In the classical theory of SDE-s, for example in [2], [4], such theorems for real functions $a(t, x)$ and $b(t, x)$ are given. Since in the iterative procedure the coefficients of the SDE (1.4) depend of the previous solution, they are random functions. In the paper [1], SDE-s of the type

$$dX(t) = f(t, \omega, X(t))dA(t) + g(t, \omega, X(t))dW(t), \quad X(0) = X_0$$

are described, where $W = (W_t, \mathscr{F}_t)$, $t \in [0, \infty]$, is a standard Wiener process and $A = (A_t, \mathscr{F}_t)$, $t \in [0, \infty)$ is a continuous increasing process. If the functions f and g satisfy some conditions, more general than (1.1) and (1.2), then the comparison theorem is proved. We shall adapt this result from [1] in accordance to our problems and using the conditions satisfied by the functions $a(t, x)$ and $b(t, x)$. We shall express this in the following lemma.

Lemma 1.1. *Let X_0 be $\mathscr{F}_0$-measurable random variable, let the functions $f_i(t, \omega, x)$, $i = 1, 2$, and $g(t, \omega, x)$ satisfy the condition* (1.1) *and let $f_1(t, \omega, x) \leqq f_2(t, \omega, x)$* for all $(t, x) \in [0, \infty) \times R$. *If the processes $(X_i(t), t \in [0, \infty))$, $i = 1, 2$,* are solutions of the equations

$$dX_i(t) = f_i(t, \omega, X_i(t))dt + g(t, \omega, X_i(t))dW(t), \quad X_i(0) = X_0, \quad i = 1, 2,$$

then $X_1(t) \leqq X_2(t)$ for all $t \in [0, \infty)$ *almost surely.*

2. Main results

Now we construct special Z-algorithms for solving the SDE (1.3), approximating only the function $a(t, x)$ by linear functions in x. So, we associate the sequences of the SDE-s

$$(2.1) \qquad dY_{n+1}(t) = u_n(t, Y_{n+1}(t))dt + b(t, Y_{n+1}(t))dW(t)$$

and

$$(2.2) \qquad dZ_{n+1}(t) = v_n(t, Z_{n+1}(t))dt + b(t, Z_{n+1}(t))dW(t),$$

$$Y_{n+1}(0) = Z_{n+1}(0) = X_0, \quad n \in N,$$

to the SDE (1.3), such that the sequences of the functions $\{u_n(t, x), n \in N\}$ and $\{v_n(t, x), n \in N\}$ to be determined sequences for the Z-algorithm, analogous to the

Chaplygin methods of chords and tangents for ordinary differential equations. These notions are used in following theorem.

Theorem 2.1. *Suppose the following conditions hold:*
(i) $a(t, x)$ *is twice differentiable in* x *and satisfies* (1.2);
(ii) $a'_x(t, x)$ *is bounded on* $[0, T]\times R$;
(iii) $a''_{xx}(t, x)$ *does not change its sign on* $[0, T]\times R$;
(iv) $b(t, x)$ *satisfies* (1.1) *and* (1.2).
Then the sequences of random processes

$$\{u_n(t, x), n\in N\}=\{a'_x(t, Y_n(t))\cdot(x-Y_n(t))+a(t, Y_n(t)), n\in N\}$$

and

$$\{v_n(t, x), n\in N\}$$

$$=\begin{cases}\{\dfrac{a(t, Z_n(t))-a(t, Y_n(t))}{Z_n(t)-Y_n(t)}\cdot(x-Y_n(t))+a(t, Y_n(t)),\\ \text{for } Z_n(t)\neq Y_n(t) \text{ almost surely, } n\in N\}\\ \{u_n(t, x), \text{ for } Z_n(t)\equiv Y_n(t) \text{ almost surely, } n\in N\}\end{cases}$$

are determined sequences for the Z-algorithm of the SDE (1.3) *and* $Y_n(t)\leqq X(t)\leqq Z_n(t)$, *or* $Y_n(t)\geqq X(t)\geqq Z_n(t)$ *for all* $t\in[0, T]$, $n\in N$, *almost surely.*

Proof. We shall prove that the sequences of random functions $\{u_n(t, x), n\in N\}$ and $\{v_n(t, x), n\in N\}$ satisfy the conditions of the Theorem 1.1., using (1.6) instead of (1.5). Indeed,

$$a(t, Y_n(t))-u_n(t, Y_n(t))=a(t, Z_n(t))-v_n(t, Z_n(t))\equiv 0, \quad n\in N.$$

So, for each $t\in[0, T]$ and all $x, y\in R$, we have

$$|u_n(t, x)-u_n(t, y)|=|a'_x(t, Y_n(t))|.|x-y|$$

$$\leqq \sup_t |a'_x(t, Y_n(t))|.|x-y|\leqq L_1.|x-y|, \quad n\in N,$$

where $L_1=\sup_t|a'_x(t, x)|$. The function $a(t, x)$ satisfies a Lipschitz condition with a constant L_2 and we find

$$|v_n(t, x)-v_n(t, y)|=\left|\frac{a(t, Z_n(t))-a(t, Y_n(t))}{Z_n(t)-Y_n(t)}\right|.|x-y|$$

$$\leqq L_2.|x-y|, \quad n\in N.$$

The condition (1.2) does not hold. Because of this fact, the following approximations can be constructed. We choose a pair of real functions $u_0(t, x)$ and $v_0(t, x)$, such that for all $(t, x)\in[0, T]\times R$, (1.1), (1.2) and $u_0(t, x)\leqq a(t, x)\leqq v_0(t, x)$ are valid. Then there exist unique solutions of the SDE-s

$$dY_1(t)=u_0(t,\ Y_1(t))dt+b(t,\ Y_1(t))dW(t),\quad Y_1(0)=X_0$$

and

$$dZ_1(t)=v_0(t,\ Z_1(t))dt+b(t,\ Z_1(t))dW(t),\quad Z_1(0)=X_0.$$

By Lemma 1.1., for each $t\in[0,\ T]$ we get

$$Y_1(t)\leqq X(t)\leqq Z_1(t)\quad \text{almost surely.}$$

If $Y_1(t)\not\equiv Z_1(t)$ for each $t\in[0,\ T]$ almost surely, we choose a positive number M and define two variables:

$$\nu_1^M=\begin{cases}\inf\{t:|Y_1(t)|>M\}\\ T \quad \text{if } |Y_1(t)|\leqq M \quad \text{for each } t\in[0,\ T]\end{cases}$$

and

$$\theta_1^M=\begin{cases}\inf\{t:|Z_1(t)|>M\}\\ T \quad \text{if } |Z_1(t)|\leqq M \quad \text{for each } t\in[0,\ T]\end{cases}.$$

Then ν_1^M and θ_1^M are stopping times with respect to $(\mathcal{F}_t)$, $t\in[0,\ T]$. Then $\tau_1^M=\inf\{\nu_1^M,\ \theta_1^M\}$ is also a stopping time with respect to the same family. Denote

$$Y_1^M(t)=\begin{cases}Y_1(t) \quad \text{if} \quad |Y_1(t)|<M\\ M\operatorname{sgn} Y_1(t) \quad \text{if} \quad |Y_1(t)|\geqq M\end{cases}$$

and

$$u_1^M(t,\ x)=a'_x(t,\ Y_1^M(t))\cdot(x-Y_1^M(t))+a(t,\ Y_1^M(t)),$$

and similarly for $Z_1^M(t)$ and $v_1^M(t,\ x)$. Now the condition (1.2) holds almost surely on the interval $[0,\ \tau_1^M]$ and there exist unique solutions of the SDE-s

$$dY_2(t)=u_1^M(t,\ Y_2(t))dt+b(t,\ Y_2(t))dW(t),\quad Y_2(0)=X_0$$

and

$$dZ_2(t)=v_1^M(t,\ Z_2(t))dt+b(t,\ Z_2(t))dW(t),\quad Z_2(0)=X_0.$$

Since for all $(t,\ x)\in[0,\ T]\times R$ the function $a''_{xx}(t,\ x)$ has the same sign, for example $a''_{xx}(t,\ x)>0$, from the geometrical interpretation of the chord and the tangent of the curve $z=a(t,\ x)$ at a fixed t, $t\in[0,\ \tau_1^M]$, we get the relation

$$a'_x(t,\ Y_1^M(t))\cdot(x-Y_1^M(t))+a(t,\ Y_1^M(t))<a(t,\ x)$$

$$<\frac{a(t,\ Z_1^M(t))-a(t,\ Y_1^M(t))}{Z_1^M(t)-Y_1^M(t)}\cdot(x-Y_1^M(t))+a(t,\ Y_1^M(t)),$$

i.e.

$$u_1^M(t,\ x)<a(t,\ x)<v_1^M(t,\ x).$$

By Lemma 1.1. for each $t\in[0,\tau_1^M]$ we have $Y_2^M(t)\leqq X(t)\leqq Z_2^M(t)$ almost surely.

By induction, using the same procedure, we come to the conclusion that on the interval $[0,\ \tau_n^M]$ the random functions $u_n^M(t,\ x)$ and $v_n^M(t,\ x)$ and the Wiener process W define solutions $Y_{n+1}(t)$ and $Z_{n+1}(t)$ respectively, such that $Y_{n+1}(t)\leqq Z_{n+1}(t)$ almost surely.

For a fixed t, $t \in [0, T]$, we can choose a sufficiently large M, such that $\tau_n^M > t$ almost surely, $n \in N$. Since there exists a stopping time $\tau^M = \inf_n \tau_n^M$ (see [7]), then on the interval $[0, \tau^M]$ all conditions of Theorem 1.1. are fulfilled. Hence $\{u_n^M(t, x), n \in N\}$ and $\{v_n^M(t, x), n \in N\}$ are determined sequences for the Z-algorithm. On this interval the relations $u_n^M(t, x) = u_n(t, x)$ and $v_n^M(t, x) = v_n(t, x)$, $n \in N$, are valid almost surely. So, by the local uniqueness theorem (see [2]) it follows that $\{Y_n, n \in N\}$ and $\{Z_n, n \in N\}$ converge almost surely, uniformly in t, $t \in [0, \tau^M]$, to the solution X of the SDE (1.3) as $n \to \infty$. Since $P\{\lim_{M \to \infty} \tau^M = T\} = 1$ (see [2]), these sequences of approximations converge almost surely, uniformly in t, $t \in [0, T]$, to the solution X as $n \to \infty$. The proof of the Theorem is completed.

Remark 2.1. Moreover, if the functions $a(t, x)$ and $a_x'(t, x)$ are continuous in t, and $b(t, x)$ is random, linear in x, i.e. $b(t, x) = q(t, \omega) . x + r(t, \omega)$, where $q(t, \omega)$ and $r(t, \omega)$ are continuous stochastic processes adapted to $(\mathscr{F}_t)$, $t \in [0, T]$, we can express explicitly the solutions of the linear SDE-s (2.1) and (2.2).

Remark 2.2. It is clear from the proof that the sequence of stochastic processes $\{Y_n, n \in N\}$ converges almost surely to the solution X from the left, and the sequence $\{Z_n, n \in N\}$ from the right. Let us mention that it cannot be proved as in ordinary differential equations, that these sequences are monotonous, because in a theory of SDE-s of Ito type we have not any theorem analogous to the Chaplygin differential inequality theorem (see [4]). Moreover, the difference $Z_n(t) - Y_n(t)$ cannot be estimated as in the case of ordinary differential equations. One mean square estimation is obtained from the proof of the Theorem 1.1. (see [3]), as

$$E\{\sup_t |Z_n(t) - Y_n(t)|^2\} \leqq K \cdot \frac{(bT)^{n-3}}{(n-3)!},$$

where K and b are some constants.

Remark 2.3. Notice that the Z-algorithm described by the determined sequence $\{v_n(t, x), n \in N\}$, i.e. by the iterative method of chords, depends on the Z-algorithm described by the determined sequence $\{u_n(t, x), n \in N\}$, i.e. by the iterative method of tangents. Moreover, the Z-algorithm based on $\{u_n(t, x), n \in N\}$ is analogous to the Newton-Kantorovich method for solving ordinary differential equations. In this case, it is not necessary to require that $a_{xx}''(t, x)$ has the same sign on $[0, T] \times R$ and the solutions $\{Y_n, n \in N\}$ can be located on both sides of the solution X.

Finally, it would be very interesting to construct other, not classical, Z-algorithms, which allow to solve the equations (1.4) effectively. Some results in this direction will be considered in a forthcoming paper.

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