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Quadratic Equation in Unital C* — Algebra

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The structure of the set of Hermitian solutions of the quadratic equation $xcx+b^*x+xb+a=0$ is studied. Under certain conditions the set of Hermitian solutions is in one-to-one correspondence with a certain set of projectors.

The purpose of this paper is to extend the classification of all the Hermitian solutions of the quadratic equation in the C*-algebra $C(K)$ of complex functions over a compact subset K of the complex field C, to all C*-algebras with unit.

In $C(K)$ it is known that

$$x_+(z)=b(z)+\sqrt{b(z)^2+a(z)} \text{ and } x_-(z)=b(z)-\sqrt{b(z)^2+a(z)}$$

are Hermitian solutions of the quadratic equation

$$-x^2+2bx+a=0 \tag{1}$$

for $b(z)$ real, and $b(z)^2+a(z)>0$, for $z\in K$. If K is not connected, then there exist nontrivial nonzero idempotents in $C(K)$. Since only values 0 and 1 are allowed for such functions, such a function is a characteristic function of a certain subset of K. Equation (1) has also solutions of the form

$$x(z)=x_+(z)\chi_E(z)+x_-(z)(1-\chi_E(z)). \tag{2}$$

Let x be a solution of the equation (1). Since x_+ and x are solutions of (1), we get

$$2b(z)(x_+(z)-x(z)) - (x_+(z)-x(z))(x_+(z)+x(z))=0.$$

Denoting the closed set $\{z:x_+(z)=x(z)\}$ by E, it follows that

$$2b(z)-(x_+(z)+x(z))=0, \text{ for } z\in E^C$$

or

$$x(z)=2b(z)-x_+(z)=x_-(z), \text{ for } z\in E^C.$$

Thus all the solutions of the equation (1) can be written in the form (2). The sets E and E^C have to be closed since $x(z)$ is continuous and $b(z)^2+a(z)>0$.

Let A be a C*-subalgebra of $B(H)$, the algebra of all bounded linear operators on a Hilbert space H. We will assume that A has a unit element e. We use [4] as a standard reference concerning C*-algebras.

The spectrum of an element $c \in A$ will be denoted by $\sigma(c)$. It is cnown that the spectrum of an element of C*-subalgebra of $B(H)$ is the same set as the spectrum of this element as an element of $B(H)$. A Hermitian element $c \in A$ will be called nonnegative (respectively nonpositive) and denoted by $c \geqq 0$ (respectively $c \leqq 0$) if $\sigma(c) \subset [0, \infty)$ ($(-\infty, 0]$, respectively). The open left half plane $\{z : \operatorname{Re} z < 0\}$ will be denoted by π_-. The C*-algebra of 2×2 matrices with entries from A will be denoted by $M_2(A)$ and it can be regarded as a C*-subalgebra of $B(H^{(2)})$, where $H^{(2)}$ denotes the Hilbert space of pairs with entries from H. We will begin with two definitions.

Definition: *For b, c in A, we say that the pair (b, c) is* s t a b i l i z a b l e *in $B(H)$ if and only if there exists x_0 in $B(H)$ such that $\sigma(b + cx_0) \subset \pi_-$.Further, we say that the pair (b, c) is* c o m p l e t e l y s t a b i l i z a b l e *in $B(H)$ if for each $\alpha < 0$ there exists an $x_\alpha \in B(H)$ such that $\sigma(b + cx_\alpha) \subset \{z : \operatorname{Re} z < \alpha\}$.*

The notions of stabilizability and complete stabilizability are well-known in the theory of mathematical systems, and they have been studied in detail. The reader is referred to R. W. Brockėtt [1], J. W. Bunce [2], O. Hijab [6], M. Megan [7], W. M. Wohnam [10], etc.

R e m a r k : If c is invertible then the pair (b, c) is completely stabilizable for any $b \in A$.

We are interested in the Riccati algebraic equation

$$xcx + b^*x + xb + a = 0 \tag{3}$$

with coefficients in A. We will consider only the case $c \leqq 0$ and $a \geqq 0$.

In the algebra $B(H)$ it is known that the stabilizability of the pair (b, c) implies the existence of a maximal solution of the equation (3). The proof for an invertible a can be found in [2], and for $a \geqq 0$ in [5]. If the pair (b, c) is completely stabilizable then there exist a maximal and a minimal solution, x_+ and x_-; that is, every Hermitian solution x of the equation (3) satisfies the relation $x_- \leqq x \leqq x_+$. The operator $\Delta = x_+ - x_-$ is an invertible element of $B(H)$ if and only if $\sigma(b + cx_+) \subset \pi_-$. If a is invertible so is Δ. The proofs of the above statements and Theorem 1 can be found in [5].

Theorem 1 is a generalization of J. C. W i l l e m's [9] and W. A. C o p p e l's [3] result on matrix algebraic Riccati equation.

Theorem 1. [5, Th. 4.] *Let A, B, and C be elements of $B(H)$, with $A = A^*$, $C \leqq 0$, and let the pair (B, C) be completely stabilizable. Let $\Delta = X_+ - X_-$ be invertible and let V_1 be an arbitrary subspace of H invariant under $B + CX_+$. If $V_2 = \Delta^{-1}(V_1^\perp)$, then $H = V_1 \oplus V_2$. If P is the corresponding projection of V onto V_1 then*

$$X = X_+P + X_-(I - P)$$

is a Hermitian solution of the equation (3). *Moreover, all Hermitian solutions of the equation* (3) *are obtained in this way, and the correspondence between V_1 and X is one-to-one.*

The next theorem is an extension of Theorem 1 to unital C*-algebras.

Theorem 2. *Let A be a unital C*-subalgebra of $B(H)$ and let $a, b, c \in A$ with $c \leqq 0$, $a \geqq 0$, a invertible, and let the pair (b, c) be completely stabilizable in $B(H)$. Then*

i) *There exist* $x_+ \in A$, *a maximal, and* $x_- \in A$, *a minimal solution of the equation* (3); *that is, every Hermitian solution* x *satisfies the relation* $x_- \leqq x \leqq x_+$.

ii) *The set of Hermitian solutions of the equation* (3) *is in one-to-one correspondence with the set of projectors* $h \in A$ *satisfying* $hrh = rh$, *where*

$$r = (x_+ - x_-)^{1/2}(b + cx_+)(x_+ - x_-)^{-1/2}.$$

Proof: From [5, Th. 1] we know that there exist a maximal solution, x_+, and a minimal solution , x_-, of the equation (3) in $B(H)$. Hence to prove (i) it suffices to show that $x_+ \in A$ and $x_- \in A$. To prove that $x_+ \in A$ we modify the proofs in [2] where the same result is obtained for $a = e$.

One can check that

$$M = \begin{bmatrix} b, & c \\ -a, & -b^* \end{bmatrix} = \begin{bmatrix} e, & 0 \\ x, & e \end{bmatrix} \begin{bmatrix} b + cx, & c \\ 0 \ , & -b^* - xc \end{bmatrix} \begin{bmatrix} e, & 0 \\ -x, & e \end{bmatrix} \quad \text{on } B(H^{(2)}),$$

for any Hermitian solution x of (3).

Since a is invertible, $\sigma(b + cx_+) \subset \pi_-$, as discussed before. Thus the spectrum of $M \in M_2(A)$ splits into two parts, one on each side of the imaginary axis. Let K be a Cauchy domain with boundary k such that $\sigma(M) \bigcap \pi_- \subset K \subset \pi_-$.

Since $M_2(A)$ is a C*-algebra, it is norm-closed and E defined by

$$E = \frac{1}{2\pi i} \int_k (zI - M)^{-1} \mathrm{d}z$$

is an element of $M_2(A)$. It is well-known that E is idempotent and that E commutes with M (E is the so-called spectral projection for M).

Since

$$(zI - M)^{-1} = \begin{bmatrix} e, & 0 \\ x_+, & e \end{bmatrix} \begin{bmatrix} (ze - (b + cx_+))^{-1}, & d \\ 0 & (ze + (b + cx_+)^*)^{-1} \end{bmatrix} \begin{bmatrix} e, & 0 \\ -x_+, & e \end{bmatrix},$$

E can be written as

$$E = \begin{bmatrix} e, & 0 \\ x_+, & e \end{bmatrix} \begin{bmatrix} p, & f \\ 0, & q \end{bmatrix} \begin{bmatrix} e, & 0 \\ -x_+, & e \end{bmatrix}, \tag{4}$$

where

$$p = \frac{1}{2\pi i} \int_k (ze - (b + cx_+))^{-1} dz$$

and

$$q = \frac{1}{2\pi i} \int_k (ze + (b + cx_+)^*)^{-1} dz.$$

But $p=e$ and $q=0$, since $\sigma(b+cx_+)\subset K$ and $\sigma(-(b+cx_+)^*)$ do not intersect with K.
Since E commutes with M, the operator matrices

$$\begin{bmatrix} b+cx_+, & c \\ 0, & -(b+cx_+)^* \end{bmatrix} \text{ and } \begin{bmatrix} e, & f \\ 0, & 0 \end{bmatrix}$$

commute. We get

$$(b+cx_+)f+f(b+cx_+)^*=c.$$

The above equation has exactly one solution [8] given by

$$f=-\int_0^\infty e^{(b+cx_+)t}c\,e^{(b+cx_+)^*t}dt,$$

because $\sigma(b+cx_+)\subset\pi_-$. Note that f is nonnegative because $c\leqq 0$. By (4) we have

$$E=\begin{bmatrix} e-fx_+, & f \\ x_+-x_+fx_+, & x_+f \end{bmatrix}.$$

Since $E\in M_2(A)$, all entries, f, x_+f, fx_+ and $x_+-x_+fx_+$ are in A. It is known that $f^{1/2}$, which exists since f is nonnegative, can be written as a limit of polynomials in f without a constant term. Hence $x_+f^{1/2}$ is in A. Since $x_+-x_+fx_+$ is in A, and $x_+fx_+=(x_+f^{1/2})(x_+f^{1/2})^*$ is in A, so is x_+.

The difference x_+-x_- satisfies the equation

$$(x_+-x_-)(b+cx_+)+(b+cx_+)^*(x_+-x_-)=(x_+-x_-)c(x_+-x_-).$$

Hence $(x_+-x_-)^{-1}$, which exists by the invertibility of a [5, Th. 3], is a solution of the equation

$$(b+cx_+)(x_+-x_-)^{-1}+(x_+-x_-)^{-1}(b+cx_+)^*=c.$$

But the equation has a unique solution f, that we already met. It follows that

$$x_-=x_+-f^{-1}.$$

Hence x_- is in A. This completes the proof of (i).
(ii) Let $x\in A$ be a Hermitian solution of the equation (3). Since the same equation can be considered also in $B(H)$, x can be written, by Theorem 1, as

$$x=x_+p+x_-(e-p),$$

where p is a corresponding idempotent in $B(H)$. Set $\Delta=x_+-x_-$. Since x_+ and x are Hermitian, it follows from

$$x_+-x=\Delta(e-p)$$

that

(5) $$\Delta p = p^* \Delta.$$

We can also write

$$p = e - \Delta^{-1}(x_+ - x),$$

which shows that p is in A. If we define h by

$$h = \Delta^{1/2} p \Delta^{-1/2}$$

it follows from (5) that $h^* = h$.

By Theorem 1, p is a projection (not necessary orthogonal) onto the invariant subspace of the operator $b + cx_+$. Thus

$$p(b + cx_+)p = (b + cx_+)p.$$

Combining this with the definition of r in (ii) we obtain

$$hrh = rh.$$

Let $h \in A$ be a projector satisfying $hrh = rh$. If we define

(6) $$p = \Delta^{-1/2} h \Delta^{1/2},$$

we shall show that

(7) $$x = x_+ p + x_-(e - p)$$

is a Hermitian solution of the equation (3).

It is a consequence of the definition of p that

(8) $$p^* \Delta = \Delta p.$$

Let $d = x_+ - x = \Delta(e - p)$. By (8), we get $d^* = d$, hence $x^* = x$. The identity

(9) $$p(b + cx_+)p = (b + cx_+)p$$

follows from $hrh = rh$.

Since x_+ and x_- are Hermitian solutions of the equation (3) it can easily be verified that

(10) $$-\Delta(b + cx_-) = (b + cx_+)^* \Delta.$$

From (8), (9) and (10), we conclude that

$$p(b + cx_-)p = p(b + cx_-).$$

Combining this with (9) we obtain

$$(e-p)(b+cx_+)=(e-p)(b+cx_+)(e-p)=(e-p)((b+cx_-)+c\Delta)(e-p)=$$
$$=(b+cx_-)(e-p)+(e-p)c\Delta(e-p).$$

Multiplying the above equality by Δ on left, and using (10) again, we get

$$\Delta(e-p)(b+cx_+)+(b+cx_+)^*\Delta(e-p)=\Delta(e-p)c\Delta(e-p),$$

from which we see that

$$x=x_+-\Delta(e-p) \tag{11}$$

is a solution of the equation

$$xcx+b^*x+xb+a=0.$$

Since Δ is invertible, (11) tells us that the correspondence between Hermitian solutions and projections h satisfying $hrh=rh$ is one-to-one. The proof is complete.

Proposition. *Let x_1 and x_2 be two Hermitian solutions of the equation* (3), *and denote the corresponding projections by h_1 and h_2. Then $x_1 \geqq x_2$ if and only if $h_1 \geqq h_2$.*

Proof. From (6) and (7) we have

$$x_1-x_2=\Delta^{1/2}(h_1-h_2)\Delta^{1/2}.$$

The proposition is obvious.

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