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Complex Hypersurfaces with Planar Geodesics in Generalized Hopf Manifolds

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Dedicated to Professor S. Ianus
on his fiftieth birthday

0. Introduction

Let M and $\bar{M}$ be Riemannian manifolds of dimension n and $n+p$, respectively. An isometric immersion of M into $\bar{M}$ is called 'with planar geodesics' when every geodesic of M is mapped, locally, into a two-dimensional totally geodesic submanifold of $\bar{M}$.

It was pointed out that planar geodesic immersions are closely related to isotropic ones in the sense of B. O'Neill [4]. S. L. Hong [1] and K. Sakamoto [6] studied the cases when $\bar{M}$ is an Euclidean space and a space form, respectively. When the ambient manifold is a complex space form, notable results are obtained by Jin Suk Pak [5].

In this note we consider planar geodesic immersion of complex hypersurfaces in locally conformal Kaehler manifolds with parallel Lee form- generalized Hopf manifolds (g. H. m.), and flat Weyl connection. These manifolds, studied mainly by I. Vaisman (e. g. [8]), are important not only as a distinguished class of Hermitian manifolds, but also in theoretical physics: it was recently proved that g. H. m. may be used as a model for Kaluza-Klein theory [3]. We shall prove that, under natural assumptions, the condition of having planar geodesics forces the hypersurface to be totally geodesic or its Weingarten operator to have only three distinct eigenvalues.

All geometric objects we consider are assumed to be differentiable of class C^∞.

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1. Preliminaries

In this section we recall briefly the basic definitions, formulas and results concerning complex hypersurfaces of g. H. m. Details can be found in [2], [8]. Next we establish some results to be used in the proof of our main results.

Let $(\bar{M}, J)$ be a complex manifold of real dimension $2n$ with complex structure J; suppose $\bar{M}$ bears a Hermitian metric $\langle , \rangle$ conformally related to a Kaehler metric in the neighbourhood of each point of $\bar{M}$. Then $\bar{M}$ is said to be locally conformal Kaehler (l. c. K). An equivalent definition requires the existence of a globally defined closed one-form ω (the Lee form) related to the fundamental two-form Ω by the equation

$$\text{(1.1)} \qquad d\Omega = \omega \wedge \Omega.$$

L. c. K. manifolds appear naturally in Gray-Hervella classification in the class W_4.

A g. H. m. is a l. c. K. manifold with parallel Lee form with respect to the Levi-Cività connection $\bar{\nabla}$ of $\langle , \rangle$. The most notable examples are the Hopf manifolds.

We suppose ω without singularities. It is easy to show [8], that a l. c. K. manifold is g. H. m. if and only if $c = |\omega|/2$ is constant on $\bar{M}$ and $\bar{\nabla} u = 0$ (where $u = \omega/|\omega|$).

Let U be the unitary vector field metrically equivalent to u (i.c.: $\langle U, X \rangle = u(X)$ for every X tangent to $\bar{M}$); we call it the Lee vector field.

On a l. c. K. manifold one has to consider the Weyl connection which appears to be just the Levi-Cività connection of the locally conformal Kaehler metrics. When the Weyl connection has vanishing curvature, the g. H. m. is called $\mathscr{P}_0\mathscr{K}$ manifold [8]. We recall that $\mathscr{P}_0\mathscr{K}$ manifolds are locally symmetric spaces [2].

We shall use the following estimation for the curvature operator of a $\mathscr{P}_0\mathscr{K}$ manifold

$$\text{(1.2)} \qquad \bar{R}(X, Y)Z = c^2 [u(X)u(Z) Y - u(Y)u(Z)X - u(X) \langle Y, Z \rangle U + u(Y) \langle X, Z \rangle U + \langle Y, Z \rangle X - \langle X, Z \rangle Y].$$

In the sequel we consider M a complex hypersurface isometrically immersed in a $\mathscr{P}_0\mathscr{K}$ manifold $\bar{M}$. We denote by the same letters the induced complex structure, and Hermitian metric. Let ∇ be the induced Levi-Cività connection, A and A' the Weingarten operators associated to the normal sections N and JN, H the second fundamental tensor of M, $\nabla^{\perp}$ the normal connection.

We recall an important result proved in [2], [7]:

Lemma 1.1 If U is everywhere tangent to M, then

i) $JA = -AJ$; $A' = JA$,
ii) $\mathrm{Tr} A = \mathrm{Tr} A' = 0$, so M is minimal.

When the ambient manifold is a (complex) space form, it is shown in [1], [6] that the second fundamental tensor of a planar geodesic immersion satisfies $\|H(X, X)\|^2 = \lambda^2$ for every unit tangent vector X, for a differentiable function λ. This property remains valid in our context (as one can see examining the proofs from the quoted papers), so we may state:

Lemma 1.2 *Let $i : M \to \bar{M}$ be a connected complex hypersurface with planar geodesics of a g. H. m. with flat Weyl connection. Then for all unit vector X tangent to M,*

$$\| H(X, X) \|^2 = \lambda^2,$$

where λ is a differentiable function on M (i. e. the immersion is isotropic).

The following result is essential. We shall adapt the proof from [5].

Lemma 1.3 *In the hypothesis of Lemma 1.2, for any pair of unit tangent vectors X, Y in $m \in M$ one has*

$$X(\lambda^2) = -2c^2 u(X) \langle nor\ U, H(X, Y) \rangle,$$

where nor U is the normal part of U.

Proof. Take m an arbitrary fixed poind of M and $\mathscr{V}$ a normal coordinate neighbourhood around m in M. Let $X, Y \in T_m M$ so that $\|X\| = \|Y\| = 1$ and $\langle X, Y \rangle = 0$. Let α be a geodesic starting at m ($\alpha(0) = m$) and with initial velocity Y. From Gauss formula we have

$$\bar{\nabla}_{i\dot{\alpha}} i\dot{\alpha} = i \nabla_{\alpha} \dot{u} + H(\dot{\alpha}, \dot{\alpha}) = H(\dot{\alpha}, \dot{\alpha}).$$

We may suppose $H(Y, Y) = H(\dot{\alpha}(0), \dot{\alpha}(0)) \neq 0$.

The geodesic α is mapped by i into a totally geodesic submanifold M^2 of $\bar{M}$. Then $\bar{\nabla}_{i\dot{\alpha}} H(\dot{\alpha}, \dot{\alpha}) \in T_{i\dot{\alpha}} M^2$ thus:

$$\bar{\nabla}_{i\dot{\alpha}} H(\dot{\alpha}, \dot{\alpha}) = a(t) i\dot{\alpha} + b(t) H(\dot{\alpha}, \dot{\alpha})$$

with a and b differentiable functions of t. Extend X to parallel vector fields $\bar{X}$, $\bar{Y}$ defined on the entire $\mathscr{V}$ (this can be done by parallel translation of X and Y along each unique geodesic from m to each point of $\mathscr{V}$). Then $\bar{X}$ and $\bar{Y}$ satisfy

$$\nabla_X \bar{X} = \nabla_Y \bar{Y} = \nabla_X \bar{Y} = \nabla_Y \bar{X} = 0 \quad \text{at } m.$$

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We now proceed to the computation of $(X \cdot \lambda^2)(m)$. We have successively

$$(X \cdot \lambda^2)(m) = X \langle H(\bar{Y}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m)$$

$$= 2 \langle \bar{\nabla}_{\bar{X}} H(\bar{Y}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m) = 2 \langle \nabla^{\perp}_{\bar{X}} H(\bar{Y}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m)$$

$$= 2 \langle (\bar{\nabla}_{\bar{X}} H)(Y, Y) + 2H(\nabla_{\bar{X}} \bar{Y}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m)$$

$$= 2 \langle (\bar{\nabla}_{\bar{X}} H)(\bar{Y}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m)$$

$$= 2 \langle (\bar{\nabla}_{\bar{Y}} H)(\bar{Y}, \bar{Y}) - c^2 u(\bar{X}) \text{ nor } U, H(\bar{Y}, \bar{Y}) \rangle (m),$$

where the last equality follows from Codazzi equation. But

$$(\bar{\nabla}_{\bar{Y}} H)(\bar{X}, \bar{Y}) = \nabla^{\perp}_{\bar{Y}} H(\bar{X}, \bar{Y}) = \bar{\nabla}_{\bar{Y}} H(\bar{X}, \bar{Y}) + A_{H(\bar{X}, \bar{Y})} \bar{Y}.$$

Hence

$$\langle (\bar{\nabla}_{\bar{Y}} H)(\bar{X}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m) = \langle \bar{\nabla}_{\bar{Y}} H(\bar{X}, \bar{Y}), H(\bar{Y}, \bar{Y}) \rangle (m)$$

$$= \langle \bar{\nabla}_{i\dot{\alpha}} H(\dot{\alpha}, \dot{\alpha}) |_{t=0}, H(\bar{Y}, \bar{Y})(m) \rangle = 0.$$

Thus

$$(X \cdot \lambda^2)(m) = -2c^2 u(X) \langle \text{nor } U, H(Y, Y) \rangle (m).$$

As m was an arbitrary point of M, the proof is complete.

Remark : The result of the previous lemma is valid for submanifolds of arbitrary codimension.

2. Main results

Lemma 2.1 *Let M be a complex hypersurface with planar geodesics everywhere tangent to the Lee vector of a $\mathcal{P}_0\mathcal{K}$ manifold. Then M has a parallel second fundamental form. In particular, the eigenvalues of A are constant.*

Proof. From Lemma 1.3 we have $X(\lambda^2)=0$ so, by connectedness, λ^2 is constant on M. Now, as above, let $m\in M$ and $X\in T_m M$ be a unit vector. Let α be a geodesic satysfying $\alpha(0)=m$ and $\dot\alpha(0)=X$. If $\lambda=0$ there is nothing more to prove. If we assume $\lambda\neq 0$, then from Lemma 1.1 we derive $\bar\nabla_{i\dot\alpha}H(\dot\alpha, \dot\alpha)=-\lambda^2 i\dot\alpha$. Taking into account the Weingarten formula

$$\bar\nabla_{i\dot\alpha}H(\dot\alpha, \dot\alpha)=-iA_{H(\dot\alpha,\dot\alpha)}+\nabla^{\perp}_{\dot\alpha}H(\dot\alpha, \dot\alpha)$$

we have $\nabla^{\perp}_{\dot\alpha}H(\dot\alpha, \dot\alpha)=0$. Thus, $(\bar\nabla H)(X, X, X)=(\bar\nabla_X H)(X, X)=0$. Now we take X, Y, Z from T_mM. The above equality implies

$$(\bar\nabla_X H)(Y, Z)+(\bar\nabla_Y H)(Z, X)+(\bar\nabla_Z H)(X, Y)=0.$$

As *nor* $U=0$ (1.2) shows that Codazzi equation reduces to

$$(\bar\nabla_X H)(Y, Z)=(\bar\nabla_Y H)(X, Z)=(\bar\nabla_Z H)(X, Y).$$

Together with (2.1) this yields to $(\bar\nabla_X H)(Y, Z)=0$ and the proof is complete.

Next we prove:

Lemma 2.2 *Let M be a complex hypersurface of a $\mathcal{P}_0\mathcal{K}$ manifold. If M is everywhere tangent to the Lee vector field U and has parallel second fundamental form then it is totally geodesic or the Weingarten operator has only three distinct eigenvalues: $-\mu$, 0, μ.*

Proof. Let $\{e_1,\dots, e_{n-1}, Je_1,\dots, Je_{n-1}\}$ be the orthonormal basis in which A has diagonal form with eigenvalues $\mu_1,\dots, \mu_{n-1}, -\mu_1,\dots, -\mu_{n-1}$ ([2], [7]).

We recall Simons' formula for locally symmetric minimal submanifolds [9]

$$(2.1)\qquad (\bar\nabla^2 H)(X, Y)=\Sigma_i\{R^{\perp}(e_i, X)H(e_i, Y)-H(R(e_i, X)e_i, Y)-H(e_i, R(e_i, X)Y)\},$$

where R and $R^{\perp}$ are respectively the Riemannian curvature tensor in the tangent bundle and the normal bundle of M.

In our case the left side of (2.1) is null. We use Gauss equation and (1.2) to compute the last two terms in (2.1)

$$H(R(e_i, H)e_i, Y)=\Sigma_i\{H(R(e_i, X)e_i, Y)+\langle Ae_i, e_i\rangle AY-\langle AY, e_i\rangle Ae_i$$
$$+\langle JAe_i, e_i\rangle JAY-\langle JAY, e_i\rangle JAe_i=c^2H(X, Y)-H(AY, AY)-H(AY, JAY),$$
$$H(R(e_i, X)Y, e_i)=-c^2H(X, Y)+H(AX, AY)+H(JAX, JAY).$$

Now (2.1) takes the form

$$(2.2)\qquad \Sigma_i R^{\perp}(e_i, X)H(e_i, Y)=H(AX, AY)+H(JAX, JAY)-H(AY, AY)-H(JAY, JAY).$$

In (2.2) we put $X=e_j$, $Y=e_k$, $j\neq k$. Thus, taking into account Lemma (1.1) we obtain

$$R^{\perp}(e_k, e_j)H(e_k, e_k)=-2\mu_k^2H(e_k, e_k). \tag{2.3}$$

But $H(e_k, e_k)=\mu_k(N-JN)$. Since $\mu_k=\text{const.}$ this implies

$$\mu_kR^{\perp}(e_k, e_j)(N-JN)=-2\mu_k^3(N-JN). \tag{2.4}$$

Interchanging j and k we also have

$$\mu_jR^{\perp}(e_j, e_k)(N-JN)=-2\mu_j^3(N-JN). \tag{2.5}$$

If all $\mu_k=0$ the hypersurface is totally geodesic. If this is not the case, suppose there exists $\mu_j\neq 0$ and $\mu_k\neq 0$, $j\neq k$. From (2.4) and (2.2) we then have

$$R^{\perp}(e_k, e_g)(N-JN)=-2\mu_k^2(N-JN), \tag{2.4}$$

$$R^{\perp}(e_j, e_k)(N-JN)=-2\mu_j^2(N-JN). \tag{2.5}$$

But $R^{\perp}(e_k, e_j)=-R^{\perp}(e_j, e_k)$ so, from the last two relations we get $\mu_k^2+\mu_j^2=0$, contradiction.
It remains that A may have only two nonzero eigenvalues: μ and $-\mu$.
We now consider the distributions

$$T_0(x)=\{X\in T_xM;\ AX=0\};$$

$$T_+(x)=\{X\in T_xM;\ AX=\mu X\};$$

$$T_-(x)=\{X\in T_xM;\ AX=-\mu X\}.$$

Note that $\dim T_+=\dim T_-=1$, $\dim T_0=2n-4$.
Obviously $T_xM=T_0(x)\oplus T_+(x)\oplus T_-(x)$.
One easily shows all of the three distributions are differentiable. They are also involutive. We shall prove this for $T_+(x)$:

$$A[X, Y]=A\nabla_XY-A\nabla_YX=\nabla_XAY-\nabla_YAX=\mu\nabla_XY-\mu\nabla_YX=\mu[X, Y]$$

for every $X, Y\in T_+(x)$.
Finally, the three distributions are parallel. We give the proof for $T_+(x)$: Let $X\in T_+(x)$ and $Y\in T_xM$. Then $A\nabla_YX=\nabla_YAX=\mu\nabla_YX$ so $\nabla_YX\in T_+(x)$.

Gathering together the two preceding lemmas we may now state:

Theorem 2.3 *Let M be a g. H. m. with a flat Weyl connection and $i: M\to\bar{M}$ a planar geodesic immersion of a connected complex hypersurface everywhere tangent to the Lee vector field.*

Then M is totally geodesic or is the Riemannian product $M_+\times M_0\times M_-$ where M_+, M_0, M_- are respectively the maximal integral manifolds of T_+, T_0, T_-.

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