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A Complete Asymptotic of the Spectral Function for Harmonic Oscillator

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1. Introduction and statement of results

Let $A=-\Delta+x^2$ be the Harmonic Oscillator, considered as a self-adjoint operator in $L^2(\mathrm{R}^n)$, $n\geqq 1$, and let $e(\lambda, x, y)$ be the spectral function of A. If $0<\lambda_1\leqq\lambda_2\leqq\ldots$ are the eigenvalues of A and $\{\varphi_k\}$ are the corresponding orthonormalized eigenfunctions (Hermit's functions), then

$$e(\lambda, x, y)=\sum_{\lambda_k\leqq\lambda}\varphi_k(x)\varphi_k(y).$$

In this paper we find a complete asymptotic of the function $e(\lambda, x, x)$ as $\lambda\to+\infty$, which is uniform with respect to parameter $x\in\mathrm{R}^n$. In particular, the parameter x may tend to infinity together with λ arbitrarily.

To explain the main idea better we consider at first the more simple problem, concerning the asymptotic behaviour of the function $N(\lambda)=\sum_{\lambda_k\leqq\lambda} 1$, which is continuous from the right. The Laplace transform, $\sigma(p)=\int_0^\infty e^{-\lambda p}\,\mathrm{d}N(\lambda)$, is holomorphic on the right halfplane $\mathrm{Re}\,p>0$ and, what is more, $\sigma(p)=(2\,\mathrm{sh}\,p)^{-n}$. In order to use the reconversion Laplace formula, we note that

$$\frac{e^{hp}-1}{h}\,\frac{\sigma(p)}{p^2}=\int_0^\infty e^{-\lambda p}N_h(\lambda)\,\mathrm{d}\lambda,$$

where $N_h(\lambda)=\frac{1}{h}\int_0^h N(\lambda+\mu)\,\mathrm{d}\mu$, $h>0$. Consequently (more details see below),

$$N(\lambda)=\frac{1}{2\pi i}\int_{\varepsilon-i\pi/2}^{\varepsilon+i\pi/2} e^{\lambda p}\sigma(p)H(\lambda+n, p)\,\mathrm{d}p, \quad \varepsilon>0,$$

where the function $s \to H(s,p)$ is 2-periodic and $H(s,p) = e^{(1-s)p}(\mathrm{sh}\,p)^{-1}$ if $0 \leqq s < 2$.

Further, the function $p \to e^{\lambda p}\sigma(p) H(\lambda+n, p)$ is $i\pi$-periodic with a pole at the point $p=0$, so the residue formula is applicable and gives

$$(1) \qquad N(\lambda) = \sum_{j=0}^{n} a_{jn}(\lambda)\lambda^{n-j}, \quad |a_{jn}| \leqq \text{const},$$

where, for example, $a_{0n}(\lambda) = 1/2^n n!$, $a_{1n}(\lambda) = h(\lambda+n)/2^n(n-1)!$, and the function h is 2-periodic, $h(s) = 1-s$ if $0 \leqq s < 2$ (the formula (1) is obvious if $n=1$).

The results for the spectral function $e(\lambda, x, x)$ are analogous but more complicated. It is convenient to write down the asymptotic of the function

$$E(\lambda, x) = e(\lambda, \sqrt{\lambda}\,x, \sqrt{\lambda}\,x), \quad \lambda \to +\infty.$$

Theorem 1. *(the case $|x^2-1| \leqq \delta$). There exists a positive number δ such that*

$$E(\lambda, x) = \sum_{j=0}^{\infty} \lambda^{n/6}[a_{jn}(\lambda,x)\lambda^{-j} + b_{jn}(\lambda,x)\lambda^{-j-1/3} + c_{jn}(\lambda,x)\lambda^{-j-2/3}],$$

uniformly with respect to parameter x, where

$$a_{jn}(\lambda,x) = \tilde{a}_{jn}(\lambda,x) f_n(\lambda^{2/3} g(x)),$$

$$b_{jn}(\lambda,x) = \tilde{b}_{jn}(\lambda,x) f'_n(\lambda^{2/3} g(x)),$$

$$c_{jn}(\lambda,x) = \tilde{c}_{jn}(\lambda,x) f''_n(\lambda^{2/3} g(x)),$$

and the functions $\tilde{a}_{jn}$, $\tilde{b}_{jn}$, $\tilde{c}_{jn}$ are bounded,

$$\tilde{a}_{on}(\lambda,x) = (2\pi)^{-n} V_n \left[\frac{x^2-1}{g(x)}\right]^{n/2},$$

$$\tilde{b}_{on}(\lambda,x) = -(2\pi)^{-n} V_n \left[\frac{g(x)}{x^2-1}\right]^{\frac{1+n}{4}} |x|^{\frac{1-n}{4}} h_1(\lambda+n, x),$$

$$\tilde{c}_{on}(\lambda,x) = (2\pi)^{-n} \frac{V_n}{g(x)} \left\{\left[\frac{g(x)}{x^2-1}\right]^{\frac{3+n}{4}} |x|^{\frac{1-n}{4}} h_2(\lambda+n, x) - \left[\frac{x^2-1}{g(x)}\right]^{n/2}\right\},$$

V_n being the volume of the unit ball in $\mathbf{R}^n$. The functions $s \to h_k(s,x)$, $k=1, 2$, are 2-periodic and

$$h_1(s,x) = (x^2-1)^{-1/2}\,\mathrm{sh}[(1-s)\,\mathrm{arsh}\sqrt{x^2-1}], \quad 0 \leqq s < 2,$$

$$h_2(s,x) = \mathrm{ch}[(1-s)\,\mathrm{arsh}\sqrt{x^2-1}], \quad 0 \leqq s < 2.$$

The function $g \in C^\infty$ and

$$g(x) = \left(\frac{3}{2}\right)^{2/3} [|x|\sqrt{x^2-1} - \mathrm{arsh}\sqrt{x^2-1}]^{2/3} > 0 \text{ if } x^2 > 1,$$

$$g(x) = -\left(\frac{3}{2}\right)^{2/3} [|x|\sqrt{1-x^2} - \operatorname{arsin}\sqrt{1-x^2}]^{2/3} < 0 \text{ if } x^2 < 1.$$

Finally,

$$f_n(s) = \int_0^\infty \sigma^{n/2} \operatorname{Ai}(\sigma+s)\,d\sigma, \text{ where } \operatorname{Ai}(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i(\sigma z + z^3/3)}\,dz$$

is the Airy function. The following recurrence relations are satisfied:

$$f_n(s) = -sf_{n-2}(s) + f''_{n-2}(s), \quad n \geqq 2,$$

$$f_0(s) = \int_s^\infty \operatorname{Ai}(\sigma)\,d\sigma,$$

$$f_1(s) = 2^{1/3}\pi\{-4^{-1/3}s[\operatorname{Ai}(4^{-1/3}s)]^2 + [\operatorname{Ai}'(4^{-1/3}s)]^2\}.$$

Corollary 1. *If* $x^2 = 1$, *then*

$$E(\lambda, x) = \sum_{j=0}^\infty \lambda^{n/6}[a_{jn}(\lambda)\lambda^{-j} + b_{jn}(\lambda)\lambda^{-j-1/3} + c_{jn}(\lambda)\lambda^{-j-2/3}],$$

where

$$a_{on}(\lambda) = (2\pi)^{-n} V_n f_n(0),$$

$$b_{on}(\lambda) = (2\pi)^{-n} V_n h_3(\lambda+n)\frac{n}{2} f_{n-2}(0),$$

$$c_{on}(\lambda) = (2\pi)^{-n} V_n [h_4(\lambda+n) - \frac{3}{20}(n+1)] f_{n+2}(0).$$

The functions $h_k(s)$, $k=3, 4$, *are 2-periodic and*

$$h_3(s) = 1-s, h_4(s) = \frac{1}{2}(1-s)^2 \text{ is } 0 \leqq s < 2.$$

The numbers $f_n(0) = \int_0^\infty \sigma^{n/2} \operatorname{Ai}(\sigma)\,d\sigma$ *are positive and can be expressed by the gamma-function:*

$$f_n(0) = \frac{1}{2\pi} 3^{n/3-1/2} \Gamma\left(\frac{n+1}{6} + \frac{1}{2}\right) \Gamma\left(\frac{n-1}{6} + \frac{1}{2}\right).$$

Theorem 2. (*the case* $x^2 \leqq 1-\delta$). *If* $\lambda \to +\infty$, *then*

$$E(\lambda, x) = \sum_{j=0}^\infty a_{jn}(\lambda, x)\lambda^{n/2-j},$$

uniformly with respect to parameter x, where the coefficients a_{jn} *are bounded and*

$$a_{on}(\lambda, x)=(2\pi)^{-n} V_n (1-x^2)^{n/2}, \quad a_{1n}=b_{1n}+c_{1n},$$

$$b_{1n}(\lambda, x)=(2\pi)^{-n} V_n (1-x^2)^{n/2-1} \frac{n}{2} h_3(\lambda+n),$$

$$c_{1n}(\lambda, x)=\frac{1}{2}(2\pi)^{-n} \int_{|\omega|=1} [1-(x\omega)^2]^{-1} h_5(\lambda+n, \lambda, x\omega)\, d\omega.$$

The function $s \to h_5(s, \lambda, x\omega)$ *is 2-periodic and*

$$h_5(s, \lambda, x\omega)=\sin[\lambda(\arccos x\omega - x\omega\sqrt{1-(x\omega)^2}-n\frac{\pi}{2}+(1-s)\arccos x\omega] \text{ if } 0\leqq s<2.$$

In particular, $c_{11}=\dfrac{(1-x^2)^{-1}}{2\pi} h_5(\lambda+1, \lambda, x)$.

Corollary 2. *If* $0<\delta_1\leqq x^2\leqq 1-\delta$ *and* $n\geqq 2$, *then* $c_{1n}(\lambda, x)=0(\lambda^{-\frac{n-1}{2}})$, *so in this case,* $a_{1n}=b_{1n}$.

Remark 1. Let us consider the spectral function $e(\lambda, x, x)$ in the classical case, when $x^2\leqq$ const. Then we have the following uniform asymptotic as a consequence of theorem 2:

$$e(\lambda, x, x)=(2\pi)^{-n} V_n\left[\lambda^{n/2}+\frac{n}{2}\sum_{j=1}^{\infty} a_{jn}(\lambda, x)\lambda^{n/2-j/2}\right], \tag{2}$$

where the functions a_{jn} are bounded and

$$a_{1n}=0, \quad a_{3n}=0, \quad a_{2n}(\lambda, x)=h_3(\lambda+n)+h_5(\lambda+n, \lambda, 0)-x^2,$$

$$a_{j,2k}(\lambda, 0)=0 \quad \text{if} \quad j>2k. \tag{3}$$

Theorem 3. (*the case* $x^2\geqq 1+\delta$). *If* $\lambda\to\infty$, *then*

$$E(\lambda, x)=\exp[-\lambda(|x|\sqrt{x^2-1}-\operatorname{arch}|x|)] \sum_{j=0}^{\infty} a_{jn}(\lambda, x)\lambda^{-1/2-j},$$

uniformly with respect to parameter x, where the coefficients a_{jn} *are bounded and*

$$a_{on}(\lambda, x)=(4\pi)^{-\frac{1+n}{2}}(x^2-1)^{-\frac{3+n}{4}}|x|^{\frac{1-n}{2}} h_6(\lambda+n, x).$$

The function $s\to h_6(s, x)$ *is 2-periodic and*

$h_6(s, x)=\exp[(1-s)\operatorname{arch}|x|]$ if $0\leqq s<2$.

2. Proof of theorem 1

First of all we establish the following basic formula:

$$e(\lambda, x, x) = \frac{1}{2\pi i} \int_{\varepsilon_0 - i\pi/2}^{\varepsilon_0 + i\pi/2} e^{\lambda p} V(p, x) H(\lambda + n, p) \, dp, \quad \varepsilon_0 > 0, \tag{4}$$

$p = \varepsilon_0 + it$, where $s \to H(s, p)$ is a 2-periodic function, $H(s, p) = e^{(1-s)p}/\operatorname{sh} p$ if $0 \leqq s < 2$, and $V(p, x)$ is the Laplace transform of the spectral function, $V(p, x) = \int_0^\infty e^{-\lambda p} de(\lambda, x, x)$, so it is holomorphic on the right halfplane $\operatorname{Re} p > 0$. The function V may be expressed explicitly. We have (see, for example, [2]):

$$V(p + ik\pi, x) = e^{ik\pi n} V(p, x), \quad k \in \mathbb{Z}, \tag{5}$$

and

$$V(p, x) = (2\pi \operatorname{sh} 2p)^{-n/2} e^{-x^2 \operatorname{th} p} \text{ if } -\frac{\pi}{2} < \operatorname{Im} p \leqq \frac{\pi}{2},$$

$\operatorname{Re} p > 0$, with the main value for the radical.

To prove (4) we note that $\int_0^\infty e^{-\lambda p} e(\lambda, x, x) \, d\lambda = \frac{V(p, x)}{p}$. Since the function $\lambda \to e(\lambda, x, x)$ is continuous only from the right, we pass to its continuous average:

$$e_h(\lambda, x, x) = \frac{1}{h} \int_0^h e(\lambda + \mu, x, x) \, d\mu, \quad h > 0.$$

Evidently, $e_h(\lambda, x, x) \to e(\lambda, x, x)$ if $h \to +0$ for every fixed (λ, x), and

$$\int_0^\infty e^{-\lambda p} e_h(\lambda, x, x) \, d\lambda = \frac{e^{hp} - 1}{h} \frac{V(p, x)}{p^2}, \quad h > 0, \ \operatorname{Re} p > 0.$$

By the reconversion Laplace formula we conclude that

$$e_h(\lambda, x, x) = \frac{1}{2\pi i} \int_{\varepsilon_0 - i\infty}^{\varepsilon_0 + i\infty} e^{\lambda p} \frac{e^{hp} - 1}{h} \frac{V(p, x)}{p^2} \, dp, \quad \varepsilon_0 > 0,$$

(the integral is absolutely summable). Using the periodicity relation (5) and the Weierstrass convergence theorem, we get

$$e_h(\lambda, x, x) = \frac{1}{2\pi i} \int_{\varepsilon_0 - i\pi/2}^{\varepsilon_0 + i\pi/2} e^{\lambda p} V(p, x) \frac{g(h, p) - g(0, p)}{h} \, dp, \tag{6}$$

where $g(s, p) = e^{ps} f(s + \lambda + n, p)$ and $f(s, p) = \Sigma \frac{e^{isk\pi}}{(p + ik\pi)^2}$. The continuous function $s \to f(s, p)$ is 2-periodic and $f(s, p) = e^{(1-s)p} (\operatorname{cth} p - 1 + s)/\operatorname{sh} p$ if $0 \leqq s < 2$, $\operatorname{Re} p > 0$.

Consequently, $\lim\limits_{h\to+0}\dfrac{g(h,p)-g(0,p)}{h}=H(\lambda+n,p)$ and the Lebesgue convergence theorem is applicable. So the formula (4) follows from (6).

For proving of theorem 1 we start from the formula

$$(7)\qquad E(\lambda,x)=\frac{1}{2\pi i}\int_{\varepsilon_0-i\pi/2}^{\varepsilon_0+i\pi/2} e^{\lambda(p-x^2\operatorname{th}p)}(2\pi\operatorname{sh}2p)^{-\frac{n}{2}}H(\lambda+n,p)\,dp.$$

Let $\varphi(p,\alpha)=p-(1+\alpha)\operatorname{th}p$, where $\alpha=x^2-1$. This function has two critical points $p_\pm=\pm\operatorname{arsh}\sqrt{\alpha}$, which degenerate if $\alpha=0$. (We take always the main value of the radical.) Since $\partial_p^2\varphi(0,0)=0$, $\partial_p^3\varphi(0,0)=2$, $\partial_{p\alpha}^2\varphi(0,0)=1$, there exists (see [1, lemma 2.3]) a holomorphic change of variables $p=p(z,\alpha)$, $p(0,\alpha)=0$, defined in some neighborhood of the point $z=0$, $\alpha=0$, such that

$$(8)\qquad \varphi(p,\alpha)=-B(\alpha)z+z^3/3,$$

where $B(\alpha)=\left(\frac{3}{2}\right)^{2/3}[\operatorname{arch}\sqrt{1+\alpha}-\sqrt{1+\alpha}\sqrt{\alpha}]^{2/3}$ and $B(\alpha)=\alpha-\frac{\alpha^2}{5}+0(\alpha^3)$, $B\in C^\infty$. The inversion holomorphic function $z=z(p,\alpha)$ is fixed by the choice $z(\varepsilon,0)>0$ if $\varepsilon>0$. Then $z(p_\pm,\alpha)=\pm\sqrt{B(\alpha)}$.

To use (8) we note that

$$(9)\qquad E(\lambda,x)\sim\frac{1}{2\pi i}\int_\gamma e^{\lambda\varphi}(2\pi\operatorname{sh}2p)^{-n/2}H(\lambda+n,p)\,dp,$$

where γ is the segment $[\varepsilon_0(1-i),\ \varepsilon_0(1+i)]$, ε_0 being sufficiently small, and the equivalence "$a(\lambda,x)\sim b(\lambda,x)$" means that $a(\lambda,x)-b(\lambda,x)=0(e^{-c\lambda})$, $c>0$.

Indeed,

$$\operatorname{Re}\varphi(p,\alpha)=\varepsilon-\frac{1+\alpha}{2}\,\frac{\operatorname{sh}2\varepsilon}{\operatorname{ch}^2\varepsilon-\sin^2 t},\quad p=\varepsilon+it,$$

therefore,
$\operatorname{Re}\varphi(\varepsilon\pm i\varepsilon,\alpha)\leqq C(\varepsilon_0)<0$ if $0<\varepsilon\leqq\varepsilon_0$, $|\alpha|<\Gamma(\varepsilon_0)$ and if the numbers ε_0 and $\Gamma(\varepsilon_0)$ are sufficiently small.

Further, using the change of variables (8) in the integral (9), we see that

$$(10)\qquad E(\lambda,x)\sim\int_{\gamma^*} e^{(-Bz+z^3/3)}z^{-n/2-1}q(z,\alpha)\,dz,$$

where the function

$$q(z,\alpha)=\frac{1}{2\pi i}\left(\frac{2\pi\operatorname{sh}2p(z,\alpha)}{z}\right)^{-n/2}p'(z,\alpha)\,zH(\lambda+n,p(z,\alpha)),$$

is holomorphic in a neighborhood of γ^*, the oriented curve $\gamma^*=(z_1, z_2)$ is the image of γ by the conformal mapping $p \to z(p, \alpha)$, and $\gamma^* \subset \{z : \operatorname{Re} z > 0\}$, $\arg z_1 \in (-\pi/2, -\pi/6)$, $\arg z_2 \in (\pi/6, \pi/2)$.

Now, to find the uniform asymptotic evaluation of the integral (10) as $\lambda \to +\infty$, we apply the Weierstrass preparation theorem [3]:

$$(11) \qquad q(z, \alpha) = \Gamma_0(\alpha) + \Gamma_1(\alpha) z + \Gamma_2(\alpha) z^2 + z(z^2 - B(\alpha)) q_1(z, \alpha).$$

An integration by parts in (10) gives the relation

$$(12) \qquad E(\lambda, x) \sim \int_{\gamma^*} e^{\lambda(-Bz + z^3/3)} z^{-n/2-1} (\Gamma_0 + \Gamma_1 z + \Gamma_2 z^2) \, dz + R,$$

where

$$R = R(\lambda, x) = \frac{1}{\lambda} \int_{\gamma^*} e^{\lambda(-Bz + z^3/3)} z^{-n/2-1} q_2(z, \alpha) \, dz,$$

and the function $q_2(z, \alpha) = \frac{n}{2} q_1(z, \alpha) - z \partial_z q_1(z, \alpha)$ is holomorphic in a neighborhood of γ^*.

The first integral in (12) can be expressed by the Airy function. To this end we note that

$$\operatorname{Ai}(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(sp + p^3/3)} \, dp = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-sz + z^3/3} \, dz,$$

therefore, by the Cauchy formula,

$$\operatorname{Ai}(s) = \frac{1}{2\pi i} \int_{\Gamma} e^{-sz + z^3/3}, \quad \Gamma = \gamma_1 \cup \gamma_2,$$

where

$$\gamma_1 : z = \rho e^{i\varphi_1}, \quad \rho \in (+\infty, 0), \quad \varphi_1 \in (-\pi/2, -\pi/6),$$

$$\gamma_2 : z = \rho e^{i\varphi_2}, \quad \rho \in (0, +\infty), \quad \varphi_2 \in (\pi/6, \pi/2).$$

Let

$$I_k(\lambda, x) = \int_{\gamma^*} e^{\lambda(-B(\alpha)z + z^3/3)} z^{-n/2-1+k} \, dz, \quad k = 0, 1, \dots$$

Since

$$z^{-n/2-1} = \pi^{-n/2} V_n \int_0^{\infty} \sigma^{n/2} e^{-\sigma z} \, d\sigma, \quad \operatorname{Re} z > 0,$$

it is not hard to see that

$$I_k(\lambda, x) \sim (-1)^k \pi^{-n/2} V_n 2\pi i \lambda^{\frac{n}{6} - \frac{k}{3}} \int_0^{\infty} \sigma^{\frac{n}{2}} \operatorname{Ai}^{(k)}(\sigma + \lambda^{2/3} B) \, d\sigma.$$

To compute the coefficients Γ_0, Γ_1, Γ_2, in (11) we use the formulas: $\Gamma_0(\alpha)=q(0,\alpha)$, $\Gamma_1(\alpha)=\frac{1}{2\sqrt{B}}[q(\sqrt{B},\alpha)-q(-\sqrt{B},\alpha)]$,

$$\Gamma_2(\alpha)=\frac{1}{2B}[q(\sqrt{B},\alpha)+q(-\sqrt{B},\alpha)-2q(0,\alpha)].$$

On the other hand, $p'(0,\alpha)=B/\alpha$ and consequently,

$$\Gamma_0(\alpha)=\frac{1}{2\pi i}\pi^{-n/2}2^{-n}\left[\frac{B(\alpha)}{\alpha}\right]^{-n/2}.$$

Further,

$$p(\pm\sqrt{B(\alpha)},\alpha)=\pm\operatorname{arsh}\sqrt{\alpha},\, p'(\pm\sqrt{B},\alpha)=\left(\frac{B}{\alpha}\right)^{1/4}(1+\alpha)^{1/4}$$

and therefore,

$$q(\pm\sqrt{B},\alpha)=\frac{1}{2\pi i}(4\pi)^{-n/2}\left(\frac{B}{\alpha}\right)^{\frac{1+n}{4}}(1+\alpha)^{\frac{1-n}{4}}(\pm\sqrt{B}H(\lambda+n,\pm\operatorname{arsh}\sqrt{\alpha})).$$

Finally,

$$\Gamma_1(\alpha)=\frac{1}{2\pi i}(4\pi)^{-n/2}\left(\frac{B}{\alpha}\right)^{\frac{1+n}{4}}(1+\alpha)^{\frac{1-n}{4}}h_1(\lambda+n,\alpha),$$

$$\Gamma_2(\alpha)=\frac{1}{2\pi i}(4\pi)^{-n/2}\frac{1}{B}\left[\left(\frac{B}{\alpha}\right)^{\frac{3+n}{4}}(1+\alpha)^{\frac{1-n}{4}}h_2(\lambda+n,\alpha)-\left(\frac{\alpha}{B}\right)^{n/2}\right],$$

where the functions $s\to h_k(s,\alpha)$, $k=1, 2$ are 2-periodic and $h_1(s,\alpha)=\alpha^{-1/2}\operatorname{sh}[(1-s)\operatorname{arsh}\sqrt{\alpha}]$, $h_2=\operatorname{ch}[(1-s)\operatorname{arsh}\sqrt{\alpha}]$ if $0\leqq s<2$.

Evidently Theorem 1 follows to within a bound of the rest $R(\lambda,x)$ in (12). Now we shall briefly explain how to estimate this rest. By iteration we may suppose that

$$R(\lambda,x)=\lambda^{-N-1}\int_{\gamma^*}e^{\lambda(-Bz+z^3/3)}z^{-n/2-1}q_N(z,\alpha)\,dz,$$

where $q_N(z,\alpha)$ is a holomorphic function in a neighborhood of γ^* and the natural number N is sufficiently large.

1st case $n=2k$. Using the Taylor formula and the bound (2.17) from [1] we obtain as before that

(13) $$R(\lambda, x) \sim \lambda^{-N-1} \sum_{j=0}^{k} \lambda^{n/6-j/3} f_k^{(j)}(\lambda^{2/3} B)$$

$$+0(\lambda^{-N-1})(\lambda^{-1/3}|\mathrm{Ai}(\lambda^{2/3} B)|+\lambda^{-2/3}|\mathrm{Ai}'(\lambda^{2/3} B)|).$$

On the other hand, the function f_n satisfies the equation $f_n'''(s)-sf_n'(s)+\frac{n}{2}f_n(s)=0$ from which it follows that

(14) $$|f_n^{(j)}(s)| \leqq C(1+|s|^{j-3})(|f_n(s)|+|f_n'(s)|+|f_n''(s)|), \quad j \geqq 3.$$

Having in mind the relation $f_{2k}^{(2k+1)}(s)=(-1)^{k+1} k!\,\mathrm{Ai}(s)$, one derives the estimate of the rest from (13), (14).

2nd case $n=2k+1$. The formula (13) is fulfilled again, but in the right hand side the Airy function $\mathrm{Ai}(s)$ must be replaced by the function

$$g(s)=\int e^{i(4^{1/3}st+t^3/3)}\, t^{1/2}\, dt = \text{const}\ \mathrm{Ai}(s)\,\mathrm{Ai}'(s).$$

The functions g and g' have separating zeros and the arguments from [1, p. 348] remain valid. Finally, $f_{2k+1}^{(2k+2)}(s)=\text{const}\,.\,g(s)$.

3. Proof of theorem 2

The main idea consists of transforming of the formula (4) so that the method of the stationary phase could be aplicable. Roughly speaking, we shall pass to the limit as $\varepsilon_0 \to 0$. An obstacle to this are the singularities at the points $p=0$, $p=\pm i\pi/2$. These singularities can be decomposed as follows:

$$V(p, x)=(4\pi \operatorname{sh} p)^{-n} \int e^{-\xi^2/4 \operatorname{cth} p + ix\xi}\, d\xi, \quad \mathrm{Re}\, p>0,$$

and

$$V(p, x)=(2\pi)^{-n}\, V_n \left(\frac{\operatorname{sh} 2p}{2p}\right)^{-\frac{n}{2}} p e^{-x^2 \operatorname{th} p} \int_0^{\infty} e^{-\sigma p} \sigma^{n/2}\, d\sigma.$$

On the other hand, to avoid the critical points on the boundary $\mathrm{Im}p = \pm\frac{\pi}{2}$, we note that the function $f(p)=e^{\lambda p}\, V(p, x)\, H(\lambda+n, p)$ is $i\pi$-periodic, therefore

$$e(\lambda, x, x)=\frac{1}{4\pi i}\int_{\Gamma} \chi(p) f(p)\, dp+\frac{1}{2\pi i}\int \chi_2(p) f(p)\, dp,$$

where Γ is the segment $[\varepsilon_0 - i\pi,\ \varepsilon_0 + i\pi]$, $\chi \in C_0^{\infty}(\Gamma)$ is $i\pi$-periodic function, $\chi(p)=\chi(\bar{p})$, $\chi=0$ near the zero, $\chi(p)=1$ if $0<\delta_1<\mathrm{Im}p<\pi-\delta_1$ ans $\chi_2=1-\chi$. Thus

$$e(\lambda, \sqrt{\lambda}\, x, \sqrt{\lambda}\, x) \sim e_1(\lambda, x)+e_2(\lambda, x),$$

where the equivalence "$a(\lambda,x)\sim b(\lambda,x)$" here means that $|a(\lambda,x)-b(\lambda,x)| \leqq C_N \lambda^{-N}$, and

$$e_1(\lambda,x)=\int_{|\omega|=1}\int e^{i\lambda\varphi} q_1(\lambda,t,\sigma,\omega)\,dt\,d\sigma\,d\omega,$$

$$\varphi(t,\sigma)=t+\frac{\sigma^2}{4}\operatorname{ctg} t - x\,\omega.\sigma,$$

$$q_1(\lambda,t,\sigma,\omega)=\frac{\lambda^{n/2}}{4\pi}(4\pi i\sin t)^{-n} H(\lambda+n,it)\,\chi(it)\,\sigma^{n-1}\varkappa(\sigma),$$

the function $\varkappa\in C_0^\infty[0,\infty)$, $\varkappa=1$ in a neighborhood of zero,

$$e_2(\lambda,x)=\int e^{i\lambda\psi} q_2(\lambda,t,\sigma)\,dt\,d\sigma,$$

$$\psi(t,\sigma)=t(1-\sigma)-x^2\operatorname{tg} t,$$

$$q_2(\lambda,t,\sigma)=\lambda^{n/2+1}(2\pi)^{1-n} V_n\left(\frac{\sin 2t}{2t}\right)^{-\frac{n}{2}} itH(\lambda+n,it)\,\chi_2(it)\,\sigma^{n/2}\varkappa(\sigma).$$

Now we may apply the method of the stationary phase. Not going into details, we only note that the critical points are nondegenerated, so the reasons are standard.

The formulas (2), (3) can be proved by the arguments from the introduction.

4. Proof of theorem 3

We use the formula (7). Since $x^2\geqq 1+\delta>1$ the phase function $\varphi(p,x)=p-x^2$ thp has one sadle point $p_0=\operatorname{arch}|x|$. Putting $\varepsilon_0=p_0$ we see that $\operatorname{Re}\varphi(p,x)\leqq\operatorname{Re}\varphi(p_0,x)\leqq -C(\delta)<0$ and therefore the function $E(\lambda,x)$ is exponentially small. To find its asymptotic, uniformly with respect to parameter x, where $x^2\geqq 1+\delta$, $\delta>0$, we shall apply the sadle-point method. Since the critical point p_0 is nondegenerated uniformly by the parameter x, $x^2\geqq 1+\delta$, we can prove that there exists a holomorphic change of variables $p=\psi(w,x)$, defined in some neighborhood $|w|\leqq C\sqrt{\delta}$, such that

$$\varphi(p,x)-\varphi(p_0,x)=-\frac{w^2}{2},\quad \psi(0,x)=p_0,$$

(15)

$$|\partial_w^k\psi(w,x)|\leqq C(k,\delta)\quad\text{if}\quad x^2\geqq 1+\delta,\quad \delta>0.$$

Here and later on all estimates are uniform with respect to parameter x, where $x^2\geqq 1+\delta$, $\delta>0$.

If U is the image of the circle $|w|\leqq C\sqrt{\delta}$ by the mapping $p=\psi(w,x)$, we have

(16) $$\{p:|p-p_0|\leqq C_1\sqrt{\delta}\}\subset U\subset\{p:|p-p_0|\leqq C_2\sqrt{\delta}\},$$

and

(17) $$\operatorname{Re}(\varphi(p_j, x)-\varphi(p_0, x)) \leqq -\rho_1(\delta)<0,$$

where $p_j=\varepsilon_0+it_j$ and $\gamma_0=(p_1, p_2)$ is the segment $U \cap (\varepsilon_0-i\pi/2, \varepsilon_0+i\pi/2)$.
To establish (15) we use the Taylor formula as usually:

$$\varphi(p, x)-\varphi(p_0, x)=(p-p_0)^2 g(x, p),$$

$$g(x, p)=\int_0^1 (1-s)\,\partial_p^2 \varphi(p_0+s(p-p_0))\,ds,$$

and note that $\operatorname{Re} g(x, p)>0$ if $\operatorname{Re} p>0$, $|\operatorname{Im} p|<\frac{\pi}{2}$, $x^2 \geqq 1+\delta$. So we may consider the holomorphic function $w=(p-p_0)\sqrt{-2g(x, p)}$. To find the inversion mapping, we solve the equation:

$$F(p, w, x)=(p-p_0)\sqrt{-2g(x, p)}-w=0, \quad F(p_0, 0, x)=0,$$

The bound $\left|\frac{\partial F}{\partial p}\right| \geqq C\delta^{1/4}$ is obtained by using the relation $|g| \geqq \operatorname{Re} g = \int_0^1 (1-s) \operatorname{Re} \varphi''(p_0+s(p-p_0))\,ds$ and the estimates:

$$\operatorname{Re} \varphi''(p, x) \geqq C_3 \sqrt{\delta},$$

$$|\partial_p^3 \varphi(p, x)| \leqq C_4 \quad \text{if} \quad |p-p_0| \leqq C\sqrt{\delta}.$$

So the solution $p=\psi(w, x)$ exists and $|\partial_w \psi(w, x)| \leqq C\delta^{-\frac{1}{4}}$. The other estimates in (15) are analogous.
Now from the formulas (7), (15), (18), (17) we get

$$E(\lambda, x)=e^{\lambda\varphi(p_0, x)} \int_{\gamma^*} e^{-\lambda\frac{w^2}{2}} f(\lambda, \psi(w, x))\,\psi'(w, x)\,dw+0(e^{-\lambda\rho_1(\delta)}),$$

where

$f(\lambda, p)=\frac{1}{2\pi i}(2\pi \operatorname{sh} 2p)^{-n/2} H(\lambda+n, p)$ and the oriented curve $\gamma_0^*=(w_1, w_2)$ is the image of the segment $\gamma_0=(p_1, p_2)$ by the conformal mapping $p=\psi(w, x)$. In particular,

$$\operatorname{Re}\frac{w_j^2}{2}=\operatorname{Re}(\varphi(p_0, x)-\varphi(p_j, x)) \geqq \rho_1(\delta)>0.$$

Further we follow the reasons on p. 170 [1]. Note only that $w=-t\sqrt{2g(x,p)}$ if $p-p_0=it$, so $w=-\sqrt{2}\,tx^{-1/2}(x^2-1)^{1/4}+0(t^2)$, and, on the other hand, $\psi'(0,x)=-i\,2^{-1/2}|x|^{1/2}(x^2-1)^{-1/4}$.

Thus theorem 3 is proved.

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