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Relations between Multivariate Rational and Spline Approximations

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In this paper we prove that multivariate rational functions are not worse than free spline functions as a tool for approximation in L_p metric, $0<p<\infty$.

Introduction

Recently A. A. Pekarskii [9] and P. Petrushev [10], [11] proved a relation in one dimensional case between rational and spline approximation in L_p metric, $0<p<\infty$, (see estimate (2)).

The purpose of the present paper is to find a relation between multivariate rational functions and free spline functions, which is similar to the estimate (2). In fact we develop the ideas used in [9], [10].

We shall denote by P_n the set of all algebraic polynomials with real coefficients of total degree n in R^d and by R_n the set of all rational functions of degree n, i.e. $R\in R_n$ if $R=P_1/P_2$, P_1, $P_2\in \mathrm{P}_n$.

First we shall give a definition of spline function in one dimensional case and after that we shall consider the multidimensional case. Assume that $[a,b]$ is a compact interval or $(-\infty, \infty)$. Let $\tilde{S}(n,k,[a,b])$ denote the set of all pise-wise polynomial functions (spline functions with defect) of degree $k-1$ with $n-1$ free knots on $[a,b]$, i.e. $\varphi\in\tilde{S}(n,k,[a,b])$ if there exist points $x_0=a<x_1<\dots<x_n=b$ such that in any interval (x_{i-1},x_i) φ is an algebraic polynomial of degree $k-1$.

There are many different definitions of multivariate free splines (see [2], [12]). Here we shall use the definition of free splines given in [4]. Let Ω be either a cube in R^d or whole space R^d. If Ω is a cube in R^d we suppose that Ω have sides parallel to the axes. Denote by $D=D(\Omega)$ the collection of all dyadic subcubes of Ω. If Ω is R^d there exist different collections D but this fact is not important for our

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consideration. Let us fix one of them and suppose that if $I \in D$, then I have sides parallel to the axes. We shall denote by $S(n, k, \Omega)$ the set of all free splines of order k and freedom n, i.e. $\varphi \in S(n, k, \Omega)$ if there is a collection $\Gamma = \{I_1, \ldots, I_{n-1}\}$ of dyadic cubes from D such that

(1) $$\varphi = \sum_{I \in \Gamma} \varphi_I \chi_I + \varphi_\Omega \chi_\Omega,$$

where $\varphi_\Omega, \varphi_I \in \mathrm{P}_{k-1}$ and χ_A is the characteristic function of the set A. It is easy to see that if $d=1$,

$$S(n, k, \Omega) \subseteq \tilde{S}(2n, k, [a, b]),$$

where $\Omega := [a, b]$.

We are interested in approximating functions in $L_p(\Omega)$ metric, $0 < p < \infty$. Denote by

$$R_n(f)_p = R_n(f, \Omega)_p = \inf\{\|f - R\|_{p(\Omega)} : R \in R_n\},$$

$$S_n^k(f)_p = S_n^k(f, \Omega)_p = \inf\{\|f - \varphi\|_{p(\Omega)} : \varphi \in S(n, k, \Omega)\},$$

$$\tilde{S}_n^k(f)_p = \tilde{S}_n^k(f, [a, b])_p = \inf\{\|f - \varphi\|_{p(\Omega)} : \varphi \in \tilde{S}(n, k, [a, b])\}$$

the errors in approximating $f \in L_p(\Omega)$ by rational functions, free splines and univariate splines, respectively, in L_p metric. In the definition of $\tilde{S}(n, k, [a, b])$ we have supposed that $f(x)$ is univariate function.

Throughout the paper c denotes a constant which may depend on any parameters but does not depend on f and n. The corresponding parameters can be written in brackets.

Suppose that $f(x)$ is univariate function. In [9], [10], [11] are proved the following estimates:

If $[a, b]$ is a compact interval, $f \in L_p[a, b]$, $0 < p < \infty$, $\alpha > 0$, $q = \min(1, p)$ and $k \geqq 1$, then for $n \geqq \max(1, k-1)$

$$R_n(f, [a, b])_p \leqq c n^{-\alpha} \left(\sum_{\nu=1}^{n} \frac{1}{\nu} (\nu^\alpha \tilde{S}_\nu^k(f, [a, b])_p)^q \right)^{1/q}.$$

(2) Moreover, if we put $f(x) \equiv 0$ for $x \in \mathrm{R}' \setminus [a, b]$ then for $n \geqq 1$,

$$R_n(f, (-\infty, \infty))_p \leqq c n^{-\alpha} (\|f\|_p^q + \sum_{\nu=1}^{n} \frac{1}{\nu} (\nu^\alpha \tilde{S}_\nu^k(f, [a, b])_p)^q)^{1/q}.$$

Also if $f \in L_p(-\infty, \infty)$, then for $n \geqq 1$

$$R_n(f, (-\infty, \infty))_p \leqq c n^{-\alpha} \left(\sum_{\nu=1}^{n} \frac{1}{\nu} (\tilde{S}_\nu^k(f, (-\infty, \infty))_p)^q \right)^{1/q},$$

where $c = c(p, k, \alpha)$.

Here we shall give a short definition of Besov spaces (see also [1], [5]). Let Θ be the unit cube in R^d. If $f\in L_p(\Theta)$, $0<p\leqq\infty$ we let $\omega_r(f,t)$, $t\geqq 0$, denote the modules of smothness of order r of f:

$$\omega_r(f,t)_p = \sup_{\|h\|\leqq t} \|\Delta_h^r(f,\cdot)\|_{p(\Theta(rh))}, \tag{3}$$

where $\|h\|$ is the Euclidean length of the vector h; Δ_h^r is the r-th order difference with step $h\in\mathsf{R}^d$; the norm in (3) is the L_p 'norm' on the set $\Theta(rh):=\{x;\ x, x+rh\in\Theta\}$. *When* $p<1$, this is not a norm, it is only a quasy-norm. If $\alpha, p, q>0$ we say f is in the Besov space B_{pq}^α whenever

$$|f|_{B_{pq}^\alpha} = \left(\int_0^\infty (t^{-\alpha}\omega_r(f,t)_p)^q \frac{dt}{t}\right)^{1/q}$$

is finite, where $r=[\alpha]+1$.

2. The main result

The basic result of this paper is the following:

Theorem 2.1. *If Ω is an arbitrary cube in R^d, $f\in L_p(\Omega)$, $0<p<\infty$, $\alpha>0$, $q=\min(1,p)$ and $k\geqq 1$, then for $n\geqq\min(1,k-1)$*

$$R_n(f,\Omega)_p \leqq cn^{-\alpha}\left(\sum_{\nu=1}^n \frac{1}{\nu}(\nu^\alpha S_\nu^k(f,\Omega)_p)^q\right)^{1/q}. \tag{4}$$

Moreover, if we put $f(x)\equiv 0$ for $x\in\mathsf{R}^d\setminus\Omega$, then for $n\geqq 1$,

$$R_n(f,\mathsf{R}^d)_p \leqq cn^{-\alpha}(\|f\|_p^q + \sum_{\nu=1}^n \frac{1}{\nu}(\nu^\alpha S_\nu^k(f,\Omega)_p)^q)^{1/q}. \tag{5}$$

Also if $f\in L_p(\mathsf{R}^d)$, then for $n\geqq 1$

$$R_n(f,\mathsf{R}^d)_p \leqq cn^{-\alpha}\left(\sum_{\nu=1}^n \frac{1}{\nu}(\nu^\alpha S_\nu^k(f,\mathsf{R}^d)_p)^q\right)^{1/q}. \tag{6}$$

In the estimates (4), (5), (6) $c=c(p,k,\alpha,d)$.

Remark 1. Clearly, the estimates (4), (5), (6) do not hold for $p=\infty$.

Corollary 2.1. *Let $f\in L_p(\Omega)$, $0<p<\infty$, and Ω be either a cube in R^d or whole space R^d. If $S_n^k(f,\Omega)_p=O(n^{-\gamma})$, $\gamma\geqq 0$, $k\geqq 1$, then*

$$R_n(f,\Omega)_p = O(n^{-\gamma}).$$

Theorem 2.1 can be used successfully in more general situations.

Corollary 2.2. *Let* $f\in L_p(\Omega)$, where $0<p<\infty$, *and* Ω *be either a cube in* R^d *or whole space* R^d. *Let* ω *be a nondecreasing and nonnegative function on* $[0, \infty)$ *such that* $\omega(2\delta)\leqq 2^\beta\omega(\delta)$ *for* $\delta\geqq 0$, $\beta\geqq 0$. *If* $S_n^k(f,\Omega)_p=0(n^{-\gamma}\omega(n^{-1}))$, $\gamma\geqq 0$, $k\geqq 1$, *then*

$$R_n(f,\Omega)_p=0(n^{-\gamma}\omega(n^{-1})).$$

The next corollary gives a result which is closely connected to the result in [3].

Corollary 2.3. *Let* Θ *be the unit cube in* R^d, $0<p<\infty$, $1/\sigma:=\alpha/d+1/p$, $\tau:=\min(1, d/(d-1)_p)$. *If* $f\in B_{\sigma\sigma}^\alpha$ *and* $0<\alpha<\tau$, *then for* $n\geqq 1$,

$$R_n(f,\Theta)_p\leqq cn^{-\alpha/d}|f|_{B_{\sigma\sigma}^\alpha}$$

with $c=c(p,\alpha,d)$.

3. Auxiliary results

The proof of Theorem 2.1 is based on the following statement:

Theorem 3.1. *Let* Ω *be a cube in* R^d *and* $\varphi\in S(n,k,\Omega)$, *where* $k\geqq 1$, $n\geqq 1$, $0<p<\infty$. *Then for any* $\lambda>0$ *there exists a rational function* R *such that*

$$\deg R\leqq cn\ln^2(e+1/\lambda)$$

$$\|\varphi-R\|_{p(\mathrm{R}^d)}\leqq\lambda\|\varphi\|_{p(\Omega)}$$

with $c=c(p,k,d)$.

To prove Theorem 3.1 we shall need another representation for φ. Let $\Gamma=\{I_1,\dots, I_n\}$ be a finite collection of dyadic cubes. If I is one of these cubes, we let B_I denote the collection of all cubes $J\in\Gamma$ such that $J\subset I$, $J\neq I$ and J is maximal (J is not contained in another cube with these properties). It may happen that some of the sets B_I have large cardinality however we can imbed Γ in a larger family $\Gamma'=\{A_1,\dots, A_m\}$, where $|B_A|\leqq 2^d$ (with respect to Γ') for all $A\in\Gamma'$.

Lemma 3.1. *Let* $\Gamma=\{I_1,\dots, I_n\}$ *be an arbitrary collection of dyadic cubes, with* $\Omega\in\Gamma$. *Then, there is a second collection* $\Gamma'=\{A_1,\dots, A_m\}$ *of dyadic cubes with the following properties:*

i) $\Gamma\subset\Gamma'$,

ii) $|B_A|\leqq 2^d$ *for all* $A\in\Gamma'$,

iii) $m\leqq 2^d n$.

This lemma is proved in [4]. In the same paper the following property is proved:

(7) If $A\in\Gamma'$ and $A_1,\dots,A_s$, $s:=2^d$, are the children of A, then any $J\in B_I$ is contained in one of the A_i, $i=1,\dots,s$. A given A_i can contain at most one $J\in B_I$.

Now let $\varphi \in S(n,k,\Omega)$ and let $\Gamma=\{\Omega, I_1,\dots,I_{n-1}\}$ be the cubes in the representation (1). Let Γ'' be the set of dyadic cubes given by Lemma 3.1. Clearly, with respect to this larger set Γ' we can also represent φ as in (1) with polynomials φ'_J. If $I\in\Gamma'$ we let $I' := I\setminus\bigcup\{J : J\in B_I\}$. The functions $\chi_{I'}$ have disjoint supports and we have

$$\varphi=\sum_{I\in\Gamma'}\varphi_{I'}\chi_{I'}\,;\qquad \varphi_{I'}=\sum_{J\in\Gamma';J\supseteq I}\varphi'_J. \tag{8}$$

If Q is a rectangle in R^l, $l\geqq 1$, with side lenght vector $\delta=(\delta_1,\dots,\delta_l)$, where $Q=\Pi_1^l\,[a_i,b_i]$ and $\delta_i=b_i-a_i>0$, then for any real $\mu>0$ let μQ denote the rectangle with side lenght vector $\mu\delta := (\mu\delta_1,\dots,\mu\delta_l)$ and the same centre as Q.

If $f\in L_1(\mathsf{R}^l)$, $l\geqq 1$, then Mf is the Hardy-Littlewood maximal function

$$Mf(x)=\sup_{J\in x}\frac{1}{|J|}\int_J |f(t)|\,dt,$$

where sup is taken over cubes J which contain x. $|A|$ is the Lebesgue measure of the set A. We shall use the following inequalities for the maximal function Mf (see [7]).

If Q is a cube in R^l with centre q, then

$$c_1\frac{|Q|}{|Q|+\|x-q\|^l}\leqq M\chi_Q(x)\leqq\frac{|Q|}{|Q|+\|x-q\|^l} \tag{9}$$

with c_1, c_2 depending only on l.

The next lemma gives an approximation of the jump function in one dimensional case.

Lemma 3.2. *Let* $0<\delta<1$, $0<\gamma<1$; $\Delta=[a,b]$ *be an arbitrary compact interval*; r – *integer. Then there exists a rational univariate function* $\sigma(x)$ *such that*

$$\deg\sigma(x)\leqq c\ln(e+1/\delta)\ln(e+1/\gamma),$$

$$0\leqq\sigma(x)\leqq 1 \quad for \quad x\in(-\infty,\ \infty),$$

$$0\leqq 1-\sigma(x)\leqq\gamma \quad for \quad x\in(1-\delta)\Delta,$$

$$0\leqq\sigma(x)\leqq\gamma(M\chi_\Delta(x))^{2r} \quad for \quad x\notin\Delta,$$

with $c=c(r)$.

This lemma is proved in [10], [11]. The next lemma is an analogue to the above lemma in several dimensional cases.

Lemma 3.3. *Let* P *be a rectangle in* R^d *and let* Q *be a cube in* R^d *which contain* P. *Then, for* $0<\delta<1$, $0<\gamma<1$, r – *integer, there exists a rational multivariate function* $\sigma(x)$ *such that*

(10) $$\deg \sigma(x) \leqq c \ln(e+1/\delta) \ln(e+1/\gamma),$$

(11) $$0 \leqq \sigma(x) \leqq 1 \qquad \textit{for} \qquad x \in \mathrm{R}^d,$$

(12) $$0 \leqq 1-\sigma(x) \leqq c\gamma \qquad \textit{for} \qquad x \in (1-\delta) P,$$

(13) $$0 \leqq \sigma(x) \leqq c\gamma (M\chi_Q(x))^{2r} \qquad \textit{for} \qquad x \notin P$$

with $c = c(r, d)$.

Proof. Suppose $P = \Pi_1^l \Delta_i$, $\Delta_i = [a_i, b_i]$, $b_i - a_i > 0$. By Lemma 3.2 we have rational univariate functions $\sigma_1(x), \ldots, \sigma_d(x)$ such that for $i = 1, \ldots, d$ the following conditions are satisfied:

(14) $$\deg \sigma_i(x) \leqq c \ln(e+1/\delta) \ln(e+1/\gamma),$$

(15) $$0 \leqq \sigma_i(x) \leqq 1 \qquad \text{for} \qquad x \in (-\infty, \infty),$$

(16) $$0 \leqq 1-\sigma_i(x) \leqq \gamma \qquad \text{for} \qquad x \in (1-\delta) \Delta_i,$$

(17) $$0 \leqq \sigma_i(x) \leqq \gamma (M\chi_{\Delta_i}(x))^{2rd} \qquad \text{for} \qquad x \notin \Delta_i.$$

Consider the rational multivariate function $\sigma(x) = \Pi_1^d \sigma_i(x_i)$, where $x = (x_1, \ldots, x_d)$. We shall prove that $\sigma(x)$ satisfies (10), (11), (12), (13). By (14), (15), we get (10), (11). To prove (12), we shall suppose that $x \in (1-\delta) P$. If $x \in (1-\delta) P$, then $x_i \in (1-\delta) \Delta_i$, $i = 1, \ldots, d$. Using (15), (16), we obtain

$$0 \leqq 1-\sigma(x) \leqq 1-\sigma_1(x_1) + \sigma_1(x_1)(1-\sigma_2(x_2) + \ldots$$
$$+ \sigma_1(x_1) \ldots \sigma_{d-1}(x_{d-1})(1-\sigma_d(x_d)) \leqq d\gamma.$$

The last inequalities implay (12).

Now let consider the case when $x \notin P$. To estimate $\sigma(x)$ it is necessary two cases to be considered. The first case is when $x \in 4Q$. Since $x \notin P$, then there exists an integer k such that $x_k \notin \Delta_k$. By (9) we obtain $1 \geqq M\chi_Q(x) \geqq c$, $c = c(d)$.

Consequently

$$\sigma(x) \leqq \sigma_k(x_k) \leqq \gamma \leqq c\gamma (M\chi_Q(x))^{2r},$$

where the second inequality uses (17).

The second case is when $x \notin 4Q$. Suppose that P and Q have central a and b, respectively. There is an integer k such that $|x_k - b_k| = \max_{1 \leq i \leq d} |x_i - b_i|$. Since $a, b \in Q$ and $x \notin Q$, then $1/2\ |x_k - b_k| \leqq |x_k - a_k| \leqq 2|x_k - b_k|$. Since $x \notin 4Q$, then $|x_k - a_k| \geqq 1/2 |x_k - b_k| \geqq 2|Q|^{1/d} > |\Delta_k|$. The last inequalities show that $x_k \notin \Delta_k$.

By (17), (9) we obtain

$$\sigma(x) \leqq \sigma_k(x_k) \leqq \gamma (M\chi_{\Delta_k}(x_k))^{2rd} \leqq c\gamma \left(\frac{|\Delta_k|}{|\Delta_k| + |x - a_k|}\right)^{2rd} \leqq$$

$$\leqq c\gamma \left(\frac{|\Delta_k|^d}{|\Delta_k|^d + |x_k - b_k|^d}\right)^{2r} \leqq c\gamma \left(\frac{|Q|}{|Q| + \|x - b\|^d}\right)^{2r} \leqq c\gamma (M\chi_Q(x))^{2r}.$$

This completes the proof.

Lemma 3.4. *Let I be a cube of the collection Γ' and $I' = I \setminus \bigcup \{J : J \in B_I\}$. Then, for $0 < \delta < 1$, $0 < \gamma < 1$, r-integer, there exists a rational function $\sigma = \sigma_{I'}(x)$ such that*

$$\deg \sigma(x) \leqq c \ln(e + 1/\delta) \ln(e + 1/\gamma), \tag{18}$$

$$0 \leqq \sigma(x) \leqq 6^d \quad \text{for} \quad x \in \mathrm{R}^d. \tag{19}$$

There exist two sets $U_1(I)$ and $U_2(I)$, which are disjoint, such that

$$\begin{aligned} &\text{i)} \quad I' = U_1(I) \cup U_2(I), \\ &\text{ii)} \quad |U_2(I)| \leqq \delta |I'|, \\ &\text{iii)} \quad |1 - \sigma(x)| \leqq c\gamma \qquad \text{for} \quad x \in U_1(I), \end{aligned} \tag{20}$$

$$0 \leqq \sigma(x) \leqq c\gamma (M\chi_{I'}(x))^{2r} \quad \text{for} \quad x \notin I' \tag{21}$$

with $c = c(r, d)$.

P r o o f. If $I' = \emptyset$, then the rational function $\sigma(x) \equiv 0$ satisfies (18), (19), (20), (21). Let us assume $I' \neq \emptyset$. There exists a collection of rectangles $F = \{J_1, \ldots, J_\mu\}$, $\mu \leqq 6^d$, which are disjoint and we have

$$I' = \bigcup_{J \in F} J.$$

Really, the existence of the collection F can be easily extracted from the property (7).

If J is a rectangle of the collection F, then by Lemma 3.3 we obtain that there exists a rational function $\sigma_J(x)$ such that

$$\deg \sigma_J(x) \leqq c \ln(e + 1/\gamma) \ln(e + 1/\delta), \tag{22}$$

$$0 \leqq \sigma_J(x) \leqq 1 \quad \text{for} \quad x \in \mathrm{R}^d, \tag{23}$$

$$0 \leqq 1 - \sigma_J(x) \leqq c\gamma \quad \text{for} \quad x \in (1 - \delta/d) J, \tag{24}$$

$$0 \leqq \sigma_J(x) \leqq c\gamma (M\chi_I(x))^{2r} \quad \text{for} \quad x \notin J. \tag{25}$$

Now let us put $\sigma(x) = \sum_{J \in F} \sigma_J(x)$.

We shall prove that $\sigma(x)$ satisfies the conditions of the lemma. By (22), (23), we get (18), (19). Let us put

$$\begin{cases} U_1(I) := \bigcup_{J\in F}(1-\delta/d)\,J, \\ U_2(I) := \bigcup_{J\in F}(J\setminus(1-\delta/d)\,J). \end{cases} \tag{26}$$

It is obvious that $U_1(I)$ and $U_2(I)$ are disjoint and we have $I' = U_1(I) \bigcup U_2(I)$ because $I' = \bigcup_{J\in F} J$. By (26) we obtain

$$|U_2(I)| = (1-(1-\delta/d)^d)\,\Sigma_{J\in F}|J| \leqq \delta\Sigma_{J\in F}|J| = \delta|I'|.$$

Now we shall estimate $|1-\sigma(x)|$ when $x\in U_1(I)$. There is an integer ν, $1\leqq\nu\leqq\mu$, such that $x\in(1-\delta/d)\,J_\nu$. If $k\neq\nu$, then $J_k\bigcap(1-\delta/d)\,J_\nu = \emptyset$, ($J_i, i=1,\dots,\mu$ are disjoint) and we have $x\notin J_k$. By (24), (25) we obtain

$$|1-\sigma(x)| \leqq |1-\sigma_{J_\nu}(x)| + \sum_{\substack{1\leqq k\leqq\mu \\ k\neq\nu}} \sigma_{J_k}(x) \leqq c\mu\gamma \leqq c\gamma.$$

The last inequalities and (26) show that (20) holds.

Let us estimate $\sigma(x)$ when $x\notin I'$. Since $x\notin I'$, then $x\notin J_i$, $i=1,\dots,\mu$. Using (25), we get

$$0 \leqq \sigma(x) \leqq c\sum_1^\mu \gamma(M\chi_I(x))^{2r} \leqq c\gamma(M\chi_I(x))^{2r}. \tag{27}$$

We shall use the following inequalities for the maximal function

$$M\chi_{I'}(x) \leqq M\chi_I(x) \leqq cM\chi_{I'}(x) \tag{28}$$

with c depending only on d. The left-hand-side inequality holds because $I'\subseteq I$. Since $I'\neq\emptyset$, by (7), we obtain $4^{-d}|I| \leqq |I'| \leqq |I|$. By the last inequalities and (9), it is not difficult to prove the right-hand-inequality in (28). Now (21) immediately follows from (27), (28).

The proof is completed.

The next lemma gives some equivalent norms for polynomials.

Lemma 3.5. *Let $\varphi\in\mathsf{P}_{k-1}$, $0<q\leqq p\leqq\infty$, k-integer, I and I' are as in Lemma 3.4. If $I'\neq\emptyset$, the following inequalities*

$$1/c\left(\frac{1}{|I|}\int_I|\varphi|^q\right)^{1/q} \leqq \left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p} \leqq c\left(\frac{1}{|I|}\int_I|\varphi|^q\right)^{1/q}, \tag{29}$$

$$1/c\left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p} \leqq \left(\frac{1}{|I'|}\int_{I'}|\varphi|^p\right)^{1/p} \leqq c\left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p} \tag{30}$$

hold with $c=c(p,q,k,d)$. *If* $\lambda>0$, *then*

$$1/c\left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p}\leqq\left(\frac{1}{|\lambda I|}\int_{\lambda I}|\varphi|^p\right)^{1/p}\leqq c\left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p} \tag{31}$$

with $c=c(p,k,\lambda,d)$. *When* p *or* q *is equal to* ∞ *the corresponding expression in* (29), (30), (31) *is replaced by* L_∞ *norm over* I, I', λI, *respectively.*

Proof. (29) and (31) are proved in [6] (see §3). We shall prove only (30). By (7), there is a cube J satisfying the following conditions:

$$J\subseteq I'\subseteq I\subseteq 8J.$$

By the last inclusions and (31), we have

$$\left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p}\leqq c\left(\frac{1}{|8J|}\int_{8J}|\varphi|^p\right)^{1/p}\leqq c\left(\frac{1}{|J|}\int_J|\varphi|^p\right)^{1/p}$$
$$\leqq c\left(\frac{1}{|I'|}\int_{I'}|\varphi|^p\right)^{1/p}\leqq c\left(\frac{1}{|I|}\int_I|\varphi|^p\right)^{1/p}.$$

The same proof works when $p=\infty$.

Lemma 3.6. *Let* I *and* I' *be as Lemma* 3.4. *If* $I'\neq\emptyset$, $\varphi\in P_{k-1}$, *k-integer, then*

$$|\varphi(x)|\leqq c\,\|\varphi\|_{\infty(I')}(M\chi_{I'}(x))^{(1-k)/d}\qquad \textit{for}\qquad x\in R^d \tag{32}$$

with $c=c(d,k)$.

Proof. This lemma is based on the following inequality for polynomials:

$$|\varphi(x)|\leqq c\,\|\varphi\|_{\infty(Q)}(M\chi_Q x))^{(1-k)/d}\qquad \text{for}\qquad x\in R^d, \tag{33}$$

with $c=c(d,k)$; Q is a cube in R^d. The inequality (33) is well-known when φ is an univariate polynomial (see [8], [9]).

Without loss of generality we can assume that $Q=[-1,1]^d$ and $x\notin Q$. The general case can be proved by simple change of variables. There is an integer l such that $|x_l|=\max_{1\leqq i\leqq d}|x_i|$. Since $x\notin Q$, we have $a:=|x_l|>1$.

We put

$$g(t)=\varphi\left(\frac{x_1}{a}t,\ \frac{x_2}{a}t,\dots,\frac{x_d}{a}t\right).$$

Clearly, g is an one variable polynomial and $g(t)\in P_{k-1}$. We shall estimate $g(t)$ for $|t|>1$.

$$|g(t)| \leqq \|g\|_{\infty[-1,1]} T_{k-1}(t) \leqq \|g\|_{\infty[-1,1]} (|t| + \sqrt{t^2-1})^{k-1}$$
$$\leqq 2^{k-1} \|g\|_{\infty[-1,1]} |t|^{k-1},$$

where $T_n(t) = \cos(n \arccos x)$ is the Chebyshev polynomial (see [8]). We put $t=a$ in the last inequalities and obtain

$$|g(a)| = |\varphi(x)| \leqq c \|g\|_{\infty[-1,1]} a^{k-1} \leqq c \|\varphi\|_{\infty(Q)} |x_l|^{k-1}.$$

Now it is easy to estimate $|x_l|^{k-1}$. Indeed,

$$|x_l|^{k-1} \leqq c \|x\|^{d(k-1)/d} \leqq c|Q|^{(k-1)/d} \left(\frac{|Q| + \|x\|^d}{|Q|}\right)^{(k-1)/d}$$
$$\leqq c(M\chi_Q(x))^{(1-k)/d}.$$

Hence (33) holds. Applying (33) with $Q := I$ and using (30), (28), we obtain that (32) holds.

This completes the proof.

Lemma 3.7. *Let* $\Gamma' = \{I_1, \ldots, I_m\}$ *be the collection of dyadic cubes given by Lemma* 3.1 *and let* $y_1, \ldots, y_m$ *be nonnegative numbers. Then, if* $v \min(1,p) > 1$, $0 < p < \infty$, *v-integer, we have*

$$\| \Sigma_1^m y_k (M\chi_{I'_k}(\cdot))^v \|_{p(\mathrm{R}^d)} \leqq c (\Sigma_1^m y_k^p |I'_k|)^{1/p} \tag{34}$$

with $c = c(p, v, d)$.

Proof. This inequality is well-known for one variable (see [9]). Without loss of generality it can be considered in (34) only these terms for which $I' \neq \emptyset$. In [7] is proved the following fundamental property of the Hatdy-Littlewood maximal function:

If $f_i \in L_p(\mathrm{R}^d)$, l-integer, $i = 1, \ldots, l$, $0 < q < \infty$, then

$$\| (\Sigma_1^l (Mf_k(\cdot))^2)^{1/2} \|_{q(\mathrm{R}^d)} \leqq c \| (\Sigma_1^l (f_k(\cdot))^2)^{1/2} \|_{q(\mathrm{R}^d)} \tag{35}$$

with $c = c(q)$.

The proof of (34), when $0 < p \leqq 1$, is based on the following well-known inequality $\| \Sigma f_i \|_p^p \leqq \Sigma \|f_i\|_p^p$. Applying (35) for $\| y_k (M\chi_{I'_k}(\cdot))^v \|_p^p$ we obtain (34). If $1 < p < \infty$, then we apply (35) with $l := m$, $q := 2p$, $f_k := \sqrt{y_k}\, \chi_{I'_k}(x)$ and obtain

$$\| \Sigma_1^m y_k (M\chi_{I'_k})^2 \|_p = \| (\Sigma_1^m (M\sqrt{y_k}\, \chi_{I'_k})^2)^{1/2} \|_{2p}^2$$
$$\leqq c \| (\Sigma_1^m (\sqrt{y_k}\, \chi_{I'_k})^2)^{1/2} \|_{2p}^2 \leqq c \| \Sigma_1^m y_k \chi_{I'_k} \|_p \leqq c (\Sigma_1^m y_k^p |I'_k|)^{1/p}.$$

Here the last equality uses that I'_k, $k = 1, \ldots, m$ are disjoint. Consequently (34) holds for $v = 2$. Since $M\chi_A(x) \leqq 1$, A is an arbitrary set, (34) holds for any $v \geqq 2$.

The proof is completed.

Proof of Theorem 3.1

If $\lambda \geqq 1$, then the rational function $R \equiv 0$ satisfies the requirements of Theorem 3.1. Let $0<\lambda<1$, r be the smallest integer such that $[2r+(1-k)/d]\min(1,p)>1$, where $[x]$ denotes the integer part of x. Let $\gamma := \lambda/a$, $\delta := (\lambda/a)^p$, where $a \geqq 1$. The exact value of a will be fixed latter. Consider the rational function

$$R = \Sigma_{I\in\Gamma'} \varphi_{I'} \sigma_{I'},$$

where Γ' is the collection of dyadic cubes from Lemma 3.1; $\varphi_{I'}$ are the polynomials from representation (8); $\sigma_{I'}$ are the rational functions from Lemma 3.4. Without loss of generality it can be assumed that if $I\in\Gamma'$, then $I' \neq \emptyset$. If $I'=\emptyset$, then we have $\chi_{I'} \equiv \sigma_{I'} \equiv 0$. Let first estimate the degree of R. By (18) and Lemma 3.1 we have

$$\deg R \leqq 2^d nc \ln\left(e+\frac{a}{\lambda}\right) \ln\left(e+\left(\frac{a}{\lambda}\right)^p\right) \leqq cn \ln^2\left(e+\frac{1}{\lambda}\right). \tag{36}$$

By (20) we obtain $I' = U_1(I) \bigcup U_2(I)$. Let us put $U(I) := U_2(I)$, $V(I) = \mathrm{R}^d \backslash U(I)$ for $I\in\Gamma'$. Now we are ready to estimate $\|\varphi - R\|_p$.

$$\varphi - R = \Sigma_{I\in\Gamma'} \varphi_{I'}(\chi_{I'} - \sigma_{I'}) = \Sigma_{I\in\Gamma'} \varphi_{I'}(\chi_{I'}-\sigma_{I'})\chi_{V(I)} + \Sigma_{I\in\Gamma'} \varphi_{I'}(\chi_{I'}-\sigma_{I'})\chi_{U(I)} = a_1 + a_2.$$

First we shall estimate a_1. By (20), (21) we get

$$|(\chi_{I'} - \sigma_{I'})\chi_{V(I)}(x)| \leqq c\gamma (M\chi_{I'}(x))^{2r}.$$

By Lemma 3.6 we obtain

$$|\varphi_{I'}(x)| \leqq c \|\varphi\|_{\infty(I')} (M\chi_{I'}(x))^{(1-k)/d}.$$

Consequenly

$$|a_1(x)| \leqq c\gamma \Sigma_{I\in\Gamma'} \|\varphi\|_{\infty(I')} (M\chi_{I'}(x))^{\nu},$$

where $\nu = [2r+(1-k)/d]$. Now by Lemma 3.7, we obtain

$$\|a_1\|_p^p \leqq c\gamma^p \Sigma_{I\in\Gamma'} \|\varphi\|_{\infty(I')}^p |I'|. \tag{37}$$

Let us estimate a_2. Since $\{I'\}_{I\in\Gamma'}$ are disjoint and $U(I) = U_2(I) \subseteq I'$, then $U(I)_{I\in\Gamma'}$ are also disjoint.

By (19), (20) we have for $\|a_2\|_p$.

$$\|a_2\|_p^p = \|\Sigma_{I\in\Gamma'} \varphi_{I'}(\chi_{I'}-\sigma_{I'})\chi_{U(I)}\|_p^p \leqq c\Sigma_{I\in\Gamma'} \|\varphi_{I'}\|_{p(U(I))}^p$$

$$\leqq c\Sigma_{I\in\Gamma'} \|\varphi_{I'}\|_{\infty(U(I))}^p |U(I)| \leqq c\delta \Sigma_{I\in\Gamma'} \|\varphi\|_{\infty(I')}^p |I'|.$$

By the last estimates and (37) we get

$$\|\varphi-R\|^p_{p(\mathbb{R}^d)}\leqq c\left(\frac{\lambda}{a}\right)^p \Sigma_{I\in\Gamma'}\|\varphi\|^p_{\varkappa(I')}|I'|$$

$$\leqq c\left(\frac{\lambda}{a}\right)^p \Sigma_{I\in\Gamma'}\|\varphi\|^p_{p(I')}\leqq c\left(\frac{\lambda}{a}\right)^p \|\varphi\|^p_{p(\Omega)}.$$

The second inequality uses the fact that φ on I' is a polynomial and Lemma 3.5.

Hence

$$\|\varphi-R\|_{p(\mathbb{R}^d)}\leqq c_0\frac{\lambda}{a}\|\varphi\|_{p(\Omega)}. \tag{38}$$

Let we put $a=\max(1,c_0)$. The estimates (36), (38) imply Theorem 3.1.

R e m a r k 2. Theorem 3.1 can be used in more general situations. Let $\varphi\in S(n,k,\mathbb{R}^d)$. If $\|\varphi\|_{p(\mathbb{R}^d)}<\varkappa$, then there exists a rational function R such that $\deg R\leqq c\ln^2(e+1/\lambda)$, $\|\varphi-R\|_{p(\mathbb{R}^d)}\leqq\lambda\|\varphi\|_{p(\mathbb{R}^d)}$.

Proof of Theorem 2.1

Let Ω be a cube in $\mathbb{R}^d$. Let $\varphi_i\in S(2^i,k,\Omega)$ and satisfies $\|f-\varphi_i\|_{p(\Omega)}=S^k_{2^i}(f)_p =S^k_{2^i}(f,\Omega)_p$ for $i\geqq 0$. Here we use the existence of the best approximating element in $S(n,k,\Omega)$ in L_p metric. This fact can be proved, but the existence is not necessary for the proof of Theorem 2.1. We use this fact only to reduce the proof. Without using the existence φ_i can be chosen such that $\|f-\varphi_i\|_{p(\Omega)}\leqq 2\ S^k_{2^i}(f)_p$, when $S^k_{2^i}(f)_p>0$. If $S^k_{2^i}(f)_p=0$, then we terminate the sequence φ_i.

Obviously, for $i\geqq 1$ we have $\varphi_i-\varphi_{i-1}\in S(2^{i+1},k,\Omega)$ and

$$\|\varphi_i-\varphi_{i-1}\|_{p(\Omega)}\leqq c(\|f-\varphi_i\|_{p(\Omega)}+\|f-\varphi_{i-1}\|_{p(\Omega)})\leqq cS^k_{2^{i-1}}(f)_p.$$

Let s be an integer. Applying Theorem 3.1 for the function $\varphi_i-\varphi_{i-1}\,(i\geqq 1)$ with $\lambda_i:=2^{(i-s)\alpha}$, we obtain that there exists a rational function R_i satisfying

$$\deg R_i\leqq c2^{i+1}\ln^2(e+2^{-(i-s)\alpha}), \tag{39}$$

$$\|\varphi_i-\varphi_{i-1}-R_i\|_{p(\Omega)}\leqq 2^{(i-s)\alpha}\|\varphi_i-\varphi_{i-1}\|_{p(\Omega)}\leqq c2^{(i-s)\alpha}S^k_{2^{i-1}}(f)_p. \tag{40}$$

Consider the rational function $R=\sum\limits_{i=0}^{s}R_i$, where $R_0=\varphi_0$, $\varphi_0\in\mathsf{P}_{k-1}$. By (39) we get for degree of R

$$N=\deg R\leqq\sum_{i=0}^{s}\deg R_i\leqq k-1+\sum_{i=1}^{s}c2^{i+1}\ln^2(e+2^{(s-i)\alpha})\leqq c\sum_{i=0}^{s}2^i(s-i)^2\leqq c2^s.$$

Hence

$$N=\deg R\leqq c2^s. \tag{41}$$

Now we estimate $\|f-R\|_{p(\Omega)}$. By (40) we get

$$\|f-R\|^q_{p(\Omega)}\leqq\|f-\varphi_s\|^q_{p(\Omega)}+\sum_{i=1}^{s}\|\varphi_i-\varphi_{i-1}-R_i\|^q_{p(\Omega)}+\|\varphi_0-R_0\|^q_{p(\Omega)}$$

$$\leqq S^k_{2^s}(f)^q_p+c\sum_{i=1}^{s}(2^{(i-s)\alpha}S^k_{2^{i-1}}(f)_p)^q$$

$$\leqq c2^{-s\alpha q}\sum_{i=0}^{s}(2^{i\alpha}S^k_{2^i}(f)_p)^q\leqq c2^{-s\alpha q}\sum_{\nu=1}^{2^s}\frac{1}{\nu}(\nu^\alpha S^k_\nu(f)_p)^q.$$

By these estimates and (4.1) we obtain that for any $s\geqq 0$

$$R_N(f)_p\leqq c_2 2-^{s\alpha}\left(\sum_{\nu=1}^{2^s}\frac{1}{\nu}(\nu^\alpha S^k_\nu(f)_p)^q\right)^{1/q}, \tag{42}$$

where $N\leqq c_1 2^s$.

Now let $n\geqq\max(1,k-1)$. If $n\leqq c_1$ then

$$R_n(f)_p\leqq R_{k-1}(f)_p\leqq S^k_1(f)_p\leqq c_1^\alpha n^{-\alpha}\left(\sum_{\nu=1}^{n}\frac{1}{\nu}(\nu^\alpha S^k_\nu(f)_p)^q\right)^{1/q}.$$

Hence (4) holds for $\max(1,k-1)\leqq n\leqq c_1$.

Let $n>c_1$. We choose $s\geqq 0$ such that $c_1 2^s<n\leqq c_1 2^{s+1}$. Then by (42) we obtain

$$R_n(f)_p\leqq c_2 2^{-s\alpha}\left(\sum_{\nu=1}^{2^s}\frac{1}{\nu}(\nu^\alpha S^k_\nu(f)_p)^q\right)^{1/q}\leqq cn^{-\alpha}\left(\sum_{\nu=1}^{n}\frac{1}{\nu}(\nu^\alpha S^k_\nu(f)_p)^q\right)^{1/q}.$$

These estimates show that (4) holds. The estimate (5) can be proved in a similar manner. In this case we choose R_0 such that $\deg R_0\leqq c\ln^2(e+2^s)\leqq c2^s$; $\|\varphi_0-R_0\|_{p(R^d)}\leqq c2^{-s}\|f\|_{p(\Omega)}$. The estimate (6) can be proved also in a similar way. It must be used the Remark 2.

The theorem is proved.

By Theorem 2.1 it is easy to obtain Corollary 2.1 and Corollary 2.2 (see [10], [11]).

Proof of Corollary 2.3

In [4] is proved the following estimate for free multivariate splines

$$S^1_n(f,\Theta)_p\leqq cn^{-\alpha/d}|f|_{B^\alpha_{\sigma\sigma}}$$

with $c=c(p,\alpha)$. By Corollary 2.1 and above estimate we obtain Corollary 2.3.

Remark 3. The restriction $\alpha<\tau=\min(1, \alpha/(d-1)_p)$ in Corollary 2.3 can be omitted. We can prove this corollary without this restriction. Corollary 2.3 holds for $0<\alpha<\varkappa$.

Remark 4. Now we shall give a new definition of a free spline.

Let Ω be either a cube in R^d or whole space R^d. φ is a free spline of order k and freedom n if there is a collection $\Gamma=\{I_1,\dots,I_m\}$ of cubes, $I_i\subset\Omega$, which are disjoint and

$$\varphi=\Sigma_{I\in\Gamma}\ \varphi_I\,\chi_I,$$

where $\varphi_I\in\mathsf{P}_{k-1}$. If $n\dot{=}1$, then $\Gamma=\{\Omega\}$. We shall denote by $\Pi(n,k,\Omega)$ the set of all free splines.

Obviously, the new definition of free splines do not use dyadic cubes, but here the cubes must be disjoint.

With the new definition of free splines Theorem 2.1 holds. It must be changed: the letter S with the letter Π.

If $d=1$, then $\tilde{S}(n,k,\Omega)\subset\Pi(n,k,\Omega)$. The last inclusion shows that for one variable we obtain exactly the result of [9], [10], [11].

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