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A Posteriori Improvement of Adaptive Spline Function Approximation

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1. Introduction

In this paper, we deal with the numerical solution of singularly perturbed 2-point boundary value problem

(1) $$-\varepsilon u''+p(x)u=f(x),\quad u(0)=\alpha_0,\quad u(1)=\alpha_1,\quad 0<\varepsilon\ll 1,\quad x\in(0,\ 1),$$

where function $p(x), f(x)$ lies in $C^2[0,\ 1]$ and satisfies the following monotonicity condition: $p(x)>\bar{p}>0$.

In [3] spline difference scheme is derived using adaptive spline function approximation. For a certain choice of arbitrary parameter q_i in [3], uniform nodal and global second order accuracy is proved in [4] $(q=\sqrt{p_i/\varepsilon}\,h,\ p_i=p(x_i))$.

Here is given a posteriori improvement of adaptive spline function approximation to numerical solution of (1), and the resulting error in uniform norm is proved to be $0(h^2)$ for uniform partition. The improvement of the approximation is achieved by introducing appropriate values for parameter q_i in [3]. Namely, setting $q_i=\sqrt{2h\,\mathrm{th}\,(p_ih/2\varepsilon)}$ we obtain the following spline difference scheme

(2) $$RV_i=Qf_i,\quad u_0=\alpha_0,\quad u_1=\alpha_1,\quad i=1\,(1)\,N-1,$$

where

$$v_{i-1}r_i^-+v_ir_i^c+v_{i+1}r_i^+=f_{i-1}q_i^-+f_iq_i^c+f_{i+1}q_i^+$$

and v_i denotes approximate solution of (1) obtained by (2) with coefficients

$$r_i^\pm=1-\frac{h^2}{\rho_{i\pm1}^2}\frac{p_{i\pm1}}{\varepsilon}\Big(1-\frac{\rho_{i\pm1}}{\mathrm{sh}\,\rho_{i\pm1}}\Big),\quad r_i^c=-2-2\frac{h^2}{\rho_i^2}\frac{p_i}{\varepsilon}(-1+\rho_i\,\mathrm{cth}\,\rho_i);$$

$$q^\pm_i=-\frac{h^2}{\varepsilon}\frac{1}{\rho^2_{i\pm1}}\Big(1-\frac{\rho_{i\pm1}}{\mathrm{sh}\,\rho_{i\pm1}}\Big),\quad q_i^c=-\frac{2h^2}{\varepsilon}\frac{1}{\rho_i^2}(-1+\rho_i\,\mathrm{cth}\,\rho_i);$$

$$p_i=p(x_i),\quad \rho_i=\sqrt{2h\,\mathrm{th}\,(p_ih/2\varepsilon)},$$

or in a shortened notation

$$r_i^- = 1 + p_{i-1} q_i^-, \quad r_i^+ = 1 + p_{i+1} q_i^+, \quad r_i^c = -2 + p_i q_i^c.$$

We suppose that mesh for the problem under consideration is uniform $\Delta: 0 = x_0 < x_1 < \ldots < x_N = 1$ with the step size $h = x_i - x_{i-1}$, $i = 1(1)N$.

Lemma 1. [1]. *Let $u(x) \in C^4[0, 1]$ and $p'(0) = p'(1) = 0$. Then the solution of* (1) *has the form*

$$u(x) = u_0(x) + w_0(x) + g(x), \tag{3}$$

where

$$u_0(x) = q_0 \exp(-x\sqrt{p(0)/\varepsilon}), \quad w_0(x) = q_1 \exp(-(1-x)\sqrt{p(1)/\varepsilon}) \tag{4}$$

q_0, q_1 are bounded functions of ε independent of x, and

$$|g^{(i)}(x)| \leqq M(1 + \varepsilon^{1-i/2}), \quad i = 0(1)4, \tag{5}$$

M is a constant independent of ε.

2. The local truncation error

The local truncation error $\tau_i(u)$ of the scheme (2) is defined by $\tau_i(u) = Ru_i - Q(Lu)_i$ for an arbitrary sufficiently smooth function u. From Lemma 1. $\tau_i(u) = \tau_i(u_0) + \tau_i(w_0) + \tau_i(g)$. As $R(u_j - v_j) = \tau_i(u)$ nodal errors are

$$\max_i |u_i - v_i| \leqq \| B \|^{-1} \max_i |\tau_i(u)|, \tag{6}$$

where B denotes matrix of coefficients in the matrix form of (2).

1°. Matrix estimate

$$\| B \| = (r_i^- + r_i^c + r_i^+) \leqq M \frac{h^2}{\varepsilon} \frac{2}{\rho_i} \operatorname{th} \frac{\rho_i}{2} \leqq M h^2/\varepsilon \tag{7}$$

for all h, ε.

We observe separately the contribution to the truncation error and consequently to the nodal errors each of the functions in Lemma 1.

2°. Contribution of the function $g(x)$ to the truncation error

When $h^2 \leqq \varepsilon$ we use the following form of truncation error

$$\tau_i(g) = \tau_i^{(2)} g_i'' + \tau_i^{(3)} g_i''' + r_i^- \frac{h^4}{4!} u^{(4)}(\eta_1) + r_i^+ \frac{h^4}{4!} u^{(4)}(\eta_2) \tag{8}$$

$$+\varepsilon q_i^- \frac{h^2}{2!} u^{(4)}(\eta_3) - p_{i-1} \frac{h^4}{4!} u^{(4)}(\eta_4) q_i^- + \varepsilon q_i^+ \frac{h^2}{2!} u^{(4)}(\eta_5) - p_{i+1} \frac{h^4}{4!} q_i^+ u^{(4)}(\eta_6),$$

$$x_{i-1} \leqq \eta_i \leqq x_{i+1}, \quad i = 0(1)6,$$

where $\tau_i^{(0)} = \tau_i^{(1)} = 0$ and

$$\tau_i^{(2)} = \frac{h^2}{2}(r_i^- + r_i^+ - q_i^- p_{i-1} - q_i^+ p_{i+1}) + \varepsilon(q_i^- + q_i^c + q_i^+);$$

$$\tau_i^{(3)} = \frac{h^3}{6}(r_i^+ - r_i^- + p_{i-1} q_i^- - p_{i+1} q_i^+) + h\varepsilon(q_i^+ - q_i^-).$$

Using shortened notation of (2)

$$\tau_i^{(2)} = h^2 + \varepsilon(q_i^- + q_i^+ + q_i^c), \quad \tau_i^{(3)} = h\varepsilon(q_i^+ - q_i^-), \quad \text{i.e.}$$

$$\tau_i^{(2)} = \mathrm{h}^2\left(1 - \frac{2}{\rho_i} \operatorname{th} \frac{\rho_i}{2}\right) + N.$$

With N is denoted the part in τ_i which obviously leads uniform $0(h^2)$ accuracy. Taylor's development of $\operatorname{th}(\rho_i/2)$ and (5) give

$$|\tau_i^{(2)} g_i''| \leqq M h^4/\varepsilon.$$

Since

$$|q_i^+ - q_i^-| \leqq M h^5/\varepsilon^2, \quad |\tau^{(3)} g_i'''| \leqq M h^4/\varepsilon$$

one can find that the remainder terms in (8) are bounded by Mh^4/ε. It means

$$|\tau_i(g)| \leqq M h^4/\varepsilon, \quad \text{then} \quad h^2 \leqq \varepsilon. \tag{9}$$

Appropriate form of the truncation error when $h^2 \geqq \varepsilon$ for shortened notation of (2) is

$$\tau_i(g) = \frac{h^2}{2}(g''(\xi_1) + g''(\xi_2)) - \varepsilon(q_i^- g_{i-1}'' + q_i^c g_i'' + q^+ g_{i+1}'') \qquad \text{(see [4])}.$$

Since $|q^{\pm c}| \leqq Mh^2/\varepsilon$, $|g''(x)| \leqq M$, then

(10) $$|\tau_i(g)| \leqq Mh^2, \quad \text{when} \quad h^2 \geqq \varepsilon.$$

3°. Contribution of the boundary layer functions to the truncation error

$$\tau_i(u_0) = u_{0i}\{r_i^- \exp(\rho_0) + r_i^c + r_i^+ \exp(-\rho_0)\} - \{(p_0 - p_{i-1})q_i^- \exp(\rho_0) + (p_0 - p_i)q_i^c + (p_0 - p_{i+1})q_i^+ \exp(-\rho_0)\}.$$

or in a shortened notation of (2)

$$\tau_i(u_0) = u_{0i}\{4\mathrm{sh}^2(\rho_0/2) + p_0(q_i^- \exp(\rho_0) + q_i^c + q_i^+ \exp(-\rho_0))\},$$

where $\rho_0 = \sqrt{p_0/\varepsilon} \cdot h$, $p_0 = p(x_0)$.

Taylor's development of these expressions gives

(11) $$\tau_i(u_0) = u_{0i}\rho_0^2(\rho_i^2 - \rho_0^2)\left\{\frac{1}{12} + \frac{1}{90}(\rho_i^2 + \rho_0^2) + \frac{7}{360}\rho_i^2 + 0(\rho^4)\right\} + N.$$

Now we must estimate the part

$$\rho_i^2 - \rho_0^2 = 2h\,\mathrm{th}(p_i h/2\varepsilon) - p_0 h^2/\varepsilon = h^2/\varepsilon\,(p_0 - p_i) - h(\rho_{0i}\,\mathrm{cth}\,\rho_{0i} - 1)/\mathrm{cth}\,\rho_{0i},$$

where $\rho_{0i} = p_i h/2\varepsilon$.

Using estimate $|t\,\mathrm{cth}\,t - 1| \leqq Mt^2/(1+t)$ we obtain

(12) $$|\rho_i^2 - \rho_0^2| \leqq M(h^2/\varepsilon)(p_0 - p_i) + N$$

(11) and (12) yield

(13) $$|\tau_i(u_{0i})| \leqq Mh^4/\varepsilon, \quad \text{for} \quad h^2 \leqq \varepsilon.$$

Appropriate form of the truncation error when $h^2 \geqq \varepsilon$ is the following one:

$$\tau_i(u_0) = u_{0i}\{(r_i^- - r_i^-(\rho_0))\exp(\rho_0) + (r_i^c - r_i^c(\rho_0)) + (r_i^+ - r_i^+(\rho_0))\exp(-\rho_0)\} - \{(p_0 - p_{i-1})q_i^- \exp(\rho_0) + (p_0 - p_i)q_i^c + (p_0 - p_{i+1})q_i^+ \exp(-\rho_0)\} \quad \text{(see [4])}.$$

Since $$|r_i^{\pm c} - r_i^{\pm c}(\rho_0)| \leqq M(\rho - \rho_0)\,h/\sqrt{\varepsilon}, \quad |q_i^{\pm c}| \leqq Mh^2/\varepsilon$$

$$|\rho - \rho_0| \leqq M(h/\sqrt{\varepsilon})(p_0 - p_i) + N, \quad \text{then}$$

$$|\tau_i(u_0)| \leqq M(h^2/\varepsilon)x_i^2 \exp(-\sqrt{p_0/\varepsilon} \cdot x_i), \quad \text{i.e.}$$

(14) $$|\tau_i(u_0)| \leqq Mh^2 \exp(-\theta\sqrt{p_0/\varepsilon}\cdot x_i) \quad \text{when} \quad h^2 \geqq \varepsilon.$$

The proof for the second boundary layer function $w_0(x)$ is similar and estimates identical

(15) $$|\tau_i(w_0)| \leqq Mh^2 \exp(-\theta(1-x_i)\sqrt{p(1)/\varepsilon}), \quad \text{when} \quad h^2 \geqq \varepsilon$$

(16) $$|\tau_i(w_0)| \leqq Mh^4/\varepsilon, \quad \text{when} \quad h^2 \leqq \varepsilon.$$

Now we can formulate the main result of this paper.

Theorem 1. *Let* $p(x)$, $f(x)$ *lie in* $C^2[0, 1]$ *and* $p(x) \geqq p > 0$, $p'(0) = p'(1) = 0$. *Let* $\{v_i\}$, $i = 0(1)n$ *denotes the approximating solution of* (1) *obtained by* (2). *Then the solution of* (1) *can be estimated by the inequality*

(17) $$|u_i - v_i| \leqq \begin{cases} Mh^2 & \text{when} \quad h^2 \leqq \varepsilon \\ M\varepsilon & \text{when} \quad h^2 \geqq \varepsilon \end{cases} \quad i = 0(1)N.$$

M is a constant independent of ε, h.

Proof. (3), (9), (13) and (16) give

(18) $$|\tau_i(u)| \leqq Mh^4/\varepsilon, \quad \text{when} \quad h^2 \leqq \varepsilon.$$

(3), (10), (14) and (15) yield

(19) $$|\tau_i(u)| \leqq Mh^2, \quad \text{when} \quad h^2 \geqq \varepsilon.$$

Matrix estimate (7), (6), (18) and (19) give the assertion of Theorem 1., which concludes the outline of the Proof.

3. Numerical evidence

Finally we illustrate the efficiency of the proposed scheme by means of results for the test problems.

Example 1.

$$-\varepsilon u'' + u = -\cos^2 \pi x - 2\varepsilon\pi^2 \cos(2\pi x), \quad u(0) = u(1) = 0$$

with exact solution

$$u(x) = (\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon}))/(1 + \exp(-1/\sqrt{\varepsilon})) - \cos^2 \pi x.$$

Table 1.
Presents rate of the uniform convergence obtained by using the well-known double mesh principle ([1], [2]).

ε \ k	1	2	3	4	5	p_ε
2^{0}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-1}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-2}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-3}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-4}	1.98	1.99	2.00	2.00	2.00	2.00
2^{-5}	1.93	1.98	1.99	2.00	2.00	1.98
2^{-6}	2.19	2.19	2.34	3.02	3.17	2.58
2^{-7}	2.42	1.59	1.79	1.93	1.98	1.94
2^{-8}	2.31	2.63	2.67	1.72	1.91	2.25
2^{-9}	2.27	2.16	2.53	3.44	2.11	2.50

The numbers shown on Table 2. represent approximately the maximum absolute errors $\max_i |u_i - v_i|$ which is attained at nodes of spline.

Table 2.

ε \ N	32	64	128	256	512	1024
2^{-6}	0.877E−02	0.184E−02	0.332E−03	0.377E−04	0.561E−05	0.949E−06
2^{-9}	0.650E−02	0.132E−02	0.210E−03	0.195E−04	0.583E−05	0.153E−06

If we compare the results presented on Table 2. with the results given in [4], the usefulness of our modification is self-evident then. This parameter yields the better approximation then a conventional $q_i = \sqrt{(p_i/\varepsilon)}h$.

This is shown by numerical computations of the next stiff examples from [1], [2]. The calculations were performed on DELTA 340 (PDP 11/34) computer.

Accuracy of the spline approximation for the problem from [2] is displayed in Table 3. The first line in each row is the maximum of the differences between two consequtive meshes.

Example 2.

$$\varepsilon u'' - \{4 + 4\varepsilon(1+x)\}/(1+x)^4 u = \{-4/(1+x)^4\} \cdot \{(1+\varepsilon(1+x) + 4\pi^2\varepsilon^2)\cos(2\pi t)$$

$$-2\pi\varepsilon^2(1+x)\sin(2\pi t) + 3(1+\varepsilon(1+x))\exp(-1/\varepsilon)(1-\exp(-1/\varepsilon))^{-1}\},$$

$$u(0) = 2.0, \quad u(1) = -1.0.$$

Table 3.

ε \ k	1	2	3	4	5
0.125	$0.274E-01$ 1.93	$0.686E-02$ 1.99	$0.172E-02$ 2.00	$0.429E-03$ 2.00	$0.107E-03$ 2.00
0.015625	$0.548E-02$ 2.25	$0.138E-02$ 1.99	$0.355E-03$ 1.96	$0.893E-04$ 1.99	$0.224E-04$ 2.00
0.0441941	$0.103E-01$ 1.98	$0.259E-02$ 1.99	$0.646E-03$ 2.00	$0.162E-03$ 2.00	$0.404E-04$ 2.00
0.001953125	$0.140E-01$ 2.26	$0.318E-02$ 2.14	$0.707E-03$ 2.17	$0.168E-03$ 2.07	$0.414E-04$ 2.02
0.001	$0.322E-01$ 1.95	$0.688E-02$ 2.24	$0.148E-02$ 2.21	$0.333E-03$ 2.15	$0.799E-04$ 2.06

The following example is taken from [1] and also confirms the theoretical accuracy $0(h)^2$ stated in Theorem 1.

Example 3.

$$\varepsilon((1+x^2)u')' - \cos(3-x)^{-3}u = 4(3x^2-3x+1)((x-0.5)^2+2)$$

$$u(0) = -1, \quad u(1) = 0.$$

Table 4

ε \ k	1	2	3	4	5	p_ε
10^0	$0.620E-04$ 2.00	$0.155E-04$ 2.00	$0.388E-05$ 2.00	$0.969E-06$ 2.00	0.242E-06 2.00	2.00
10^{-1}	$0.558E-03$ 2.00	$0.140E-03$ 2.00	$0.349E-04$ 2.00	$0.872E-05$ 2.00	$0.218E-05$ 2.00	2.00
10^{-2}	$0.285E-02$ 2.00	$0.712E-03$ 2.00	$0.178E-03$ 2.00	$0.445E-04$ 2.00	$0.111E-04$ 2.00	2.00
10^{-3}	$0.133E-01$ 2.38	$0.213E-02$ 2.64	$0.398E-03$ 2.42	$0.917E-04$ 2.12	$0.225E-04$ 2.03	2.32
10^{-4}	$0.207E+00$ 1.98	$0.452E-01$ 2.20	$0.101E-01$ 2.16	$0.195E-02$ 2.38	$0.259E-02$ 2.91	2.33
10^{-6}	$0.416E+01$ 2.22	$0.273+01$ 0.611	$0.118E+01$ 1.21	$0.341E+00$ 1.79	$0.733E-01$ 2.22	1.61

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