Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

Now Series Vol. 3, 1989, Fasc. 3-4

Barrelledness in Locally Convex Riesz Spaces

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Presented by P. Kenderov

In this paper we define and study order-σ-quasibarrelled and order-sequentially quasibarrelled Riesz spaces. We obtain a dual characterization of these locally convex Riesz spaces in terms of the order structure. We also give some conditions for order-σ-quasibarrelled. Riesz space to be σ-barrelled or order-countably quasibarrelled. We show that an order-DF-Riesz space is a DF-Riesz space. Finally, we obtain that the property of being a DF-Riesz space is preserved under passage to any 1-ideal.

For basic concepts and definitions we refer to T. Husain and S. M. Khaleelulla [4], H. H. Schaefer [7], and Y. Wong and K. F. Ng [8] with only difference that convex circled sets will be called absolutely convex, and an absolutely convex subset of E which absorbs every bounded subset of E will be called bornivorous.

We begin with the following definitions:

Definition 1. Let (E, C, t) be a locally convex Riesz space (for definition see [8]). A solid d-barrel (resp. solid σ-barrel) in (E, C, t) is a barrel of the form $\cap_{n=1}^{\infty} V_n$, where (V_n) is a sequence of closed absolutely convex solid t-neighbourhoods (resp. solid σ_S (E, E')-neighbourhoods) of 0.

Definition 2. Locally convex Riesz space (E, C, t) is called order-σ-quasibarrelled if each order-bornivorous σ-barrel is a t-neighbourhood of 0.

Definition 3. Locally convex Riesz space (E, C, t) is called order-sequentially quasibarrelled if every sequence in E' which is $\sigma_S(E', E)$-convergent to 0, is t-equicontinuous.

Theorem 1. *Let (E, C, t) be a locally convex Riesz space with the topological dual E'. Then the following statements are equivalent:*

(a) *(E, C, t) is order-σ-quasibarrelled;*

(b) *Each solid σ-barrel in (E, C, t) is a t-neighbourhood of* 0;

(c) *Each $\sigma_S(E', E)$-bounded sequence in E' is t-equicontinuous;*

(d) *Each positive $\sigma(E', E)$-bounded sequence in E' is t-equicontinuous;*

Proof. We know that C' is a strict $\mathcal{B}$-cone in $(E', C', \sigma_S(E', E))$ and that each positive $\sigma(E', E)$-bounded subset of E' is σ_S(E', E)-bounded; then it follows that (c) and (d) are equivalent. The proof that (a) is equivalent to (c) follows easily by using the following result: A subset V of E is an order — bornivorous σ-barrel in $(E, C,$

t), if and only if the polar V^0 of V, taken in E', is $\sigma_S(E', E)$-bounded and $V = \cap_{n=1}^{\infty} V_n$, where $V_n = \{x'_n\}^0$. Obvious, a subset V of E is a solid σ-barrel in (E, C, t), if and only if $V = \cap_{n=1}^{\infty} V_n$, where $V_n = [-x'_n, x'_n]^0 = \{-x'_n, x'_n\}^0$, $x'_n \in C' \subset E'$. From this it follows that (b) is equivalent to (d). The proof of the Theorem is completed.

The following theorem is easily verified and its proof is omitted (see [9]).

Theorem 2. *For any locally convex Riesz space (E, C, t) the following statements are equivalent:*

(a) *(E, C, t) is order-sequentially quasibarrelled;*

(b) *Each positive sequence in E' which is $\sigma(E', E)$-convergent to 0, is t-equicontinuous;*

(c) *For each sequence (V_n) of closed absolutely convex $\sigma(E, E')$-neighbourhoods of 0, such that every order-bounded subset of E is contained in $\cap_{n=m}^{\infty} V_n$ for some m, the set $\cap_{n=1}^{\infty} V_n$ is a t-neighbourhood of 0.*

Corollary 1. *An order-σ-quasibarrelled Riesz space is order-sequentially quasibarrelled and a σ-barrelled (resp. sequentially barrelled) Riesz space is an order-σ-quasibarrelled (resp. order-sequentially quasibarrelled) Riesz space.*

Using the previous remarks and ([6], Chapter VIII) we conclude that the following implications hold in the class of locally convex Riesz spaces:

barrelledness	$\Rightarrow$	order-quasibarrelledness	$\Rightarrow$	quasibarrelledness
countably barrelledness	$\Rightarrow$	order-countably quasibarrelledness	$\Rightarrow$	countably quasibarrelledness
σ-barrelledness	$\Rightarrow$	order-σ-quasibarrelledness	$\Rightarrow$	σ-quasibarrelledness
sequentially barrelledness	$\Rightarrow$	order-sequentially quasibarrelledness	$\Rightarrow$	sequentially quasibarrelledness.

Remark. From the Corollary 3 and by ([6], Chapter VIII, example 3) it follows that there exists an order-σ-quasibarrelled (resp. order-sequentially quasibarrelled) Riesz space which is not σ-barrelled (resp. sequentially barrelled). Similarly, we know [8] that there exist a quasibarrelled Riesz space which is not order-quasibarrelled, i.e. by Corollary 4 it is countably quasibarrelled (resp. σ-quasibarrelled; sequentially quasibarrelled) but is not order-countably quasibarrelled (resp. order-σ-quasibarrelled; order-sequentially quasibarrelled).

Therefore it is natural to ask under what conditions on E (or E') the corresponding converse implications hold.

Theorem 3. *Let (E, C, t) be an order-sequentially quasibarrelled Riesz space. Then each $\sigma_S(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded.*

Proof. It is known [8] that if (E, C, t) is a locally convex Riesz space with the topological dual E' and if $\sigma_S(E', E)$ is the locally solid topology on E' associated with $\sigma(E', E)$, then $\sigma_S(E', E)$ is coarser than the strong topology $\beta(E', E)$ (each order-bounded subset of E is t-bounded). Let A be a $\sigma_S(E', E)$-bounded subset of E'. In view of Definition 3, it follows that locally convex Riesz space $(E', C', \sigma_S(E', E))$ is semicomplete, i.e., A is $\sigma_S(E', E)$-strongly bounded subset of E'. Since $\sigma(E', E)$ is coarser than $\sigma_S(E', E)$ it follows that A is $\sigma(E', E)$-strongly bounded (that is $\beta(E', E)$-bounded). The proof is complete.

Corollary 2. *Locally convex Riesz space (E, C, t) is order-σ-quasibarrelled (resp. order-countably quasibarrelled) if and only if it is order-sequentially quasibarrelled and σ-quasibarrelled (resp. order-σ-quasibarrelled and countably quasibarrelled).*

Corollary 3. *Locally convex Riesz space (E, C, t) is σ-barrelled (resp. countably barrelled) if and only if it is sequentially barrelled and order-σ-quasibarrelled (resp. σ-barrelled and order-countably quasibarrelled).*

It is clear that the concepts of countably barrelledness and σ-barrelledness (resp. countably quasibarrelledness and σ-quasibarrelledness) for the space $(E, \sigma(E, E'))$ coincide. For the space $(E, C, \sigma_S(E, E'))$ we have.

Proposition 1. *Locally convex Riesz space $(E, C, \sigma_S(E, E'))$ is order-countably quasibarrelled if and only if it is order-σ-quasibarrelled.*

Proof. It is an immediate consequence of the Definitions 1 and 2 and ([6], Chapter VIII, Theorem 1).

Let (E, C, t) be a locally convex Riesz space and suppose that u is in C. It is easy to see that $E_u = \cup_{n=1}^{\infty} n[-u, u]$ is an l-ideal in E generated by u. If p_u denotes the gauge of $[-u, u]$ on E and $C_u = C \cap E_u$, then (E_u, C_u, p_u) is a normed Riesz space for which the relative topology on E_u induced by t is coarser than the norm topology p_u on E_u.

Now, we have the following result:

Theorem 4. *For any locally convex Riesz space (E, C, t) with topological dual E', if t' is any locally solid topology on E', then (E', C', t') is barrelled (resp. countably barrelled; σ-barrelled) if and only if it is order-quasibarrelled (resp. order-countably quasibarrelled; order-σ-quasibarrelled).*

Proof. For any locally convex Riesz space (E, C, t) with the topological dual E', we have that $\sigma_S(E, E')$ is the coarsest locally solid topology on E compatible with the dual pair $\langle E, E' \rangle$. From this it follows that for all $u' \in C'$, $[-u', u']$ is $\sigma(E', E)$-compact, i.e., $E'_{u'}$ is complete for the norm $p_{u'}$, i.e, $E'_{u'}$ is a barrelled space. Let V be any barrel (resp. d-barrel; σ-barrel) in locally convex Riesz space (E', C', t') and let A be an order-bounded subset of E'. There exists u' in C' such that $A \subset [-u', u']$. It is clear that $V \cap E'_{u'}$ is a barrel in $(E'_{u'}, C'_{u'}, p_{u'})$ since the relative topology on $E'_{u'}$, induced by t' is coarser than the norm topology $p_{u'}$. From this it follows that $V \cap E'_{u'}$ is a neighbourhood of 0 in the norm topology $p_{u'}$, i.e, $V \cap E'_{u'}$ absorbs $[-u', u']$. Hence, V absorbs the subset A and then V is a t'-neighbourhood of 0 in E', i.e, (E', C', t') is barrelled (resp. countably barrelled; σ-barrelled) if (E', C', t') is an order-quasibarrelled (resp. order-countably quasibarrelled; order-σ-quasibarrelled) Riesz space.

Let (E, t) be a locally convex space, and let $\mathscr{B}$ be a family consisting of some t-bounded sets in E. $\mathscr{B}$ is called a fundamental system of t-bounded sets, if each

t-bounded set in E is contained in some member of $\mathscr{B}$. Locally convex Riesz space (E, C, t) is an order-DF-space (resp. DF-space) if it is order-countably quasibarrelled (resp. countably quasibarrelled) and if the family all order-bounded (resp. t-bounded) sets has a countable fundamental system (for order-DF spaces see [6]).

Now, we have the following interesting result.

Theorem 5. *Let (E, C, t) be an order-countably quasibarrelled Riesz space which contains a countable fundamental system of order-bounde sets, then (E, C, t) contains a countable fundamental system of t-bounded sets, i. e., each order-DF-Riesz space is a DF-Riesz space.*

Proof. Since (E, C, t) contains a countable fundamental system of order-bounded sets, it follows that the locally convex Riesz space $(E', C', \sigma_S(E', E))$ is metrisable, i. e., bornological. It remains to show that the topology $\beta(E', E)$ is coarser than $\sigma_S(E', E)$, i. e, that each t-bounded set is order-bounded. From ([7], exercise 17, p. 99) it follows that if (x'_n) is a null sequence in $(E', C', \sigma_S(E', E))$, it is t-equicontinuous by Theorem 1.(c), hence (x'_n) is $\beta(E', E)$-bounded. From this it follows that $\sigma_S(E', E)=\beta(E', E)$, i. e. the family of all bounded sets of (E, C, t) contains fundamental system. The proof is complete.

In terms of the order structure, we are to give some characterizations of countably quasibarrelled and σ-quasibarrelled Riesz spaces similar to those given in Theorem 1 for the order σ-quasibarrelled, resp. for bornological, order-quasibarrelled and quasibarrelled Riesz spaces from [8].

Theorem 6. *For any locally convex Riesz space (E, C, t) the following statements are equivalent:*

(a) *(E, C, t) is countably quasibarrelled;*

(b) *Each solid d-barrel which absorbs all t-bounded subsets of E is a t-neighbourhood of* 0;

(c) *Each solid d-barrel which absorbs all positive t-bounded subsets of E is a t-neighbourhood of* 0;

(d) *Each $\beta(E', E)$-bounded subset of E'-which is the countable union of t-equicontinuous subsets of E' is t-equicontinuous;*

(e) *Each positive $\beta(E', E)$-bounded subset of E' which is the countable union of t-equicontinuous subsets of E' is t-equicontinuous.*

Proof. The implication (a)$\Rightarrow$(b) is trivial. Therefore, the statements (b)–(e) are mutually equivalent by ([8], Proposition (11.2) (c), (d)) and ([3], Theorem 2). The implication (b)$\Rightarrow$(a) follows from the fact that $\mathrm{sk}(V)=\cap_{n=1}^{\infty}\mathrm{sk}(V_n)$ is a bornivorous solid d-barrel, if $V=\cap_{n=1}^{\infty} V_n$ is a bornivorous d-barrel. Since a subset of E absorbs each t-bounded set in (E, C, t), if and only if the solid kernel $\mathrm{sk}(V)$ of V absorbs all t-bounded sets in (E, C, t), it follows that (b) implies (a). The proof of the theorem is completed.

Theorem 7. *For any locally convex Riesz space (E, C, t) the following statements are equivalent:*

(a) *(E, C, t) is σ-quasibarrelled Riesz space;*

(b) *Each solid σ-barrel which absorbs all t-bounded subsets of E is a t-neighbourhood of* 0;

(c) *Each $\beta(E', E)$-bounded sequence in E' is t-equicontinuous*;

(d) *Each positive $\beta(E', E)$-bounded sequence in E' is t-equicontinuous.*

It is well-known that subspaces of a DF-locally convex space are, in general, not DF-spaces with respect to the relative topologies. From [8] it follows that each 1-ideal in a quasibarrelled (resp. bornological) Riesz space is quasibarrelled (resp. bornological) with respect to the relative topology. Therefore, it is natural to ask whether any 1-ideal in a DF-Riesz space with respect to the relative topology is a DF-Riesz space. We give an affirmative answer to this question using the following result which should be compared with the lemma from ([7], Chapter II, 6). Our proof is a modification of the proof from [8] for bornological and quasibarrelled spaces.

Theorem 8. *Let F be an* 1*-ideal in a locally convex Riesz space* (E, C, t) *and let V be an absolutely convex solid neighbourhood of* 0 *in F with respect to the relative topology; then there exists an absolutely convex solid t-neighbourhood U of O in E such that* $U \cap F = V$.

Proof. Let $U = \{x \in E : y \in V$ whenever $0 \leqq y \leqq |x|$ and $y \in F\}$. Then U is an absolutely convex solid set in E (see [8]) such that $U \cap F = V$. We have only to show that U is a t-neighbourhood of 0. Suppose, on the contrary, that we have $W \not\subset U$ for every absolutely convex solid t-neighbourhood W of 0. For each $W \in \mathcal{U}$, where $\mathcal{U}$ denotes the family of all absolutely convex solid t-neighbourhood of 0, there exists $x_w \in W$ but $x_w \notin U$. Hence, there is $y_w \in F$ with $0 \leqq y_w \leqq |x_w|$ such that $y_w \notin V$. Notice that $\{x_w : W \in \mathcal{U}, \supseteq\}$ is a net in (E, C, t), converging to 0. Since W is solid, it follows that $y_w \in W$ and then the net $\{y_w : W \in \mathcal{U}, \supseteq\}$, also converging to 0 in (E, C, t), does not converge to 0 in F with respect to the relative topology, contrary to the fact that $y_w \in W \cap F$, for each $W \in \mathcal{U}$.

Corollary 4. *Any l-ideal in a DF (resp. countably quasibarrelled) locally convex Riesz space is the space of the same type with respect to the relative topology.*

Proof. Let F be an 1-ideal in a locally convex Riesz space (E, C, t) and let $V = \cap_{n=1}^{\infty} V_n$ be a bornivorous solid d-barrel in the subspace F. Let, for each n in N

$U_n = \{x \in E : y \in V_n$ whenever $0 \leqq y \leqq |x|$ and $y \in F\}$ and

$U = \{x \in E : y \in V$ whenever $0 \leqq y \leqq |x|$ and $y \in F\}$.

It is easily verified that $U = \cap_{n=1}^{\infty} U_n$; hence U is a bornivorous solid d-barrel in E, by the preceding Theorem and ([8], Corollary (15.4)), such that $U \cap F = V$. It follows that V is an neighbourhood of 0 in F with respect to the relative topology. If (E, C, t) contains a countable fundamental system of bounded sets then obviously F contains a countable fundamental system of bounded sets with respect to the relative topology.

Corollary 5. *Any l-ideal in a σ-quasibarrelled Riesz space is a σ-quasibarrelled Riesz space with respect to the relative topology.*

Proof. $\sigma_S(E, E')$ induces $\sigma_S(F, F')$.

References

1. M. De Wilde, C. Houet. On increasing sequence of absolutely convex sets in locally convex spaces. *Math. Ann.* **192**, 1971, 257-261.
2. A. Grothendieck. Sur les espaces (F) et (DF). *Summa Brasil. Math.* **3**, 1954, 57-123.
3. T. Husain. Two new classes of locally convex spaces. *Math. Ann.*, **166**, 1966, 289-299.
4. T. Husain, S. M. Khaleelula. Countably order-quasibarrelled vector lattices and spaces. *Math. Jap.*, **20**, 1975, 3-15.
5. Order-quasibarrelled vector lattices and spaces. *Periodica Math. Hungarica*, vol. **6**, 1975, 363-371.
6. Barrelledness in topological and Ordered Vector Spaces. Lecture Notes in Math. No 692, 1978.
7. H. H. Schaefer. Topological vector spaces. New-York-Berlin, 1971.
8. K. F. Ng. Wong. Partially ordered topological vector spaces. Oxford, 1973.
9. J. H. Webb. Sequentially barrelled spaces. Math. Colloq. U.C.T. VIII, 1973, 73-87.

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Received 28. 12. 1988