Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol. 3, 1989, Fasc. 3-4

An Optimal Control Problem for the Heat Equation

Tsvetomir Tsachev

Presented by P. Kenderov

An optimal control problem for the heat equation with constrained boundary condition as control function is considered. The constraints on the control are of L_2-type and the cost functional is quadratic. A constructive method for finding an ε-solution is developed. The uniqueness of the solution and its dependence on the problem data are studied.

1. Introduction

Let $l>0$, $T>0$ be given and $Q_T=(0, l)\times(0, T)$. For any given $\varphi(t)\in L_2(0, T)$ we denote by $u(x, t; \varphi)$ the solution of the following boundary-value problem for the heat equation

$$\begin{aligned} &u_t(x, t; \varphi)-u_{xx}(x, t; \varphi)=0, && (x, t)\in Q_T \\ &u_x(0, t; \varphi)=0; \ u_x(l, t; \varphi)=\varphi(t) && t\in(0, T) \\ &u(x, 0; \varphi)=0, && x\in(0, l). \end{aligned}$$

According to [2] $u(x, t_0; \varphi)\in L_2(0, l)$ for each $t_0\in[0, T]$. Using remark 8.7 on p. 199 in [11] we have

$$(1)\quad u(x, t; \varphi)=\frac{1}{\sqrt{l}}.\left\{\int_0^t \varphi(\tau)d\tau\right\}.v_1(x)+\sum_{k=2}^{\infty}(-1)^{k-1}\frac{\sqrt{2}}{\sqrt{l}}.\left\{\int_0^t\varphi(\tau)e^{\lambda_k(t-\tau)}d\tau\right\}.v_k(x),$$

where λ_k are the eigenvalues and $v_k(x)$ are the eigenfunctions of the elliptic problem

$$v''(x)-\lambda.v(x)=0,$$

$$v'(0)=v'(l)=0,$$

i.e.

$$\lambda_1=0, \qquad \lambda_k=-(k-1)^2\pi^2/l^2 \qquad \text{for} \quad k=2, 3,\dots,$$

$$v_1(x)=1/\sqrt{l}, \quad v_k(x)=\frac{\sqrt{2}}{\sqrt{l}}\cos(k-1)\frac{\pi x}{l} \quad \text{for} \quad k=2, 3,\dots .$$

It is well-known that $\{v_k(\cdot)\}_{k=1}^{\infty}$ form an orthonormal basis of $L_2(0, l)$.

We shall use the following notations:

— N — the set of the natural numbers;

— $[A\varphi](x) \stackrel{\text{def}}{=} u(x, T; \varphi)$ for any given $\varphi(t) \in L_2(0, T)$. It follows from (1) that

(2)
$$[A\varphi](x) = \frac{1}{\sqrt{l}}\left\{\int_0^T \varphi(T-t)dt\right\} . v_1(x)$$
$$+ \sum_{k=2}^{\infty} (-1)^{k-1} . \frac{\sqrt{2}}{\sqrt{l}}\left\{\int_0^T \varphi(T-t)e^{\lambda_k t}dt\right\} . v_k(x)$$

— $B_1^T = \{\varphi(t) \in L_2(0, T);\ \|\varphi\|_{L_2(0,T)} \leqq 1\}$ — the closed unit ball of $L_2(0, T)$

— $A(B_1^T) = \{f(x) \in L_2(0, l);\ f(x) = [A\varphi](x),\ \varphi(t) \in B_1^T\}$ — the image of B_1^T

(3)
$$H = \mathrm{cl}\,\{\mathrm{span}\,\{e^{\lambda_k t}\}_{k=1}^{\infty}\} \subset L_2(0, T),$$

i.e. H is the smallest closed subspace of $L_2(0, T)$, containing the set $\{e^{\lambda_k t}\}_{k=1}^{\infty}$. Since $\Sigma_{k=2}^{\infty} 1/|\lambda_k| < \infty$, we have $L_2(0, T)\backslash H \neq \emptyset$ (cf. [3], [4], [8]). Since H is closed in $L_2(0, T)$, it is a Hilbert space endowed with the scalar product of $L_2(0, T)$.

— $\{\psi_k(t)\}_{k=1}^{\infty}$ — the set of functions obtained after orthonormalizing the set $\{e^{\lambda_k t}\}_{k=1}^{\infty}$ by the Gramm-Schmidt method. We have

(4)
$$e^{\lambda_k t} = \sum_{i=1}^{k} a_{ki} . \psi_i(t) \quad \text{for each } k \in \mathrm{N},$$

where $a_{kk} \neq 0$ for each $k \in \mathrm{N}$.

— $l_2 = \{\vec{x} = (x_1, x_2, \ldots, x_n, \ldots);\ x_i \in \mathrm{R}$ for each $i \in \mathrm{N}$ and $\Sigma_{i=1}^{\infty} x_i^2 < \infty\}$;

— $(\vec{x}, \vec{y})_{l_2} = \Sigma_{i=1}^{\infty} x_i y_i$ — the scalar product of l_2;

— $B_1 = \{\vec{x} \in l_2;\ \|\vec{x}\|_{l_2} \leqq 1\}$ — the closed unit ball of l_2;

— $\vec{x}^n = (x_1, x_2, \ldots, x_n) \in \mathrm{R}^n$ for each $\vec{x} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in l_2$;

— $(\vec{x}, \vec{y})_{\mathrm{R}^n} = \Sigma_{i=1}^{n} x_i y_i$ for each $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathrm{R}^n$ and $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathrm{R}$;

(5) — $c_k(\varphi) = \int_0^T \varphi(t) . e^{\lambda_k t}dt$ for each $\varphi(\cdot) \in L_2(0, T)$ and each $k \in \mathrm{N}$;

(6) — $\vec{c}(\varphi) = (c_1(\varphi), c_2(\varphi), \ldots, c_n(\varphi), \ldots)$ for each $\varphi(\cdot) \in L_2(0, T)$.

Since $\Sigma_{k=2}^{\infty} 1/|\lambda_k| < \infty$, we have $\vec{c}(\varphi) \in l_2$ for each $\varphi(\cdot) \in L_2(0, T)$;

(7) — $U_F = \{\vec{c}(\varphi);\ \varphi \in \mathrm{F}\}$ for each subset F of $L_2(0, T)$;

— $(f, g)_{L_2(0,l)} = \int_0^l f(x) . g(x)dx$ — the scalar product of $L_2(0, l)$;

— $\mathrm{Is}_1[f(\cdot)] = ((f, v_1)_{L_2(0,l)}, \ldots, (f, v_n)_{L_2(0,l)}, \ldots) \in l_2$ for each $f(\cdot) \in L_2(0, l)$.

Since $\{v_k(\cdot)\}_{k=1}^{\infty}$ form an orthonormal basis of $L_2(0, l)$, the mapping $\mathrm{Is}_1 : L_2(0, l) \to l_2$ is an isomorphism between $L_2(0, l)$ and l_2, i.e. the mapping $\mathrm{Is}_1^{-1}(\vec{x})$ is well defined for each $\vec{x} \in l_2$.

$-\mathrm{Is}_2[\varphi(\cdot)] = ((\varphi, \psi_1)_{L_2(0,T)}, \ldots, (\varphi, \psi_n)_{L_2(0,T)}, \ldots) \in l_2$ for each $\varphi(\cdot) \in H$. Since $\{\psi_k(\cdot)\}_{k=1}^{\infty}$ form an orthonormal basis of H, the mapping $\mathrm{Is}_2 : H \to l_2$ is an isomorphism between H and l_2, i.e. the mapping $\mathrm{Is}_2^{-1}(\vec{x})$ as well defined for each $\vec{x} \in l_2$.

$-\mathrm{Pr}_K x$ — the metric projection of $x \in \mathrm{X}$ on $K \subset \mathrm{X}$, where X is a Hilbert space, K is closed convex subset of X, and $\|\cdot\|_{\mathrm{X}}$ is the norm in X, i.e. $\|\mathrm{Pr}_K x - x\|_{\mathrm{X}} = \min_{z \in K} \|z - x\|_{\mathrm{X}}$.

2. Statement of the problem and finite-dimensional approximation

For any given $y(\cdot) \in L_2(0, l)$ let us denote $J(\varphi, y) = \int_0^l |u(x, T; \varphi) - y(x)|^2 dx$. Here we are interested in the problem

(P0) given $y(\cdot) \in L_2(0, l)$, minimize $J(\varphi, y)$ over the closed unit ball of $L_2(0, T)$

and in a constructive method for finding an ε-solution to it. So, denoting $\mathrm{I}(\mathrm{y}) = \inf_{\|\varphi\|_{L_2(0,T)} \leq 1} J(\varphi, y)$, we have the problem

(P1) given $y(\cdot) \in L_2(0, l)$ and $\varepsilon > 0$, find $\varphi_{\varepsilon,y}(\cdot)$ from the closed unit ball of $L_2(0, T)$, such that $J(\varphi_{\varepsilon,y}, y) < I(y) + \varepsilon$.

Lemma 2.1. *Let* $\vec{f}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i,i-1}, \alpha_{ii}, 0, 0, \ldots) \in l_2$, *where* $\alpha_{ii} \neq 0$ *for each* $i \in \mathrm{N}$,

$$M_n = \begin{pmatrix} (\vec{f}_1, \vec{f}_1)_{l_2} & (\vec{f}_1, \vec{f}_2)_{l_2} & \cdots & (\vec{f}_1, \vec{f}_n)_{l_2} \\ (\vec{f}_2, \vec{f}_1)_{l_2} & (\vec{f}_2, \vec{f}_2)_{l_2} & \cdots & (\vec{f}_2, \vec{f}_n)_{l_2} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ (\vec{f}_n, \vec{f}_1)_{l_2} & (\vec{f}_n, \vec{f}^2)_{l_2} & \cdots & (\vec{f}_n, \vec{f}_n)_{l_2} \end{pmatrix}$$

for each $n \in \mathrm{N}$, $b_i(\vec{x}) = (\vec{x}, \vec{f}_i)_{l_2}$ *for each* $\vec{x} \in l_2$ *and* $i \in \mathrm{N}$ *and let* $\vec{b}^n(\vec{x}) = (b_1(\vec{x}), b_2(\vec{x}), \ldots, b_n(\vec{x})) \in \mathrm{R}^n$ *for each* $n \in \mathrm{N}$. *Then* $\|\vec{x}\|_{l_2} \leq 1$ *if* $(M_n^{-1}\vec{b}^n(\vec{x}), \vec{b}^n(\vec{x}))_{\mathrm{R}^n}$ *for each* $n \in \mathrm{N}$.

Proof. Let $n\in\mathsf{N}$ be fixed and $\vec{f}_i^n=(\alpha_{i1}, \alpha_{i2},\dots, \alpha_{ii}, 0,\dots, 0)\in\mathsf{R}^n$ for $i=1, 2,\dots, n$. Since $\alpha_{ii}\neq 0$ for each $i\in\mathsf{N}$, $\{\vec{f}_i^n\}_{i=1}^n$ form a basis in R^n. Hence, because of $(\vec{f}_i, \vec{f}_j)_{l_2}=(\vec{f}_i^n, \vec{f}_j^n)_{\mathsf{R}^n}$ for $1\leqq i, j\leqq n$, the matrix M_n^{-1} is well defined.

Let $\vec{x}=(x_1, x_2,\dots, x_n,\dots)\in l_2$. Then $\vec{x}^n=\Sigma_{i=1}^n\xi_i\vec{f}_i^n$ and for $j=1, 2,\dots, n$ we have

$$b_j(\vec{x})=(\vec{x}, \vec{f}_j)_{l_2}=(\vec{x}^n, \vec{f}_j^n)_{\mathsf{R}^n}=\left(\sum_{i=1}^n \xi_i\vec{f}_i^n, \vec{f}_j^n\right)_{\mathsf{R}^n}=\sum_{i=1}^n \xi_i(\vec{f}_i, \vec{f}_j)_{l_2}.$$

Hence, $\vec{b}^n(\vec{x})=M_n\vec{\xi}$, where $\vec{\xi}=(\xi_1, \xi_2,\dots, \xi_n)\in\mathsf{R}^n$, i.e. $\vec{\xi}=M_n^{-1}\vec{b}^n(\vec{x})$. But

$$\sum_{i=1}^n x_i^2=\|\vec{x}^n\|^2_{\mathsf{R}^n}=\left(\sum_{i=1}^n \xi_i\vec{f}_i^n, \sum_{j=1}^n \xi_j\vec{f}_j^n\right)_{\mathsf{R}^n}=(M_n\vec{\xi}, \vec{\xi})_{\mathsf{R}^n}=(M_n^{-1}\vec{b}^n(\vec{x}), \vec{b}^n(\vec{x}))_{\mathsf{R}^n},$$

i.e. for each $n\in\mathsf{N}$ we have $\Sigma_{i=1}^n x_i^2\leqq 1$ iff $(M_n^{-1}\vec{b}^n(\vec{x}), \vec{b}^n(\vec{x}))_{\mathsf{R}^n}\leqq 1$, which proves the lemma.

Corollary. *Let*

$$\hat{M}_n=\begin{pmatrix} \int_0^T e^{\lambda_1 t}.e^{\lambda_1 t}dt & \int_0^T e^{\lambda_1 t}.e^{\lambda_2 t}dt & \dots & \int_0^T e^{\lambda_1 t}.e^{\lambda_n t}dt \\ \int_0^T e^{\lambda_2 t}.e^{\lambda_1 t}dt & \int_0^T e^{\lambda_2 t}.e^{\lambda_2 t}dt & \dots & \int_0^T e^{\lambda_2 t}.e^{\lambda_n t}dt \\ \dots & \dots & \dots & \dots \\ \int_0^T e^{\lambda_n t}.e^{\lambda_1 t}dt & \int_0^T e^{\lambda_n t}.e^{\lambda_2 t}dt & \dots & \int_0^T e^{\lambda_n t}.e^{\lambda_n t}dt \end{pmatrix}$$

for each $n\in\mathsf{N}$ *and* $V_n=\{(x_1, x_2,\dots, x_n,\dots)\in l_2;\ (\hat{M}_n^{-1}\vec{x}^n, \vec{x}^n)_{\mathsf{R}^n}\leqq 1\}\subset l_2$ *for each* $n\in\mathsf{N}$. *Then*

$$U_{B_1^T}=U_{\{\varphi(\cdot)\in H;\,\|\varphi\|_H\leqq 1\}}=\bigcap_{n=1}^{\infty} V_n, \tag{8}$$

where $U_{B_1^T}$ *is defined by* (7).

Proof. It follows immediately from (5), (6), and (7) that $U_{B_1^T}=U_{\{\varphi(\cdot)\in H;\,\|\varphi\|_H\leqq 1\}}$. Using (4) and the mapping $\mathrm{Is}_2: H\to l_2$ we finish the proof by directly applying lemma 2.1.

Lemma 2.2. *Let* $E_1 = 1/\sqrt{l}$ *and*

$$E_n = \begin{matrix} 1/\sqrt{l} & 0 & \dots & 0 \\ 0 & -\sqrt{2}/\sqrt{l} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (-1)^{n-1}\sqrt{2}/\sqrt{l} \end{matrix} \quad \text{for } n \geqq 2,$$

$$W_n = \{(x_1, x_2, \dots, x_n, \dots) \in l_2;\ ([E_n \hat{M}_n E_n]^{-1} \vec{x}^n, \vec{x}^n)_{\mathsf{R}^n} \leqq 1\} \tag{9}$$

for each $n \in \mathsf{N}$ *and*

$$W_\infty = \{\vec{x} \in l_2;\ \vec{x} = \mathrm{Is}_1[f(\cdot)] \text{ for some } f(\cdot) \in A(B_1^T)\}.$$

Then $W_\infty = \cap_{n=1}^\infty W_n$.

Proof. Let $\vec{\theta} = (\theta_1, \theta_2, \dots, \theta_n, \dots) \in W_\infty$. Then, according to (2) and (5), there exists $\varphi(t) \in B_1^T$, such that $\theta_1 = c_1(\hat{\varphi})/\sqrt{l}$, $\theta_n = (-1)^{n-1}\sqrt{2}\, c_n(\hat{\varphi})/\sqrt{l}$ for $n \geqq 2$, where $\hat{\varphi}(t) = \varphi(T-t) \in B_1^T$. Hence, for each $n \in \mathsf{N}$ we have $\vec{\theta}^n = E_n \vec{c}^n(\hat{\varphi})$, i.e. $\vec{c}^n(\hat{\varphi}) = E_n^{-1} \vec{\theta}^n$. It follows from (8) that $(\hat{M}_n^{-1} E_n^{-1} \vec{\theta}^n, E_n^{-1} \vec{\theta}^n)_{\mathsf{R}^n} \leqq 1$ holds true for each $n \in \mathsf{N}$, i.e. $([E_n \hat{M}_n E_n]^{-1} \vec{\theta}^n, \vec{\theta}^n)_{\mathsf{R}^n} \leqq 1$ for each $n \in \mathsf{N}$. Thus $\vec{\theta} \in W_n$ for each $n \in \mathsf{N}$, i.e. $W_\infty \subset \cap_{n=1}^\infty W_n$.

Now let $\vec{\theta} = (\theta_1, \theta_2, \dots, \theta_n, \dots) \in \cap_{n=1}^\infty W_n$. Let $c_1 = \sqrt{l}.\theta_1$ and $c_n = (-1)^{n-1} \times \theta_n \sqrt{l}/\sqrt{2}$ for $n \geqq 2$, i.e. $\vec{c}^n = E_n^{-1} \vec{\theta}^n$ for each $n \in \mathsf{N}$. Then

$$(\hat{M}_n^{-1} \vec{c}^n, \vec{c}^n)_{\mathsf{R}^n} = (\hat{M}_n^{-1} E_n^{-1} \vec{\theta}^n, E_n^{-1} \vec{\theta}^n)_{\mathsf{R}^n} = ([E_n \hat{M}_n E_n]^{-1} \vec{\theta}^n, \vec{\theta}^n)_{\mathsf{R}^n} \leqq 1$$

for each $n \in \mathsf{N}$. Because of (8) there exists $\hat{\varphi}(t) \in B_1^T$ such that $\vec{c} = (c_1, c_2, \dots, c_n, \dots) = \vec{c}(\hat{\varphi})$ ($\vec{c}(\varphi)$ is defined by (6)). Let $\varphi(t) = \hat{\varphi}(T-t) \in B_1^T$ and $\theta(x) = \Sigma_{i=1}^\infty \theta_i . v_i(x)$. Then (2) yields $[A\varphi](x) = \theta(x)$, i.e. $\vec{\theta} \in W_\infty$. Thus $W_\infty \supset \cap_{n=1}^\infty W_n$.

Lemma 2.3. *Let* W_n *be defined by* (9). *Then* $W_n \supset W_{n+1}$ *for each* $n \in \mathsf{N}$.

Proof. Let $\vec{c} = (c_1, c_2, \dots, c_n, \dots) \in l_2$ be fixed and let $\vec{x} = (x_1, x_2, \dots, x_n, \dots) \in l_2$ be defined by

$$\begin{aligned} &a_{11}x_1 &&= c_1 \\ &a_{21}x_1 + a_{22}x_2 &&= c_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &&= c_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

where $\{a_{ij}\}_{i=1}^{\infty}{}_{j=1}^{i}$ are the coefficients in (3). Then as in the proof of lemma 2.1 we obtain $\Sigma_{i=1}^{n} x_i^2 = (\hat{M}_n^{-1}\vec{c}^{\,n}, \vec{c}^{\,n})_{\mathrm{R}^n}$ for each $n \in \mathrm{N}$. Hence $(\hat{M}_{n+1}^{-1}\vec{c}^{\,n+1}, \vec{c}^{\,n+1})_{\mathrm{R}^{n+1}} = \Sigma_{i=1}^{n+1} x_i^2 \geqq \Sigma_{i=1}^{n} x_i^2 = (\hat{M}_n^{-1}\vec{c}^{\,n}, \vec{c}^{\,n})_{\mathrm{R}^n}$ for each $\vec{c} \in l_2$. Thus $W_n \supset W_{n+1}$ because of (9).

Lemma 2.4. *The set W_n defined by* (9) *is a closed convex subset of l_2.*

Proof. This fact is clear because the matrix $\hat{M}_n^{-1}$ is positive for each $n \in \mathrm{N}$.

Let $\hat{W}_n = \{f(\cdot) \in L_2(0, l);\ f(\cdot) = \mathrm{Is}_1^{-1}(\vec{x})$ for some $\vec{x} \in W_n\} \subset L_2(0, l)$ for each $n \in \mathrm{N}$. Lemmata 2.2, 2.3 and 2.4 yield

(10)
i) $A(B_1^T) = \cap_{n=1}^{\infty} \hat{W}_n$;
ii) $\hat{W}_n \supset \hat{W}_{n+1}$ for each $n \in \mathrm{N}$;
iii) $\hat{W}_n$ is closed convex subset of $L_2(0, l)$ for each $n \in \mathrm{N}$.

It follows immediately from (10) that the sequence of sets W_n converges in the sense of Mosco to the set $A(B_1^T)$ (cf. Definition 3 on p. 403 in [6]). In particular i) and iii) yield that $A(B_1^T)$ is closed convex subset of $L_2(0, l)$. Hence, for each $y(\cdot) \in L_2(0, l)$. $\|\mathrm{Pr}_{\hat{W}_n} y(\cdot) - \mathrm{Pr}_{A(B_1^T)} y(\cdot)\|_{L_2(0,l)} \xrightarrow[]{n\to\infty} 0$ (theorem 4 on p. 403 in [6]).

Let us denote $y^n(x) = \Sigma_{k=1}^{n} (y, v_k)_{L_2(0,l)} \cdot v_k(x)$ for each $y(\cdot) \in L_2(0, l)$. Since $\|y^n(\cdot) - y(\cdot)\|_{L_2(0,l)} \xrightarrow[n\to\infty]{} 0$ and $\|\mathrm{Pr}_K x - \mathrm{Pr}_K y\|_{\mathrm{X}} \leqq \|x - y\|_{\mathrm{X}}$ for each $x, y \in \mathrm{X}$, where X is a Hilbert space and K is a closed convex subset of X (cf., e. g. [10]), we see that

$$\|\mathrm{Pr}_{\hat{W}_n} y^n(\cdot) - \mathrm{Pr}_{A(B_1^T)} y(\cdot)\|_{L_2(0,l)} \leqq \|\mathrm{Pr}_{\hat{W}_n} y^n(\cdot) - \mathrm{Pr}_{\hat{W}_n} y(\cdot)\|_{L_2(0,l)}$$

$$+ \|\mathrm{Pr}_{\hat{W}_n} y(\cdot) - \mathrm{Pr}_{A(B_1^T)} y(\cdot)\|_{L_2(0,l)} \leqq \|y^n(\cdot) - y(\cdot)\|_{L_2(0,l)}$$

$$+ \|\mathrm{Pr}_{\hat{W}_n} y(\cdot) - \mathrm{Pr}_{A(B_1^T)} y(\cdot)\|_{L_2(0,l)} \xrightarrow[n\to\infty]{} 0.$$

So we have

$$\|\mathrm{Pr}_{\hat{W}_n} y^n(\cdot) - y(\cdot)\|^2_{L_2(0,l)} \xrightarrow[n\to\infty]{} \|\mathrm{Pr}_{A(B_1^T)} y(\cdot) - y(\cdot)\|^2_{L_2(0,l)} = I(y). \tag{11}$$

Since $(y^n, v_k)_{L_2(0,l)} = (y, v_k)_{L_2(0,l)}$ for $k = 1, 2, \ldots, n$ and $(y^n, v_k)_{L_2(0,l)} = 0$ for $k \geqq n+1$, the problem of computing $\mathrm{Pr}_{\hat{W}_n} y^n(\cdot)$ is finite-dimensional, namely

find the projection of the point $((y, v_1)_{L_2(0,l)}, \ldots, (y, v_n)_{L_2(0,l)}) \in \mathrm{R}^n$ on the set $W_n^* = \{\vec{x}^n = (x_1, x_2, \ldots, x_n) \in \mathrm{R}^n;\ ([E_n \hat{M}_n E_n]^{-1} \vec{x}^n, \vec{x}^n)_{\mathrm{R}^n} \leqq 1\}$.

Let $(v_1, v_2, \ldots, v_n) \in W_n^*$ be the solution of this problem, i.e.

(12) $$[\mathrm{Pr}_{\hat{W}_n} y^n(\cdot)](x) = \sum_{k=1}^{n} v_k \,.\, v_k(x).$$

Using the method developed in [1], §3 we can find a function $\hat{\varphi}(t) \in H$, such that $\int_0^T \hat{\varphi}(t)dt = v_1\sqrt{l}$, $\int_0^T \hat{\varphi}(t)e^{\lambda_k t}dt = (-1)^{k-1} v_k \sqrt{l}/\sqrt{2}$ for $k = 2, 3, \ldots, n$, $\int_0^T \hat{\varphi}(t)e^{\lambda_k t}dt = 0$ for $k \geqq n+1$. Let $\varphi_{y,n}(t) = \hat{\varphi}(T-t)$. Then (2) and (12) yield

(13) $$[A\varphi_{y,n}](x) = \sum_{k=1}^{n} v_k, \ v_k(x) = [\mathrm{Pr}_{\hat{W}_n} y^n(\cdot)](x).$$

It is easily seen that $\varphi_{y,n} \in B_1^T$. Because of (13) we can rewrite (11) as

$$J(\varphi_{y,n}, y) \underset{n\to\infty}{\longrightarrow} I(y).$$

Thus, when solving (P1), we may take $\varphi_{y,n}$ for sufficiently large n as $\varphi_{\varepsilon,y}$, i.e. as an ε-solution to the problem (P0).

3. Uniqueness of the solution to the problem (P0)

Lemma 3.1. *Let H be defined by* (3) *and*

(14) $$H_1 = \{\varphi(t) \in L_2(0, T)\,;\ \hat{\varphi}(t) = \varphi(T-t) \in H\}.$$

Then (P0) has unique solution belonging to H_1.

Proof. The function $\varphi(t) \in B_1^T$ is solution of (P0) iff $[A\varphi](x) = [\mathrm{Pr}_{A(B_1^T)} y(\cdot)](x)$. Let $\eta(t) \in B_1^T$ be a solution of (P0) and $\hat{\eta}(t) = \eta(T-t)$. Let $\hat{\mu}(t) \in H$ be such that $\| \hat{\mu}(\cdot) - \hat{\eta}(\cdot) \|_{L_2(0,T)} = \min_{\varphi(\cdot) \in H} \| \varphi(\cdot) - \hat{\eta}(\cdot) \|_{L_2(0,T)}$. Then $\mu(t) = \hat{\mu}(T-t) \in H_1$ and $(\hat{\eta} - \hat{\mu}, \ \varphi)_{L_2(0,T)} = 0$ for each $\varphi(\cdot) \in H$, i.e. $(\eta - \mu, \ \varphi)_{L_2(0,T)} = 0$ for each $\varphi(\cdot) \in H_1$. Because of (2) $[A\mu](x) = [A\eta](x) = [\mathrm{Pr}_{A(B_1^T)} y(\cdot)](x)$, i.e. $\mu(t)$ is the unique solution of (P0) belonging to H_1.

Lemma 3.2. *The set $\Xi_1 = \{f(x) \in L_2(0, l)\,; f(x) = [A\varphi](x)$ for some $\varphi(t) \in H_1\}$ is dense in $L_2(0, l)$.*

Proof. It is clear that the set

$$\Xi_2 = \{f(x) \in L_2(0, l); \ \ f(x) = \sum_{k=1}^{n} z_k \,.\, v_k(x), \ \ n \in \mathbf{N}, \ \ (z_1, z_2, \ldots, z_n) \in \mathbf{R}^n\}$$

is dense in $L_2(0, l)$. According to corollary 3.2, §3 in [1] (p. 279) we have $\Xi_2 \subset \Xi_1$, which proves the lemma.

Lemma 3.3. *Let* $\overset{\circ}{B}{}_1^T = \{\varphi(\cdot) \in L_2(0, T);\ \|\varphi\|_{L_2(0,T)} < 1\}$ *and* $A(\overset{\circ}{B}{}_1^T) = \{f(x) \in L_2(0, l);\ f(x) = [A\varphi](x)$ *for some* $\varphi(t) \in \overset{\circ}{B}{}_1^T\}$. *Let* $y(\cdot) \notin A(\overset{\circ}{B}{}_1^T)$ *and* $\varphi_y(t) \in B_1^T \cap H_1$ *be a solution of* (P0). *Then* $\|\varphi_y\|_{L_2(0,T)} = 1$.

Proof. This proposition follows directly from the Kuhn-Tucker theorem ([7]; pp. 261-262).

Theorem 3.1. *Let* $\overset{\circ}{B}{}_1^T = \{\varphi(\cdot) \in L_2(0, T);\ \|\varphi\|_{L_2(0,T)} < 1\}$, $A(\overset{\circ}{B}{}_1^T) = \{f(x) \in L_2(0, l);\ f(x) = [A\varphi](x)$ *for some* $\varphi(t) \in \overset{\circ}{B}{}_1^T\}$, *$H$ be defined by* (3), *H_1 be defined by* (14), *and* $H^{\perp} = \{\varphi(\cdot) \in L_2(0, T);\ (\varphi, \eta)_{L_2(0,T)} = 0$ *for each* $\eta(\cdot) \in H\}$.

a) *If* $y(\cdot) \notin A(\overset{\circ}{B}{}_1^T)$ *the solution of* (P0) *is unique.*

b) *Let* $y(\cdot) \in A(\overset{\circ}{B}{}_1^T)$ *and* $\varphi_y(t)$ *be the unique solution of* (P0) *from* H_1. *Then* $\zeta \overset{\text{def}}{=} 1 - \|\varphi_y\|^2_{L_2(0,T)}$ *is positive and*

$$(15)\quad S_{y(\cdot)} \overset{\text{def}}{=} \{\varphi_y(t) + h(t);\ \hat{h}(t) = h(T-t) \in H^{\perp},\quad \|h\|^2_{L_2(0,T)} \leqq \zeta\}$$

is the set of all solutions of (P0).

Proof. a) Let $\varphi_y(t) \in B_1^T \cap H_1$ be a solution of (P0) (lemma 3.1) and $\eta(t) \in B_1^T$ be an arbitrary solution of (P0). Because of (2) and $[A\varphi_y](x) = [A\eta](x) = [\mathrm{Pr}_{A(B_1^T)} y(\cdot)](x)$ we have $\hat{\eta}(t) - \hat{\varphi}_y(t) = \eta(T-t) - \varphi_y(T-t) \in H^{\perp}$. Hence

$$\|\eta\|^2_{L_2(0,T)} = \|\hat{\eta}\|^2_{L_2(0,T)} = \|\hat{\varphi}_y\|^2_{L_2(0,T)} + \|\hat{\eta} - \hat{\varphi}_y\|^2_{L_2(0,T)}$$

$$= \|\varphi_y\|^2_{L_2(0,T)} + \|\hat{\eta} - \hat{\varphi}_y\|^2_{L_2(0,T)}.$$

Since $\|\varphi_y\|^2_{L_2(0,T)} = 1$ (lemma 3.3), we have $\hat{\eta}(\cdot) = \hat{\varphi}_y(\cdot)$, i.e. $\eta(\cdot) = \varphi_y(\cdot)$.

b) Let $y(\cdot) \in A(\overset{\circ}{B}{}_1^T)$ and $\varphi_y \in B_1^T \cap H_1$ be such that $[A\varphi_y](x) = [\mathrm{Pr}_{A(B_1^T)} y(\cdot)](x) = y(x)$. Then $\zeta = 1 - \|\varphi_y\|^2_{L_2(0,T)} > 0$. Let $h(t)$ be such that $\hat{h}(t) = h(T-t) \in H^{\perp}$ and $\|h\|^2_{L_2(0,T)} = \|\hat{h}\|^2_{L_2(0,T)} < \zeta$. Because of (2) we have $[A(\varphi_y + h)](x) = [A\varphi_y](x) = y(x)$. Since $\varphi_y \in H$,

$$\|\varphi_y + h\|^2_{L_2(0,T)} = \|\hat{\varphi}_y + \hat{h}\|^2_{L_2(0,T)} = \|\hat{\varphi}\|^2_{L_2(0,T)} + \|\hat{h}\|^2_{L_2(0,T)} \leqq 1.$$

Hence $S_{y(\cdot)}$ (defined by (15)) consists of solutions of (P0).

Now let $\varphi_y(t) + h(t)$ be a solution of (P0). Since $[A(\varphi_y + h)](x) = y(x) = [A\varphi_y](x)$, (2) yields $\hat{h}(t) = h(T-t) \in H^{\perp}$. Since $\hat{\varphi}_y(t) = \varphi_y(T-t) \in H$, we have

$$\|\varphi_y+h\|^2_{L_2(0,T)}=\|\hat{\varphi}_y+\hat{h}\|^2_{L_2(0,T)}=\|\hat{\varphi}_y\|^2_{L_2(0,T)}+\|\hat{h}\|^2_{L_2(0,T)}=\|\varphi_y\|^2_{L_2(0,T)}+\|h\|^2_{L_2(0,T)}.$$

Because of $\|\varphi_y+h\|^2_{L_2(0,T)}\leqq 1$, $\|h\|^2_{L_2(0,T)}\leqq 1-\|\varphi_y\|^2_{L_2(0,T)}=\zeta$ holds true. Thus each solution of (P0) belongs to $S_{y(\cdot)}$.

4. The dependence of the solution to the problem (P0) on $y(\cdot)\in L_2(0, l)$

The principal result in this section is theorem 4.1. Its proof will be prepared by five lemmata.

Lemma 4.1. *Let* $\{a_{ij}\}_{i=1}^{\infty}\,{}_{j=1}^{i}$ *be the coefficients in* (4), *the operator* $\tilde{A}: l_2\to l_2$ *be defined by*

$$(16)\quad \tilde{A}[(x_1, x_2,\dots, x_n,\dots)]=(a_{11}x_1,\ a_{21}x_1+a_{22}x_2,\dots,\ \sum_{i=1}^{n} a_{ni}x_i,\dots)$$

for each $\vec{x}=(x_1, x_2,\dots, x_n,\dots)\in l_2$ *and* $\mathcal{R}(\tilde{A})=\{\vec{y}\in l_2\,;\,\vec{y}=\tilde{A}\vec{x}$ *for some* $\vec{x}\in l_2\}$. *Then the operator* $\tilde{A}: l_2\to l_2$ *is compact and injective and* $\mathcal{R}(\tilde{A})$ *is dense in* l_2.

Proof. Using the mapping $\mathrm{Is}_2: H\to l_2$ we have

$$\sum_{i=1}^{\infty}\left[\sum_{j=1}^{i} a_{ij}^2\right]=\sum_{i=1}^{\infty}\int_0^T e^{2\lambda_i t}dt<T+\frac{1}{2}\sum_{i=2}^{\infty} 1/|\lambda_i|<\infty.$$

Defining $\tilde{A}_n: l_2\to l_2$ for each $n\in\mathbf{N}$ by

$$\tilde{A}_n[(x_1, x_2,\dots, x_n,\dots)]=(a_{11}x_1,\ a_{21}x_1+a_{22}x_2,\dots,\ \sum_{i=1}^{n} a_{ni}x_i,\ 0,\ 0,\dots,\ 0,\dots),$$

we have

$$\sup_{\|\vec{x}\|_{l_2}\leqq 1}\|\tilde{A}\vec{x}-\tilde{A}_n\vec{x}\|^2_{l_2}=\sup_{\|\vec{x}\|_{l_2}\leqq 1}\sum_{i=n+1}^{\infty}\left[\sum_{j=1}^{i} a_{ij}x_j\right]^2\leqq\sum_{i=n+1}^{\infty}\left[\sum_{j=1}^{i} a_{ij}^2\right]\underset{n\to\infty}{\longrightarrow}0.$$

Then $\tilde{A}$ is compact according to the theorem on p. 290 (Ch. 6, §3) in [9].

Since $a_{ii}\neq 0$ for each $i\in\mathbf{N}$, $\tilde{A}$ is injective.

Lemma 3. 2 and the mapping $\mathrm{Is}_2: H\to l_2$ yield the fact that $\mathcal{R}(\tilde{A})$ is dense in l_2.

Lemma 4. 2. *Let* $\tilde{A}: l_2\to l_2$ *be defined by* (16), $\tilde{A}^*: l_2\to l_2$ *be the adjoint operator of* $\tilde{A}$ *and* $D: l_2\to l_2$ *be defined by* $D=\tilde{A}\,.\,\tilde{A}^*$. *Then*

a) there is unique positive operator $\sqrt{D}: l_2\to l_2$ *such that* $\sqrt{D}\,.\sqrt{D}=D$.

b) there is an orthonormal set $\{\vec{g}_k\}_{k=1}^{\infty}\subset l_2$ *of eigenvectors of* $\sqrt{D}$ *which form a basis of* l_2.

c) the operator $\sqrt{D}$ *is injective.*

P r o o f. a) Since $D: l_2 \to l_2$ is positive and self-adjoint, according to theorem 2 on p. 246 (Ch. VII, § 3) in [12] there is unique positive operator $\sqrt{D}: l_2 \to l_2$ such that $\sqrt{D}.\sqrt{D}=D$.

b) Since $\tilde{A}: l_2 \to l_2$ is compact, $\tilde{A}^*: l_2 \to l_2$ is compact (theorem 4.19 on p. 119 in [13]). Then $D=\tilde{A}.\tilde{A}^*$ is compact (theorem 4.18 f) on p. 118 in [13]). Since D is compact and self-adjoint, there is an orthonormal basis $\{\vec{g}_k\}_{k=1}^{\infty} \subset l_2$ of eigenvectors of D ([12], [14]) with corresponding eigenvalues $\xi_k \geqq 0$ (D is positive).

For each $k \in \mathbf{N}$ we have

$$(\sqrt{D}+\sqrt{\xi_k}.\mathrm{Id})(\sqrt{D}-\sqrt{\xi_k}.\mathrm{Id})\vec{g}_k=(D-\xi_k.\mathrm{Id})\vec{g}_k=0,$$

where $\mathrm{Id}\vec{x}=\vec{x}$ for each $\vec{x} \in l_2$. Since $\sqrt{D}$ is positive, $(\sqrt{D}-\sqrt{\xi_k}.\mathrm{Id})\vec{g}_k=0$ holds true for each $k \in \mathbf{N}$. Hence $\vec{g}_k$ is eigenvector of $\sqrt{D}$ with corresponding eigenvalue $\sqrt{\xi_k}$ for each $k \in \mathbf{N}$.

c) Since $\mathscr{R}(\tilde{A})$ is dense in l_2, $\tilde{A}^*$ is injective (theorem 12.10 on p. 335 in [13]). Since $\tilde{A}$ is injective, $D=\tilde{A}.\tilde{A}^*$ is injective too. Hence $\xi_k>0$ for each $k \in \mathbf{N}$. Since

$$\sqrt{D}\vec{x}=\sqrt{D}\left(\sum_{k=1}^{\infty}(\vec{x},\vec{g}_k)_{l_2}.\vec{g}_k\right)=\sum_{k=1}^{\infty}\sqrt{\xi_k}.(\vec{x},\vec{g}_k)_{l_2}.\vec{g}_k \tag{17}$$

for each $\vec{x} \in l_2$, $\sqrt{D}.\vec{x}=0$ implies $\vec{x}=\vec{0}$, because $\sqrt{\xi_k}>0$ for each $k \in \mathbf{N}$. Hence $\sqrt{D}$ is injective.

Lemma 4.3. *Let $\tilde{A}: l_2 \to l_2$ be defined by* (16) *and $\sqrt{D}: l_2 \to l_2$ be defined as in lemma* 4.2. *Let $\sqrt{D}(B_1)=\{\vec{y} \in l_2;\ \vec{y}=\sqrt{D}\vec{x},\ \|\vec{x}\|_{l_2} \leqq 1\}$ and $\tilde{A}(B_1)=\{\vec{y} \in l_2;\ \vec{y}=\tilde{A}\vec{x},\ \|\vec{x}\|_{l_2} \leqq 1\}$. Then $\sqrt{D}(B_1)=\tilde{A}(B_1)$.*

P r o o f. Let $\mathscr{R}(D)=\{\vec{y} \in l_2;\ \vec{y}=D\vec{x},\ \vec{x} \in l_2\}$. It is easily seen that

$$\sqrt{D}(B_1) \cap \mathscr{R}(D)=\tilde{A}(B_1) \cap \mathscr{R}(D). \tag{18}$$

Let $\vec{c}=\tilde{A}\vec{x}$, $\|\vec{x}\|_{l_2}<1$. Since $\mathscr{R}(\tilde{A}^*)=\{\vec{y} \in l_2;\ \vec{y}=\tilde{A}^*\vec{x},\ \vec{x} \in l_2\}$ is dense in l_2($\tilde{A}$ is injective), we can find $\{\tilde{A}^*\vec{y}_n\}_{n=1}^{\infty}$, $\|\tilde{A}^*\vec{y}_n\|_{l_2}<1$ for $n \in \mathbf{N}$, such that $\|\tilde{A}^*\vec{y}_n-\vec{x}\|_{l_2} \underset{n\to\infty}{\longrightarrow} 0$. Then if $\vec{c}_n=\tilde{A}\tilde{A}^*\vec{y}_n=D\vec{y}_n$ we have $\vec{c}_n \in \sqrt{D}(B_1)$ for $n \in \mathbf{N}$ (see (18)) and $\|\vec{c}_n-\vec{c}\|_{l_2} \underset{n\to\infty}{\longrightarrow} 0$. Since $\sqrt{D}(B_1)$ is closed $\vec{c} \in \sqrt{D}(B_1)$ holds true. Now let $\vec{c}=\tilde{A}\vec{x}$, $\|\vec{x}\|_{l_2}=1$ and $\vec{c}_n=\tilde{A}\left(\frac{n}{n+1}\vec{x}\right)$ for $n \in \mathbf{N}$. Then $\vec{c}_n \in \sqrt{D}(B_1)$ and $\|\vec{c}_n-\vec{c}\|_{l_2} \underset{n\to\infty}{\longrightarrow} 0$, i.e. $\vec{c} \in \sqrt{D}(B_1)$. Hence $\tilde{A}(B_1) \subset \sqrt{D}(B_1)$. The proof of $\sqrt{D}(B_1) \subset \tilde{A}(B_1)$ is the same.

Lemma 4.4. *Let $B_\delta(\vec{y}_0)=\{\vec{y}\in l_2\,;\ \|\vec{y}-\vec{y}_0\|_{l_2}\leqq\delta\}$ for $\vec{y}_0\in l_2$ and $\delta>0$, let $\tilde{A}$ be defined by* (16), *let $\tilde{A}(B_1)=\{\vec{y}\in l_2\,;\ \vec{y}=\tilde{A}\vec{x},\ \|\vec{x}\|_{l_2}\leqq 1\}$ and $\vec{x}(\vec{y})=\tilde{A}^{-1}(\mathrm{Pr}_{\tilde{A}(B_1)}\vec{y})$ for each $\vec{y}\in l_2$. Then for each $B_\delta(\vec{y}_0)\subset l_2\backslash\tilde{A}(B_1)$ there exists a constant $C_0(\delta,\ \vec{y}_0)$, such that*

$$\|\vec{x}(\vec{y}_1)-\vec{x}(\vec{y}_2)\|_{l_2}\leqq C_0(\delta,\ \vec{y}_0)\,.\,\|\vec{y}_1-\vec{y}_2\|_{l_2} \tag{19}$$

for each $\vec{y}_1$ and $\vec{y}_2$ belonging to $B_\delta(\vec{y}_0)$.

Proof. Let $\vec{z}(\vec{y})=\sqrt{D}^{-1}(\mathrm{Pr}_{\sqrt{D}(B_1)}\vec{y})$ for each $\vec{y}\in l_2$. Since $\sqrt{D}(B_1)=\tilde{A}(B_1)$ (lemma 4.3) we have $\vec{x}(\vec{y})=\tilde{A}^{-1}\,(\mathrm{Pr}_{\tilde{A}(B_1)}\vec{y})=\tilde{A}^{-1}\,(\mathrm{Pr}_{\sqrt{D}(B_1)}\vec{y})=\tilde{A}^{-1}\bullet\sqrt{D}\,\vec{z}(\vec{y})$.

Now let $\vec{y}_0\in l_2$ and $\delta>0$ be such that $B_\delta(\vec{y}_0)\subset l_2\backslash\tilde{A}(B_1)=l_2\backslash\sqrt{D}(B_1)$. Because of (17) we can apply theorem 3 a) from [5], i.e. there exists a constant $C_1(\delta,\ \vec{y}_0)$ such that

$$\|\vec{z}(\vec{y}_1)-\vec{z}(\vec{y}_2)\|_{l_2}\leqq C_1(\delta,\ \vec{y}_0)\|\vec{y}_1-\vec{y}_2\|_{l_2}$$

for each $\vec{y}_1$ and $\vec{y}_2$ belonging to $B_\delta(\vec{y}_0)$.

It is easily seen that the operator $\tilde{A}^{-1}\,.\sqrt{D}:l_2\to l_2$ is defined for each $\vec{y}\in l_2$ and is bounded. Hence

$$\|\vec{x}(\vec{y}_1)-\vec{x}(\vec{y}_2)\|_{l_2}=\|\tilde{A}^{-1}\bullet\sqrt{D}[\vec{z}(\vec{y}_1)-\vec{z}(\vec{y}_2)]\|_{l_2}\leqq\|\tilde{A}^{-1}\bullet\sqrt{D}\|\,.\,C_1(\delta,\ \vec{y}_0)\,.\,\|\vec{y}_1-\vec{y}_2\|_{l_2}$$

for each $\vec{y}_1$ and $\vec{y}_2$ belonging to $B_\delta(\vec{y}_0)$, i.e. (19) holds true with $C_0(\delta,\ \vec{y}_0)=\|\tilde{A}^{-1}\,.\sqrt{D}\|\bullet C_1(\delta,\ \vec{y}_0)$.

Lemma 4.5. *Let $B_\delta(\vec{y}_0)=\{\vec{y}\in l_2\,;\ \|\vec{y}-\vec{y}_0\|_{l_2}\leqq\delta\}$ for $\vec{y}_0\in l_2$ and $\delta>0$, $N:l_2\to l_2$ be defined by*

$$N[(x_1,\ x_2,\dots,\ x_n,\dots)]=(x_1/\sqrt{l},-\sqrt{2}\,x_2/\sqrt{l},\dots,\ (-1)^{n-1}x_n\sqrt{2}/\sqrt{l},\dots) \tag{20}$$

for each $(x_1,\ x_2,\dots,\ x_n,\dots)\in l_2$, let $\tilde{A}$ be defined by (16) *and $N\bullet\tilde{A}(B_1)=\{\vec{y}\in l_2;\ \vec{y}=N\bullet\tilde{A}\vec{x},\ \|\vec{x}\|_{l_2}\leqq 1\}$. Then*

a) *$N\bullet\tilde{A}:l_2\to l_2$ is injective operator;*

b) *for each $B_\delta(\vec{y}_0)\subset l_2\backslash N\bullet\tilde{A}(B_1)$ there exists a constant $C(\delta,\ \vec{y}_0)$ such that*

(21) $\|[N\bullet\tilde{A}]^{-1}(\mathrm{Pr}_{N\bullet\tilde{A}(B_1)}\vec{y}_1)-[N\bullet\tilde{A}]^{-1}(\mathrm{Pr}_{N\bullet\tilde{A}(B_1)}\vec{y}_2)\|_{l_2}\leqq C(\delta,\vec{y}_0)\|\vec{y}_1-\vec{y}_2\|_{l_2}$

for each $\vec{y}_1$ *and* $\vec{y}_2$ *belonging to* $B_\delta(\vec{y}_0)$.

Proof. a) Since $\tilde{A}$ and N are injective, $N\bullet\tilde{A}$ is injective.

b) Since $\sqrt{D}(B_1)=\tilde{A}(B_1)$ (lemma 4.3) and $\tilde{A}:l_2\to l_2$ is compact (lemma 4.1), $\sqrt{D}:l_2\to l_2$ is compact too. Since $N:l_2\to l_2$ is bounded, $N\bullet\sqrt{D}:l_2\to l_2$ is compact (theorem 4.18 f) on p. 118 in [13]). The operator $NDN=[N\bullet\sqrt{D}]\bullet[\sqrt{D}\bullet N]=[N\bullet\sqrt{D}]\bullet[N\bullet\sqrt{D}]^*$ is self-adjoint, injective, compact and positive. Using the operator $\sqrt{NDN}$ as in lemma 4.3 we obtain $N\bullet\sqrt{D}(B_1)=\sqrt{NDN}(B_1)$, where $N\bullet\sqrt{D}(B_1)=\{\vec{y}\in l_2;\ \vec{y}=N\bullet\sqrt{D}\vec{x},\ \|\vec{x}\|_{l_2}\leqq 1\}$ and $\sqrt{NDN}(B_1)=\{\vec{y}\in l_2;\ \vec{y}=\sqrt{NDN}\vec{x},\ \|\vec{x}\|_{l_2}\leqq 1\}$. Now let $\vec{y}_0\in l_2$ and $\delta>0$ be such that $B_\delta(\vec{y}_0)\subset l_2\setminus N\bullet\tilde{A}(B_1)=l_2\setminus N\bullet\sqrt{D}(B_1)$. As in lemma 4.4 we can prove that there exists a constant $C_2(\delta,\vec{y}_0)$ such that

$$\|[N\bullet\sqrt{D}]^{-1}(\mathrm{Pr}_{N\bullet\sqrt{D}(B_1)}\vec{y}_1)-[N\bullet\sqrt{D}]^{-1}(\mathrm{Pr}_{N\bullet\sqrt{D}(B_1)}\vec{y}_2)\|_{l_2}\leqq C_2(\delta,\vec{y}_0).\|\vec{y}_1-\vec{y}_2\|_{l_2}$$

for each $\vec{y}_1$ and $\vec{y}_2$ from $B_\delta(\vec{y}_0)$. Since the operator $\tilde{A}^{-1}\bullet\sqrt{D}:l_2\to l_2$ is bounded, we have

$$\|[N\bullet\tilde{A}]^{-1}(\mathrm{Pr}_{N\bullet\sqrt{D}(B_1)}\vec{y}_1)-[N\bullet\tilde{A}]^{-1}(\mathrm{Pr}_{N\bullet\sqrt{D}(B_1)}\vec{y}_2)\|_{l_2}$$

$$=\|[\tilde{A}^{-1}\bullet\sqrt{D}]\{[N\bullet\sqrt{D}]^{-1}(\mathrm{Pr}_{N\bullet\sqrt{D}(B_1)}\vec{y}_1)\}-[\tilde{A}^{-1}\bullet\sqrt{D}]\{[N\bullet\sqrt{D}]^{-1}(\mathrm{Pr}_{N\bullet\sqrt{D}(B_1)}\vec{y}_2)\}\|_{l_2}$$

$$\leqq\|\tilde{A}^{-1}\bullet\sqrt{D}\|.C_2(\delta,\vec{y}_0).\|\vec{y}_1-\vec{y}_2\|_{l_2}$$

for each $\vec{y}_1$ and $\vec{y}_2$ from $B_\delta(\vec{y}_0)$, i.e. (21) holds true with $C(\delta,\vec{y}_0)=\|\tilde{A}^{-1}\bullet\sqrt{D}\|.C_2(\delta,\vec{y}_0)$, because $N\bullet\tilde{A}(B_1)=N\bullet\sqrt{D}(B_1)$.

In theorem 4.1 we shall need the following

Definition. Let X be a linear normed space and $B=\{\vec{x}\in X,\ \|\vec{x}\|_X<1\}$. For each two subsets V_1 and V_2 of X the number

$$d(V_1,V_2)=\inf\{\varepsilon>0;\ V_1\subset V_2+\varepsilon.B \text{ and } V_2\subset V_1+\varepsilon.B\}$$

is called Hausdorff distance between V_1 and V_2.

Theorem 4.1. *Let* $A(B_1^T)=\{f(x)\in L_2(0,\ l);\ f(x)=[A\varphi](x),\ \|\varphi\|_{L_2(0,T)}\leqq 1\}$ *and* $A(\mathring{B}_1^T)=\{f(x)\in L_2(0,\ l);\ f(x)=[A\varphi](x),\ \|\varphi\|_{L_2(0,T)}<1\}$.

a) *Let* $y_0(\cdot)\in L_2(0,\ l)$ *and* $\delta>0$ *be such that*

$$B_\delta(y_0(\cdot))=\{f(\cdot)\in L_2(0,\ l);\ \|f(\cdot)-y_0(\cdot)\|_{L_2(0,l)}\leqq\delta\}\subset L_2(0,\ l)\backslash A(B_1^T)$$

and let $\varphi_y(t)$ *denote the unique solution of (PO) for given* $y(\cdot)\in B_\delta(y_0(\cdot))$. *(Then there exists a constant* $C(\delta,\ y_0(\cdot))$ *such that*

$$(22)\qquad \|\varphi_{y'}(\cdot)-\varphi_{y''}(\cdot)\|_{L_2(0,T)}\leqq C(\delta,\ y_0(\cdot)).\ \|y'(\cdot)-y''(\cdot)\|_{L_2(0,l)}$$

for each $y'(\cdot)$ *and* $y''(\cdot)$ *from* $B_\delta(y_0(\cdot))$.

b) *Let* $S_{y(\cdot)}$ *denote the set of all solutions to (PO) for given* $y(\cdot)\in L_2(0,\ l)$ *(according to theorem 3.1a), if* $y(\cdot)\notin A(\mathring{B}_1^T)$ *the set* $S_{y(\cdot)}$ *consists of one element). Then for each* $y_0(\cdot)\in A(B_1^T)\backslash A(\mathring{B}_1^T)$ *and for each* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $\|y(\cdot)-y_0(\cdot)\|_{L_2(0,l)}<\delta$ *implies* $d(S_{y(\cdot)},\ S_{y_0(\cdot)})<\varepsilon$.

c) *For each* $y_0(\cdot)\in A(\mathring{B}_1^T)$ *there exists a sequence* $\{y_n(\cdot)\}_{n=1}^\infty\subset A(\mathring{B}_1^T)$ *such that* $\|y_n(\cdot)-y_0(\cdot)\|_{L_2(0,l)}\xrightarrow[n\to\infty]{}0$ *but nevertheless* $d(S_{y_n(\cdot)},\ S_{y_0(\cdot)})$ *does not tend to zero.*

Proof. a) It follows from (2), (3), (16), (20) and the definitions of $\mathrm{Is}_1: L_2(0,\ l)\to l_2$ and $\mathrm{Is}_2: H\to l_2$ that $[A\varphi](x)=[\mathrm{Is}_1^{-1}\bullet N\bullet\tilde{A}\bullet\mathrm{Is}_2\hat{\varphi}(\cdot)](x)$ for each $\varphi(t)\in H_1$, where $\hat{\varphi}(t)=\varphi(T-t)$. Since $[A\varphi_y](x)=[\mathrm{Pr}_{A(B_1^T)}y(\cdot)](x)$ for each $y(\cdot)\in B_\delta(y_0(\cdot))$ and $\varphi_y(\cdot)\in B_1^T\cap H_1$ (lemma 3.1), we have $[\mathrm{Is}_1^{-1}\bullet N\bullet\tilde{A}\bullet\mathrm{Is}_2\hat{\varphi}_y(\cdot)](x)=[\mathrm{Pr}_{A(B_1^T)}y(\cdot)](x)$, i.e. $\mathrm{Is}_2\hat{\varphi}_y(\cdot)=[N\bullet\tilde{A}]^{-1}\{\mathrm{Is}_1[(\mathrm{Pr}_{A(B_1^T)}y(\cdot))(x)]\}$ for each $y(\cdot)\in B_\delta(y_0(\cdot))$. Using (2) and (14) we have $A(B_1^T)=A(B_1^T\cap H_1)$. Hence $A(B_1^T)=\mathrm{Is}_1^{-1}\bullet N\bullet\tilde{A}\bullet\mathrm{Is}_2(B_1^T\cap H)$, i.e. $\mathrm{Is}_1[(\mathrm{Pr}_{A(B_1^T)}y(\cdot))(x)]=\mathrm{Is}_1[(\mathrm{Pr}_{\mathrm{Is}_1^{-1}\bullet N\bullet\tilde{A}(B_1)}y(\cdot))(x)]=\mathrm{Pr}_{N\bullet\tilde{A}(B_1)}\mathrm{Is}_1y(\cdot)$, i.e.

$$\mathrm{Is}_2\hat{\varphi}_y(\cdot)=[N\bullet\tilde{A}]^{-1}\bullet\mathrm{Pr}_{N\bullet\tilde{A}(B_1)}\mathrm{Is}_1y(\cdot).$$

According to lemma 4.5 there exists a constant $C(\delta,\ y_0(\cdot))$ such that

$$\|\mathrm{Is}_2\hat{\varphi}_{y'}(\cdot)-\mathrm{Is}_2\hat{\varphi}_{y''}(\cdot)\|_{l_2}\leqq C(\delta,\ y_0(\cdot)).\ \|\mathrm{Is}_1y'(\cdot)-\mathrm{Is}_2y''(\cdot)\|_{l_2}$$

for each $y'(\cdot)$ and $y''(\cdot)$ from $B_\delta(y_0(.))$, i.e.

$$\|\hat{\varphi}_{y'}(\cdot)-\hat{\varphi}_{y''}(\cdot)\|_{L_2(0,T)}\leqq C(\delta,\ y_0(\cdot)).\ \|y'(\cdot)-y''(\cdot)\|_{L_2(0,l)}$$

for each $y'(\cdot)$ and $y''(\cdot)$ from $B_\delta(y_0(\cdot))$. Since $\|\varphi_{y'}(\cdot)-\varphi_{y''}(\cdot)\|_{L_2(0,T)}=\|\hat{\varphi}_{y'}(\cdot)-\hat{\varphi}_{y''}(\cdot)\|_{L_2(0,T)}$, (22) holds true.

b) Let $y_0(\cdot)\in A(B_1^T)\setminus A(\mathring{B}_1^T)$ and the sequence $\{y_n(\cdot)\}_{n=1}^{\infty}\subset L_2(0, l)$ be such that $\|y_n(\cdot)-y_0(\cdot)\|_{L_2(0,l)}\xrightarrow[n\to\infty]{}0$. Let $\mu_{y_n}\in S_{y_n}(\cdot)$ for each $n\in\mathrm{N}$. It can be proved by standard argument that

(23) $\{\mu_{y_n}\}_{n=1}^{\infty}$ is converging to φ_{y_0} weakly in $L_2(0, T)$

(where φ_{y_0} is the unique solution of (PO) for $y(\cdot)=y_0(\cdot)$) and that

$$\|\mu_{y_n}\|_{L_2(0,T)}\xrightarrow[n\to\infty]{}\|\varphi_{y_0}\|_{L_2(0,T)}. \tag{24}$$

Then (23) and (24) yield

$$\|\mu_{y_n}(\cdot)-\varphi_{y_0}(\cdot)\|_{L_2(0,T)}\xrightarrow[n\to\infty]{}0. \tag{25}$$

Let us assume that there exist $\varepsilon_0>0$ and a subsequence of $\{y_n(\cdot)\}_{n=1}^{\infty}$ which we will denote also by $\{y_n(\cdot)\}_{n=1}^{\infty}$ such that $d(S_{y_n(\cdot)}, S_{y_0(\cdot)})\geqq\varepsilon_0$. This means that for each $n\in\mathrm{N}$ we have either $S_{y_0(\cdot)}\not\subset S_{y_n(\cdot)}+\varepsilon_0\,.\,\mathring{B}_1^T$, or $S_{y_n(\cdot)}\not\subset S_{y_0(\cdot)}+\varepsilon_0\,.\,\mathring{B}_1^T$. It follows from here that either

(26) $S_{y_0(\cdot)}\not\subset S_{y_{n_k}(\cdot)}+\varepsilon_0\,.\,\mathring{B}_1^T$ for a whole subsequence $\{y_{n_k}(\cdot)\}_{k=1}^{\infty}$,

or

(27) $S_{y_{n_k}(\cdot)}\not\subset S_{y_0(\cdot)}+\varepsilon_0\,.\,\mathring{B}_1^T$ for a whole subsequence $\{y_{n_k}(\cdot)\}_{k=1}^{\infty}$.

Since $S_{y_0(\cdot)}=\{\varphi_{y_0}(\cdot)\}$, (26), as well as (27), contradicts (25).

c) Let $y_0(\cdot)\in A(\mathring{B}_1^T)$ and φ_{y_0} be the unique solution to (PO) from H_1 (lemma 3.1). Since H is infinite dimensional, (14) implies that H_1 is infinite dimensional too. Let $\{\eta_n(t)\}_{n=1}^{\infty}$ be an orthonormal basis of H_1. Since $\varphi_{y_0}\in\mathring{B}_1^T\cap H_1$, we can find $\varepsilon>0$ such that $\varphi_n(t)=\varphi_{y_0}(t)+\varepsilon\,.\,\eta_n(t)\in\mathring{B}_1^T\cap H_1$ for each $n\in\mathrm{N}$. Then

(28) $\{\varphi_n(t)\}_{n=1}^{\infty}$ is converging to $\varphi_{y_0}(t)$ weakly in $L_2(0, T)$

and $\varphi_n(t)=\varphi_{y_n}(t)$, where $\varphi_{y_n}(t)$ is the unique solution to (PO) for $y(x)=y_n(x)$, belonging to H_1 (lemma 3.1). Let us define $y_n(x)=[A\varphi_n(\cdot)](x)$. Since $A:H_1\to L_2(0, l)$ is compact, (28) yields

$$\|y_n(\cdot)-y_0(\cdot)\|_{L_2(0,l)}=\|[A\varphi_n](\cdot)-[A\varphi_{y_0}](\cdot)\|_{L_2(0,l)}\xrightarrow[n\to\infty]{}0.$$

Now let $\varphi_{y_n}(t)+h_n(t)\in S_{y_n(\cdot)}$ for each $n\in\mathrm{N}$ and $\varphi_{y_0}(t)+h_0(t)\in S_{y_0(\cdot)}$. Since $\hat{h}_n(t)=h(T-t)\in H^{\perp}$ and $\hat{h}_0(t)=h_0(T-t)\in H^{\perp}$ (theorem 3.1 b)), we have

$$\|\varphi_{y_n}+h_n-(\varphi_{y_0}+h_0)\|^2_{L_2(0,T)}=\|\varphi_n-\varphi_{y_0}\|^2_{L_2(0,T)}+\|h_n-h_0\|^2_{L_2(0,T)}$$

$$\geqq\|\varphi_n-\varphi_{y_0}\|^2_{L_2(0,T)}=\varepsilon^2.$$

Hence $d(S_{y_n(\cdot)}, S_{y_0(\cdot)})\geqq\varepsilon$.

References

1. H. O. Fattorini, D. I. Russel. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, **43**, 1971, 272-292.
2. J.-L. Lions, E. Magenes. Problèmes aux limites non homogènes et applications, v. 2. Paris, 1968.
3. Ch. Müntz. Über den Approximationssatz von Weierstrass. *Math. Abh.* (*Sch. Festschr.*), 1914, 303-312.
4. R. M. Redheffer. Elementary remarks on completeness. *Duke Math. J.*, **35**, 1968, 103-116.
5. Ts. Tsachev. An l_2 constrained best approximation problem. *C. R. Acad. bulg. sci.*, **42**, 1989 (to appear).
6. T. Zolezzi. Some approximation and convergence problems in optimization. *Serdica*. **9**, 1983, 400-406.
7. В. М. Алексеев, В. М. Тихомиров, С. В. Фомин. Оптимальное управление. М., 1979.
8. Н. И. Ахиезер. Лекции по теории аппроксимации. Москва – Ленинград, 1947.
9. Т. Г. Генчев. Частни диференциални уравнения (второ издание). София, 1976.
10. Д. Киндерлерер, Г. Стампаккья. Введение в вариационные неравенства и их приложения. М., 1983.
11. Ж.-Л. Лионс. Оптимальное управление системами, описываемыми уравнениями с частными производными. М., 1972.
12. Л. А. Люстерник, В. И. Соболев. Краткий курс функционального анализа. М., 1982.
13. У. Рудин. Функциональный анализ. М., 1975.
14. Г. Е. Шилов. Математический анализ. Специальный курс. М., 1961.

Higher Institute for
Mining and Geology
Sofia
BULGARIA

Received 05.01.1989