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# Spectral Continuity in Complex Interpolation

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*Presented by M. Putinar*

Let $[\mathscr{X}_0, \mathscr{X}_1]$ be an interpolation pair in the sense of A. P. Calderón, and let $A$ be a linear map from $\mathscr{X}_0+\mathscr{X}_1$ into itself such that $A\mathscr{X}_j \subset \mathscr{X}_j$ and $A|\mathscr{X}_j$ is a bounded map for $j=0, 1$. For $0<t<1$, let $A_t$ be the restriction of $A$ to the interpolation space $\mathscr{X}_t$ (obtained by Calderón's complex method), and let $\sigma(A_t)$ denote the spectrum of this operator. It is shown that the mapping $t \to \sigma(A_t)^{\wedge}$ is continuous at every point $t_0 \in (0, 1)$ such that $\sigma(A_{t_0})$ has empty interior. The analogou result true for the essential spectrum. All these results suggest that $t \to \sigma(A_t)(t \in (0, 1))$ is always a continuous mapping. Some examples illustrate the difficulties involved in the solution of this problem.*

## 1. Introduction

Recall that an interpolation pair $[\mathscr{X}_0, \mathscr{X}_1]$ is a pair of complex Banach spaces ($\mathscr{X}_0$ and $\mathscr{X}_1$) continuously embedded in a Hausdorff topological vector space $v$. For each $0<t<1$, let $\mathscr{X}_t=[\mathscr{X}_0, \mathscr{X}_1]_t$ be the interpolation space obtained via Calderón's Complex Method applied to the pair $[\mathscr{X}_0, \mathscr{X}_1]$ [3]. Throughout this article, it will always be assumed that $\mathscr{X}_0 \cap \mathscr{X}_1$ is (norm) dense in $\mathscr{X}_t$, $0 \leqq t \leqq 1$.

Suppose that $A$ is a linear map from $\mathscr{X}_0+\mathscr{X}_1$ into itself such that $A\mathscr{X}_j \subset \mathscr{X}_j$ and $A|\mathscr{X}_j$ is a bounded map $(j=0, 1)$. Let $A_t \in \mathscr{L}(\mathscr{X}_t)$ (=the algebra of all bounded linear operators acting on $\mathscr{X}_t$) be the restriction of $A$ to $\mathscr{X}_t (0<t<1$; $\|A_t\| \leqq \|A_0\|^{1-t}\|A_1\|^t)$.

Let $\sigma(B)$ denote the spectrum of an operator $B$. It is well-known that the mapping $t \to \sigma(A_t)(t \in [0, 1])$ can be discontinuous at $t=0$ and at $t=1$, with respect to the Hausdorff metric on the compact subsets of the complex plane $\mathbf{C}$; for instance, it can happen that $\sigma(A_0)=D^-$ and $\sigma(A_t)=\{0\}$ for $0<t \leqq 1$, or $\sigma(A_0)=\sigma(A_1)=\partial D \cup \{0\}$ and $\sigma(A_t)=D^-$ for $0<t<1$, where $D$ denotes the open unit disk [7, ë8].

On the other hand, I. Ya. Šneiberg has shown that the mapping $t \to \sigma(A_t)$ is upper semicontinuous on (0, 1) [11]. An immediate consequence is that

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$$\bigcup \{\{t\} \times \sigma(A_t) : 0 < t < 1\}$$

is a closet subset (in the relative topology) of $(0, 1) \times \mathbf{C}$. (The second example indicated above shows that $t \to \sigma(A_t)$ is not upper semicontinuous on $[0, 1]$ and that $\cup \{\{t\} \times \sigma(A_t) : 0 \leqq t \leqq 1\}$ is not a closed subset of $[0, 1] \times \mathbf{C}$, in general).

**Conjecture.** *$t \to \sigma(A_t)$ is a continuous mapping on* $(0, 1)$.

In this article, several partial results are given, which support this conjecture.

We recall that minor modifications of Theorem 1 of C. J. A. Halberg's article [7] indicate that either the spectral radius, $sp(A_t)$, of $A_t$ is identically zero, or the mapping $t \to \log sp(A_t)$ is a convex function of $t \in (0, 1)$. In either case, $t \to sp(A_t)$ is a continuous function.

**Theorem 1.1.** *If $\sigma(A_{t_0})$ has empty interior (for some $t_0 \in (0, 1)$), then $t \to \sigma(A_t)$ is continuous at $t = t_0$.*

Thus, if the conjecture is false, a counterexample will necessarily show some pathological behaviour associated with one of the "holes" of the spectrum of $\sigma(A_t)$. (A "hole" in a compact subset $\sigma$ of $C$ is a bounded component of $C \setminus \sigma$).

It is impossible, for instance, that $\sigma(A_t) = \partial D$ for $0 < t < \frac{1}{2}$ and $\sigma(A_t) = \partial D \cup \{0\}$ for $\frac{1}{2} \leqq t < 1$. However, the results of Theorem 1.1 do not rule out the possibility of an example with $\sigma(A_t) = \partial D$ for $0 < t < \frac{1}{2}$ and $\sigma(A_t) = D^-$ for $\frac{1}{2} \leqq t < 1$, or $\sigma(A_t) = \partial D$ for $t \neq \frac{1}{2}$ and $\sigma(A_{\frac{1}{2}}) = D^-$.

In the second case, $A_r^{-1} | \mathcal{X}_0 \cap \mathcal{X}_1$ cannot coincide with $A_s^{-1} | \mathcal{X}_0 \cap \mathcal{X}_1$ for $0 < r < \frac{1}{2} < s < 1$. Indeed, it follows from Šneiberg's article that if $A_r - \lambda$ and $A_s - \lambda$ are invertible for some $\lambda \in \mathbf{C}$ and $0 \leqq r < s \leqq 1$, then the following are equivalent:

(1) $A_t - \lambda$ is invertible for $r \leqq t \leqq s$,

(2) $\ker(A - \lambda | \mathcal{X}_r + \mathcal{X}_s) = \{0\}$ and $(A - \lambda)$ maps $\mathcal{X}_r \cap \mathcal{X}_s$ onto itself; moreover, in this case $(A_t - \lambda)^{-1} | \mathcal{X}_r \cap \mathcal{X}_s$ is independent of $t \in [r, s]$ (see also [1]).

It can also happen that $\sigma(A_t)$ has no holes for $0 < t \leqq 1/2$, but it does have a hole for $1/2 < t < 1$, etc. (see Section 4).

In Section 3, we shall use this result to analyze the behavior of the mapping $t \to \sigma_e(A_t)$, where $\sigma_e(A_t)$ denotes the essential spectrum of $A_t$. (If $B \in \mathcal{L}(\mathcal{X})$ and $\mathcal{K}(\mathcal{X})$ denotes the ideal of all compact operators acting on $\mathcal{X}$, then $\sigma_e(B)$ is the spectrum of the coset of $B$ in the quotient Calkin algebra $\mathcal{L}(\mathcal{X})/\mathcal{K}(\mathcal{X})$. The essential spectral radius of $B$ will be denoted by $\mathrm{sp}_e(B)$.)

Recall that $B$ is a semi-Fredholm operator if $\operatorname{ran} B$ is closed and $\min\{\dim\ker B, \dim\ker B^*\}$ is finite. In this case we define $\operatorname{ind} B = \dim\ker B - \dim\ker B^*$ [6] [11]. Šneiberg's results indicate that if $A_t - \lambda$ is semi-Fredholm for all $t \in [r, s] (\subset [0, 1])$ then $\operatorname{ind}(A_t - \lambda)$ is constant on $[r, s]$, while $\min\{\dim\ker(A_t - \lambda), \dim\ker(A_t - \lambda)^*\}$ is upper semicontinuous.

The analogous of Theorem 1.1 for the essential spectrum is the following:

**Theorem 1.2.** *If $\sigma_e(A_{t_0})$ has empty interior (for some $t_0 \in (0, 1)$), then $t \to \sigma_e(A_t)$ is continuous at $t = t_0$.*

The results also include several illustrating examples, as well as some observations on the behaviour of the normal eigenvalues of $A_t$, $\tau \in (0, 1)$.

## 2. Stafney's éstimates on the size of the interpolated spectrum

If $f$ is a complex-valued function and $r$ a positive real number, let $\Delta(r, f)$ denote the set of elements $\lambda$ in the domain of $f$ such that $|f(\lambda)| \leqq r$. Let $\sigma_0$, $\sigma_1$ be nonempty compact subsets of the complex plane $\mathbf{C}$ such that $\sigma_0 \subset \sigma_1$. For $0 \leqq t \leqq 1$ we define the set $\mathbf{I}_t(\sigma_0, \sigma_1)$ to be $\cap \Delta(r^t, f)$, where the intersection is taken over all pairs $(r, f)$ such that (a) $r$ is a real number $> 1$, (b) $f$ is analytic on a neighborhood of $\sigma_1$, (c) $\Delta(r, f)$ is a compact subset of the domain of $f$, and (d) $\sigma_j$ is a subset of the interior of $\Delta(r^j, f)$, $j = 0, 1$.

In [14], J. D. Stafney has shown that

$$\rho(A) = \{\lambda \in \mathbf{C} : \lambda \notin \sigma(A_0) \cup \sigma(A_1) \text{ and } (\lambda - A_0)^{-1},\ (\lambda - A_1)^{-1} \text{ agree on } \mathscr{X}_0 \cap \mathscr{X}_1\}$$

is an open union of components of $\mathbf{C} \setminus [\sigma(A_0) \cup \sigma(A_1)]$ including the unbounded component of this latter set. Moreover, if $\sigma(A_0) \subset \sigma(A_1)$ and $\sigma'(A) = \mathbf{C} \setminus \rho(A)$, then

$$\sigma(A_t) \subset \mathbf{I}_t(\sigma(A_0),\ \sigma'(A)),\quad 0 \leqq t \leqq 1$$

(see Lemma 1.7 and Theorem 1.9 in the above reference).

Of course, in most cases $\sigma(A_0)$ is not a subset of $\sigma(A_1)$. Nevertheless, we can still obtain important information from Stafney's results. Instead of $[\mathscr{X}_0, \mathscr{X}_1]$ and $A$, consider $[\mathscr{X}_0 \oplus \mathscr{X}_0,\ \mathscr{X}_1 \oplus \mathscr{X}_0]$ and $A \oplus A_0$; then for $0 < t < 1$, $[\mathscr{X}_0 \oplus \mathscr{X}_0,\ \mathscr{X}_1 \oplus \mathscr{X}_0]_t = \mathscr{X}_t \oplus \mathscr{X}_0$ and $(A \oplus A_0)_t = A_t \oplus A_0$. Clearly, $\sigma(A_0 \oplus A_0) = \sigma(A_0) \subset \sigma(A_1 \oplus A_0) = \sigma(A_1) \cup \sigma(A_0)$ and $\sigma'(A \oplus A_0) = \sigma'(A)$; moreover, the roles of 0 and 1 can be easily reversed.

Hence, we have obtained the following:

**Corollary 2.1.** *For* $0 \leqq t \leqq 1$,

$$\sigma(A_t) \subset \mathbf{I}_t(\sigma(A_0),\ \sigma'(A))$$
$$\sigma(A_t) \subset \mathbf{I}_{1-t}(\sigma(A_1),\ \sigma'(A)).$$

We can go another step further. Indeed, according to [3, ë12.3, p. 121], if $\mathscr{X}_r = [\mathscr{X}_0, \mathscr{X}_1]_r$, $\mathscr{X}_s = [\mathscr{X}_0, \mathscr{X}_1]_s$ $(0 \leqq r \leqq s \leqq 1)$ and $t = (1-\tau)r + \tau s$ $(0 \leqq \tau \leqq 1)$, then $[\mathscr{X}_0, \mathscr{X}_1]_t$ and $[\mathscr{X}_r, \mathscr{X}_s]_\tau$ and their norms coincide. This means that the operators $A_t$ (on $[\mathscr{X}_0, \mathscr{X}_1]_t$) and $A_\tau$ (on $[X_r, X_s]_\tau$) coincide, up to a suitable identification of the underlying spaces.

From these observations and Corollary 2.1, we obtain the following:

**Corollary 2.2.** *Let* $0 \leqq r \leqq s \leqq 1$, *and let* $A(r, s) = A|\mathscr{X}_r + \mathscr{X}_s$. *If* $0 \leqq t, \tau \leqq 1$ *satisfy*

$$t = (1-\tau)r + \tau s,$$

*then*

$$\sigma(A_t) \subset \{\mathbf{I}_\tau[\sigma(A_r), \sigma'(A(r, s))]\} \bigcap \{\mathbf{I}_{1-\tau}[\sigma(A_s), \sigma'(A(r, s))]\}.$$

Given a compact subset $\sigma$ of $\mathbf{C}$ and $r > 0$, let $\sigma_r = \{\lambda \in \mathbf{C} : d[\lambda, \sigma] \leqq r\}$. Recall that the Hausdorff distance between two nonempty compact subsets $\sigma_1$ and $\sigma_2$ is defined as

$$d_H[\sigma_1, \sigma_2] = \min \{r \geqq 0 : \sigma_1 \subset (\sigma_2)_r \text{ and } \sigma_2 \subset (\sigma_1)_r\}.$$

Given a family $\{\sigma_t\}_{0<t<1}$ of nonempty compact sets included in a fixed compact subset $\sigma$ of $\mathbf{C}$, for $t_0 \in (0, 1)$ we define

$\liminf_{t\to t_0} \sigma_t = \{\lambda \in \mathbf{C}$: for each $t \in (0, 1)$ *there exists* $\lambda_t \in \sigma_t$ such that $|\lambda - \lambda_t| \to 0 (t \to t_0)\}$ and

$\limsup_{t\to t_0} \sigma_t = \{\lambda \in \mathbf{C}$: there exist a sequence $\{t_k\}_{k=1}^\infty \subset (0, 1) \setminus \{t_0\}$ converging to $t_0$ and $\lambda(t_k) \in \sigma_{t_k} (k \geqq 1)$ such that $|\lambda - \lambda(t_k)| \to 0 (k \to \infty)\}$.

Clearly, $\lim_{t\to t_0} \sigma_t$ and $\limsup \sigma_t$ are compact subsets of $\sigma$ such that

$$\liminf_{t\to t_0} \sigma_t \subset \limsup_{t\to t_0} \sigma_t;$$

moreover

$$\sigma_0 = \lim_{t\to t_0} \sigma_t$$

exists (in the sense of Hausdorff distance) if and only if $\liminf_{t\to t_0} \sigma_t = \limsup_{t\to t_0} \sigma_t = \sigma_0$.

**Lemma 2.3.** *For each* $t_0 \in (0, 1)$,

$$\partial\sigma(A_{t_0})^\wedge \subset \liminf_{t\to t_0} \sigma(A_t).$$

P r o o f. Assume not; then there exist $\mu \in \partial\sigma(A_{t_0})^\wedge$, a sequence $\{r_k\}_{k=1}^\infty$ in $(0, 1)$ converging to $t_0$ and $\eta > 0$ such that

$$\sigma(A_{r_k}) \bigcap D(\mu, \eta) = \emptyset$$

for all $k \geqq 1$, where $D(\mu, \eta)$ denotes the open disk of radius $\eta$ entered at $\mu$. Let $\gamma$ be a Jordan arc from $\mu$ to $\infty$ such that $\gamma \cap \sigma(A_{t_0}) = \{\mu\}$.

By passing, if necessary to a subsequence we can directly assume that $\{r_k\}_{k=1}^\infty$ is a n i n c r e a s i n g s e q u e n c e (if $r_k > t_0$ for all $k$, then we consider a decreasing sequence).

Let $0 < \varepsilon < \eta/4$ and let $\delta > 0$ be such that

$$|t_0 - s| < \delta \Rightarrow \sigma(A_s)\hat{} \subset [\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \qquad \text{(use [12]).}$$

We shall assume that $\varepsilon$ is so small that

$$\gamma \bigcap ([\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \setminus D(\mu, \eta/2)) = \emptyset.$$

Let $s_0 = t_0 + \delta/2$. By applying Corollary 2.2 to $[r_k, s_0]$, we infer that

$$\sigma(A_{t_0})\hat{} \subset \mathbf{I}_{\tau_k}[\sigma(A_{r_k}), \sigma'(A(r_k, s_0))]\hat{},$$

where $t_0 = (1-\tau_k)_{r_k} + \tau_k s_0$, $k = 1, 2, \dots$ .

Observe that

$$\sigma(A_{s_0})\hat{} \subset [\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \text{ and } \sigma(A_{r_k})\hat{} \subset [\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \setminus D(\mu, \eta).$$

In particular, $\sigma(A_{r_k})\hat{} \cap \gamma = \emptyset$. Thus, by using Runge's theorem (see, e. g., [5]), it is easy to construct a polynomial $f$ such that for some $r > 1$

$$[\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \subset \text{ interior } \Delta(r, f),$$

$$[\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \setminus D(\mu, \eta) \subset \text{ interior } \Delta(1, f)$$

and

$$[D(\mu, \eta/2) \cup \gamma] \cap \Delta(1, f) = \emptyset.$$

This implies that

$$\sigma(A_{t_0})\hat{} \subset \mathbf{I}_{\tau_k}([\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \setminus D(\mu, \eta), [\sigma(A_{t_0})\hat{}]_{\varepsilon/2} \subset \Delta(r^{\tau_k}, f)$$

for all $k \geqq 1$.

But this is utterly impossible because $r_k \to t$ implies that $\tau_k \to \infty$ and therefore $d_H[\Delta(r^{\tau_k}, f), \Delta(1, f)] \to 0$ $(k \to \infty)$, so that

$$\mu \in \Delta(1, f),$$

a contradiction. ■

Now we are in a position to prove the main result.

P r o o f o f t h e o r e m 1.1 Assume that interior $\sigma(A_{t_0}) = \emptyset$. By Lemma 2.3,

$$\partial\sigma(A_{t_0})\hat{} \subset \liminf_{t \to t_0} \sigma(A_t).$$

Let $\Phi$ be a hole in $\sigma(A_{t_0})$ and let $\zeta$ be any point of $\Phi$. According to [12], $A_t - \zeta$ is invertible for all $t$ close enough to $t_0$. By applying Lemma 2.3 to $(A_t - \zeta)^{-1}$ ($t$ in some neighborhood of $t$), we deduce that

$$\partial\Phi \subset \liminf_{t \to t_0} \sigma(A_t).$$

Since this applies to every hole of $\sigma(A_{t_0})$, by using these observations and the results of [12], we conclude that

$$\limsup_{t\to t_0} \sigma(A_t) \subset \sigma(A_{t_0})$$

$$=(\partial\sigma(A_{t_0})^{\wedge} \bigcup [\bigcup \{\partial\Phi : \Phi \textit{ is a hole in } \sigma(A_{t_0})\}]^{-}$$

$$\subset \liminf_{t\to t_0} \sigma(A_t).$$

It readily follows that

$$\lim_{t\to t_0} \sigma(A_t) = \sigma(A_{t_0})$$

(in the Hausdorff distance).

Hence, $t\to\sigma(A_t)$ is continuous at $t=t_0$. ■

## 3. Normal eigenvalues and essential spectrum

A point $\zeta\in\sigma(B)$ is a normal eigenvalue of the operator $B\in\mathscr{L}(\mathscr{X})$ if $\zeta$ is an isolated point of $\sigma(B)$ and the associated Riesz spectral invariant subspace $\mathscr{X}(B;\zeta)$ is finite dimensional. (This is equivalent to saying that $\zeta$ is an isolated point of $\sigma(B)\backslash\sigma_e(B)$.) The following auxiliary result has some interest in itself.

**Proposition 3.1.** *Suppose that $\zeta$ is a normal eigenvalue for $A_{t_0}$, for some $t_0\in[r, s]\subset[0, 1]$. Assume, moreover, that for some $\eta>0$, $D(\zeta, \eta)\cap\sigma(A_{t_0})=\{\zeta\}$ and $\partial D(\zeta, \eta)\cap\sigma(A_t)=\emptyset$ for $r\leqq t\leqq s$.*

*Then $D(\zeta, \eta)\cap\sigma(A_t)=\{\zeta\}$, $\zeta$ is a normal eigenvalue of $A_t$, $\mathscr{X}(A_t; \zeta)=\mathscr{X}(A_{t_0}; \zeta)$ and $A_t|\mathscr{X}(A_t; \zeta)=A_{t_0}|\mathscr{X}(A_{t_0}; \zeta)$ for all $t\in[r, s]$.*

Proof. By using Calderón's result [3, p. 421], we can directly assume that $r=0$ and $s=1$.

Recall that $\mathscr{X}_\Sigma:=\mathscr{X}_0+\mathscr{X}_1$ is a Banach space with the norm

$$\|f\|_\Sigma=\inf\{\|f_0\|_0+\|f_1\|_1;\ f=f_0+f_1,\ f_j\in\chi_j,\ j=0, 1\},$$

and $\mathscr{X}_\Delta:=\mathscr{X}_0\cap\mathscr{X}_1$ is a Banach space with the norm

$$\|f\|_\Delta=\max\{\|f\|_0, \|f\|_1\}.$$

If $A_\Sigma=A|\mathscr{X}_\Sigma$ and $A_\Delta=A|\mathscr{X}_\Delta$, then the union of two of the three sets

$$\sigma(A_0)\bigcup\sigma(A_1),\ \sigma(A_\Sigma),\ \sigma(A_\Delta)$$

includes the third (and similarly for the case when $\sigma(\cdot)$ is replaced by $\sigma_e(\cdot)$ [1, Theorem 2.3]). Therefore, $\partial D(\zeta, \eta)\cap[\sigma(A_\Sigma)\cup\sigma(A_\Delta)]=\emptyset$, and we can define the idempotent

$$B_\Sigma=\frac{1}{2\pi i}\int_{\partial D(\zeta,\eta)}(\lambda-A_\Sigma)^{-1}d\lambda\in\mathscr{L}(\mathscr{X}_\Sigma).$$

Our hypotheses imply that $B_{t_0}$ is a finite rank idempotent. A fortiori, so is $B_\Delta$ because every solution of $B_\Delta x=x(\in\mathscr{X}_\Delta)$ is necessarily a solution of $B_{t_0}x=x$.

Our hypothesis about $\mathscr{X}_0$, $\mathscr{X}_1$ implies that $\mathscr{X}_\Delta$ is dense in $\mathscr{X}_\Sigma$. Since $B_\Sigma$ is idempotent and maps $\mathscr{X}_j$ into $\mathscr{X}_j$ $(j=0, 1)$, it is not difficult to check that the finite dimensonal subspace $ran\, B_\Delta$ is dense in $ran\, B_\Sigma$, whence we readily deduce that

$$ran\, B_\Delta=ran\, B_\Sigma=ran\, B_t\,(\leqq t\leqq 1).$$

In other words, $\mathscr{X}(A_t\,;\zeta)=\mathscr{X}(A_{t_0};\zeta)$ for all $t\in[0, 1]$. By interpolating $A|ran\, B_\Delta$ between these two identical finite dimensional subspaces, we conclude that

$$A_t|\mathscr{X}(A_{t_0};\zeta)=A_{t_0}|\mathscr{X}A_{t_0};\zeta)\,(t\in[0, 1])\ \blacksquare$$

**Lemma 3.2.** *Suppose that $A_{t_0}-\lambda$ is a Fredholm operator of index zero. There exists a finite rank operator $F\in\mathscr{L}(\mathscr{X}_0+\mathscr{X}_1)$ such that $F(\mathscr{X}_j)\subset\mathscr{X}_j$ and $F/\mathscr{X}_j$ is a bounded map $(j=0, 1)$, and $(A-F)_{t_0}-\lambda$ is invertible.*

Proof. Since $A_{t_0}-\lambda$ is Fredholm of index zero, we can find a finite rank operator $F_0\in\mathscr{L}(\mathscr{X}_{t_0})$ such that $A_{t_0}-F_0-\lambda$ is invertible (see, e.g. [10]). Thus,

$$F_0=\sum_{j=1}^{m}x_j\oplus\Phi_j,$$

where $x_j\in\mathscr{X}_{t_0}$ and $\Phi_j\in(\mathscr{X}_{t_0})^*$ $(j=1, 2,\ldots, m)$, and $x\oplus\Phi(y):=\Phi(y)x(y\in\mathscr{X}_{t_0})$.

Since $\mathscr{X}_0\cap\mathscr{X}_1$ is dense in $\mathscr{X}_{t_0}$, $(\mathscr{X}_0\cap\mathscr{X}_1)^*$ is $w^*$-dense in $(\mathscr{X}_{t_0})^*$, and we can uniformly approximate the $x_j$'s and approximate the $\Phi_j$'s in the $w^*$-topology of $(\mathscr{X}_{t_0})^*$, in order to construct

$$F=\sum_{j=1}^{m}x'_j\oplus\Phi'_j\in\mathscr{L}(\mathscr{X}_{t_0})$$

such that $\|F_0-F\|<(2\|A_{t_0}-F_0-\lambda)^{-1}\|)^{-1}$, with $x'_j\in\mathscr{X}_0\cap\mathscr{X}_1$ and $\Phi_j\in(\mathscr{X}_0\cap\mathscr{X}_1)^*$.

Clearly, $F\in\mathscr{L}(\mathscr{X}_0+\mathscr{X}_1)$, $F(\mathscr{X}_j)\subset\mathscr{X}_j(j=0, 1)$ and $(A-F)_{t_0}-\lambda$ is invertible. $\blacksquare$

**Lemma 3.3** *For each $t_0\in(0, 1)$,*

$$\partial\sigma_e(A_{t_0})\,\hat{}\subset\liminf_{t\to t_0}\sigma_e(A_t).$$

Proof. By Lemma 2.3, $\partial\sigma(A_{t_0})\hat{}\subset\liminf_{t\to t_0}\sigma(A_t)$. But

$$\partial\sigma(A_{t_0})\hat{}=[\partial\sigma_e(A_{t_0})\hat{}]\bigcup[\sigma_0(A_{t_0})\backslash\sigma_e(A_{t_0})\hat{}]$$

(where $\sigma_0(A)$ denotes the set of all normal eigenvalues of the operator $A$) and, by Proposition 3.1,

$$\sigma_0(A_{t_0})\subset\liminf_{t\to t_0}\sigma_0(A_t).$$

Since $\sigma_0(A_t)\cap\sigma_e(A_t)=\emptyset$, we conclude that

$$\partial\sigma_e(A_{t_0})\hat{}\subset\liminf_{t\to t_0}\sigma_e(A_t).$$ ■

From this observation and Halberg's result [7], we obtain the following.

**Corollary 3.4**. *Either the essential spectral radius, $sp_e(A_t)$, of $A_t$ is identically zero, or $t\to\log sp_e(A_t)$ is a convex function of $t\in(0, 1)$. In either case, $t\to sp_e(A_t)$ is a continuous function of $t\in(0, 1)$.*

**Proof of Theorem 1.2.** By combining Lemma 3.3 and Corollary 3.2, we deduce (as in the proof of Theorem 1.1) that $\liminf_{t\to t_0}\sigma_e(A_t)$ includes $\partial\sigma_e(A_{t_0})\hat{}$ and $\partial\Phi$ for each component $\Phi$ of $\{\lambda\in\mathbf{C}:A_{t_0}-\lambda$ is a Fredholm operator of index $0\}$.

Now assume that $\Phi$ is a (necessarily bounded) component of the Fredholm domain of $A_{t_0}$ such that $\operatorname{ind}(A_t-\lambda)=m>0$ for all $\lambda\in\Phi$. Let $\zeta$ be any point of $\Phi$. According to [12], $A_t-\zeta$ is a Fredholm operator of index $m$ for all $t\in[r,s]$ for some subinterval $[r,s]$ of (0, 1) containing $t_0$ in its interior.

Clearly, we can assume that $\sigma(A_t)\subset D(0, 1)$ for all $t\in[0, 1]$. If $m>0$, then we replace $[\mathscr{X}_0, \mathscr{X}_1]$ and $A$ by $[\mathscr{X}_0\oplus H^2(\partial D)^{(m)}, \mathscr{X}_1\oplus H^2(\partial D)^{(m)}]$ and, respectively, by $A\oplus S^{(m)}$, where $S^{(m)}$ denotes the direct sum of $m$ copies of the unilateral shift operator "multiplication by $\lambda$" on the Hardy space $H^2(\partial D)$; then $(A\oplus S^{(m)})_t=A_t\oplus S^{(m)}(\in\mathscr{L}(\mathscr{X}_t\oplus H^2(\partial D)^{(m)}$, $(A\oplus S^{(m)})_t-\zeta$ is a Fredholm operator of index zero, and the component of $\zeta$ in $\mathbf{C}\backslash\sigma_e((A\oplus S^{(m)})_{t_0})$ coincides with $\Phi$.

If $m<0$, then we replace $A$ by $A\oplus S^{*(-m)}$, where $S^*$ denotes the adjoint of $S$.

Once again, we see that $\partial\Phi$ is included in $\liminf_{t\to t_0}\sigma_e(A_t)$. The remainder of the proof follows exactly as in the case of Theorem 1.1. ■

Let $B\in\mathscr{L}(\chi)$ and let $\Omega$ be a component of the semi-Fredholm domain of $B$, $\{\lambda\in\mathbf{C}:\lambda-B$ is a semi-Fredholm operator$\}$. The function $\lambda\to\min\{\dim\ker(B-\lambda), \dim\ker(B-\lambda)^*\}$ is continuous on $\Omega$, except for an exceptional subset of points of $\Omega$ that can only accumulate on $\partial\Omega$. This is the set of singular points of $B$. T. Kato has shown that if $\zeta$ is a singular point of $B$, then $\mathscr{X}=\mathscr{X}_\zeta\oplus\mathscr{X}'_\zeta$, where

$\mathscr{X}_\zeta$ and $\mathscr{X}'_\zeta$ are invariant subspace of $B$, $\mathscr{X}_\zeta$ is finite dimensional and $\zeta$ is not singular for $B|\mathscr{X}'_\zeta$ [9] (see also [4]). In particular, the normal eigenvalues of $B$ are singular points of the operator.

Proposition 3.1 strongly suggests the following:

**Conjecture.** *If $\zeta$ is a singular point of the semi-Fredholm domain of $A_{t_0}$ for some $t_0 \in [0, 1]$, then*

$$\mathscr{X}_0 + \mathscr{X}_1 = \mathscr{X}_\zeta \oplus \mathscr{X}'_\zeta,$$

*where $\mathscr{X}_\zeta$ is a finite dimensional subspace of $\mathscr{X}_0 \cap \mathscr{X}_1$, $\mathscr{X}_\zeta$ and $\mathscr{X}'_\zeta$ are invariant under $A$, and $\zeta$ is not singular for any $A_t|(\mathscr{X}'_\zeta)_t = (A|\mathscr{X}'_\zeta)_t$ such that $A_t|(\mathscr{X}'_\zeta)_t - \zeta$ is semi-Fredholm. Furthermore, if $\Phi$ is the component of*

$$\{(\lambda, t);\ A_t - \lambda \text{ is semi-Fredholm}\},$$

*then* $\dim\ker(A_t - \lambda)$ and $\dim\ker(A_t - \lambda)^*$ *are constant on $\Phi$, except at the singularities.*

By combining the results of Theorem 1.1 and 1.2 (and their proofs) with the stability properties of the semi-Fredholm index in interpolation spaces studied in [12], we obtain the following.

**Corollary 3.5.** *If*

$$\sigma_e(A_{t_0}) \backslash \{\lambda : A_{t_0} - \lambda \text{ is a semi-Fredholm operator of index equal to } \infty \text{ or } -\infty\}$$

*has empty interior, then $t \to \sigma_e(A_t)$ is continuous at $t = t_0$.*

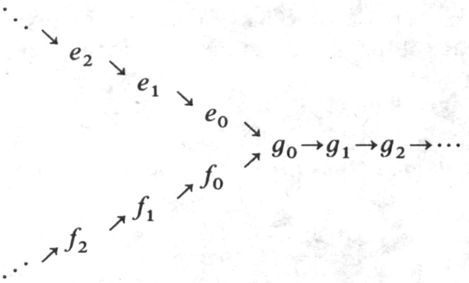
*If, in addition $\{\lambda : A_{t_0} - \lambda$ is a Fredholm operator of index $0\}$ also has empty interior, then $t \to \sigma(A_t)$ is continuous at $t = t_0$.*

Assume that $t \to \sigma(A_t)$ is not continuous at $t = t_0$, and let $B = A \oplus 1$, where 1 is the identity on an infinite dimensional Hilbert space $\mathscr{H}$. ($B$ acts on $(\mathscr{X}_0 + \mathscr{X}_1) \oplus \mathscr{H}$); then $t \to \sigma_e(B_t) = \sigma(B_t) = \sigma(A_t)$ is not continuous at $t = t_0$.

Thus, in order to prove our conjecture that $t \to \sigma(A_t)$ is always a continuous mapping on (0, 1), it suffices to prove that $t \to \sigma_e(A_t)$ is always continuous.

## 4. The "sleeping *Y*" and other examples

The articles [8] and [9] include several examples that indicate that $t \to \sigma(A_t)$ can have a very irregular behavior at $t = 0$, or $t = 1$, as well as the fact that the operators $A_t$ can be Fredholm (or semi-Fredholm) of different indices for different values of $t \in (0, 1)$. For instance, Example 8.2 of [8] is based on Hlbert spaces with orthogonal bases $\{e_n, f_n, g_n\}_{n=0}^{\infty}$ ordered as a "sleeping *Y*" and the operator acting as follows:




That is, $Ae_n = e_{n-1}$ and $Af_n = f_{n-1}$ for $n > 0$, $Ae_0 = Af_0 = g_0$ and $Ag_n = g_{n+1}$ $(n \geqq 0)$. $\mathscr{X}_0$ is the Hilbert space with this orthogonal basis and the norm

$$\| \sum_{n-0}^{\infty} (a_n \varepsilon_n + B_n f_n + c_n g_n) \|_0 = \{\Sigma(|\alpha_n^0 a_n|^2 + |\beta_n^0 b_n| + \gamma_n^0 c_n|^2)\}^2$$

for suitable sequences $\{\alpha_n^0\}$, $\{\beta_n^0\}$and $\{\gamma_n^0\}$ of positive reals, and $\mathscr{X}_1$ is the Hilbert space with the same orthogonal basis and similarly defined norm, with $\alpha_n^0$, $\beta_n^0$ and $\gamma_n^0$ replaced by $\alpha_n^1$, $\beta_n^1$ and resepctively, $\gamma_n^1$.

If the $\alpha$'s, $\beta$'s and $\gamma$'s are carefully chosen, then $A\mathscr{X}_j \subset \mathscr{X}_j$ and $A_j = A|\mathscr{X}_j \in \mathscr{L}(\mathscr{X}_j)$ $(_j = 0,\ 1)$. For $0 < t < 1$, $\mathscr{X}_t$ is a Hilbert space with the same orthogonal basis and norm

$$\| \sum_{n-0}^{\infty} (a_n e_n + B_n f_n + c_n g_n) \|_f = \{ \sum_{n=0}^{\infty} (|(\alpha_n^0)^{1-t} (\alpha_n^1)^t a_n|^2 + |(\beta_n^0)^{1-t} (\beta_n^1)^t b_n|^2$$
$$+ |(\gamma_n^0)^{1-t} (\gamma_n^1)^t c_n|^2)\}^{1/2}.$$

In [8, Example 8.2], the weights were chosen $\alpha_n^0 = \beta_n^1 = \gamma_n^0 = \gamma_n^1 = 1$ and $\alpha_n^1 = \beta_n^0 = (n!)^{-1}$, and it was shown that in this case $A_0$ and $A_1$ are both unitarily equivalent to compact perturbations of $U \oplus 0$ (where $U$ is the bilateral shift on $l_Z^2$) and $\sigma(A_0) = \sigma(A_1) = \sigma_e(A_0) = \sigma_e(A_1) = \partial D(0,\ 1) \cup \{0\}$, while $A_t$ is unitarily equivalent to a compact perturbation of $S \oplus 0$ (where $S$ is the uniateral shift on $l_N^2$), $\sigma(A_t) = D(0,\ 1)^-$, $\sigma_e(A_t) = D(0,\ 1) \backslash \{0\}$, and $A_t - \lambda$ is a Fredholm operator of trivial kernel and index $-1$ for $0 < t < 1$ and $\lambda \in D(0,\ 1) \backslash \{0\}$.

If we choose $\alpha_n^0 = \beta_n^1 = \gamma_n^0 = \gamma_n^1 = 1$ and $\alpha_n^1 = \beta_n^0 = r^n$ $(0 < r < 1)$, then a similar computation shows that $A_0$ and $A_1$ are unitarily equivalent to compact perturbations of $U \oplus rS^*$, $\sigma(A_0) = \sigma(A_1) = \partial D(0,\ 1) \cup D(0,\ r)^-$, $\sigma_e(A_0) = \sigma_e(A_1) = \partial D(0,\ 1) \cup \partial D(0,\ r)$, and $A_j - \lambda$ is a Fredholm operator of trivial cokernel and index 1 for $|\lambda| < 1/2$ $(j = 0,\ 1)$. On the other hand, for $0 < t < 1$, $\sigma(A_t) = \{\lambda \in \mathbb{C} : |\lambda| \leqq \min[r^t,\ r^{1-t}]$ or $\max[r^t,\ r^{1-t}] \leqq |\lambda| \leqq 1\}$, $\sigma_e(A_t) = \partial\sigma(A_t)$, and $A_t - \lambda$ is a Fredholm operator of trivial cokernel and index 1 for $\lambda \in \sigma(A_t) \backslash \sigma_e(A_t)$. In particular, for $t = 1/2$ the spectrum of $A_{1/2}$ is the closed unit disk and $\sigma_e(A_{1/2}) = \partial D(0,\ 1) \cup \partial D(0,\ r^{1/2})$.

In this case, $A_t - r^{1/2}$ is invertible for all $t \in [0, 1]$, $t \neq 1, 2$. $(A_t - r^{1/2})^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1$ is independent of $t$ for $0 \leqq t < 1/2$, or for $1/2 < t \leqq 1$. However, $(A_{t_1} - r^{1/2})^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1$ does not coincide with $(A_{t_2} - r^{1/2})^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1$ for $0 \leqq t_1 < 1/2 < t_2 \leqq 1$!

(Similar examples can be constructed on the same model, with $A$ "pushing" the coordinates in the other direction: $Ag_n = g_{n-1}$ $(n > 0)$, $Ag_0 = e_0 + f_0$, $Ae_n = e_{n+1}$ and $Af_n = f_{n+1}$ $(n \geqq 0)$, see Examples 8.1 of [8].)

An easier example can be constructed as follows: $\mathscr{X}_0$ and $\mathscr{X}_1$ are Hilbert spaces with the same orthogonal basis $\{e_n\}_{n \in Z}$, $\|\Sigma \alpha_n e_n\|_0 = (\Sigma |R^n a_n|^2)^{1/2}$ and $\|\Sigma a_n e_n\|_1 = (\Sigma |r^n a_n|^2)^{1/2}$ for suitable constants $R > r > 0$, and $Ae_n = e_{n+1}$ $(n \in Z$; then $A_0(A_1)$ is unitarily equivalent to $\frac{1}{R} U$ $(\frac{1}{r} U$, resp.), so that $\sigma(A_0) = \partial D(0, \frac{1}{R})$ and $\sigma(A_1) = \partial D(0, \frac{1}{r})$. Similarly, $A_t$ is unitarily equivalent to $(R^{t-1} r^{-t}) U$ and $\sigma(A_t) = \partial D(0, R^{t-1} r^{-t})$ for $0 < t < 1$.

Clearly, $\sigma(A_t) \cap \sigma(A_s) = \emptyset$ for $0 \leqq t < s \leqq 1$; moreover, for $\lambda \notin \sigma(A_t) \cup \sigma(A_s)$, $(A_t - \lambda)^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1$ and $(A_s - \lambda)^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1$ coincide if and only if either $|\lambda| > R^{s-1} r^{-s}$ or $|\lambda| < R^{t-1} r^{-t}$.

Let $\Omega = \cup_{k=1}^{\infty} (a_k, b_k)$ (disjoint union) be an open subset of $(0, 1)$. For each $k \geqq 0$ it is possible to construct $[\mathscr{X}_0(k), \mathscr{X}_1(k)]$ and $A(k)$ as above, in such a way that if $\mathscr{X}_j = \Sigma \oplus_{k=0}^{\infty} \mathscr{X}_j(k)$ (orthogonal direct sum, $j = 0, 1$) and $A = \Sigma \oplus_{k=0}^{\infty} [A(k) - c_k]$ (for suitably chosen $c_k$, $\frac{1}{R_k} < c_k < \frac{1}{r_k}$), then $A_t$ is invertible if and only if $t \in \Omega$, but

$$A_{t_1}^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1 = A_{t_2}^{-1} | \mathscr{X}_0 \cap \mathscr{X}_1$$

if and only if $t_1, t_2 \in (a_k, b_k)$ for some $k$.

The details of the construction are left to the reader (as usual, see [9]).

Let $\mathscr{X}_0$, $\mathscr{X}_1$ and $A$ be constructed as above, so that $A_0 = \frac{1}{R} U$ and $A_1 = \frac{1}{r} U$, where $r = 1$ and $R = 4$. Let $H =$ "multiplication by $x$" on $L^2 = L^2([0, 1], dx)$, and let $B = A \oplus H$ (acting on $(\mathscr{X}_0 + \mathscr{X}_1) \oplus L^2$); then

$$\sigma(B_t) = \{\lambda \in \mathbf{C} : B_t - \lambda \text{ is not semi-Fredholm}\}$$
$$= \{\lambda \in \mathbf{C} : |\lambda| \leqq 4^{t-1}\} \quad (0 \leqq t \leqq 1)$$

is simply connected.

If $C = e^{2\pi i B}$, then $\sigma(C_t)$ is a "croissant" for $0 \leqq t < 1/2$, but $\sigma(C_t)$ is a "doughnut" for $1/2 \leqq t \leqq 1$.

This example shows that $\sigma(C_t)$ can be simply connected for some values of $t \in (0, 1)$ and doubly connected for other values of $t$; moreover, it also shows that the mappings

$$t \to \sigma(C_t)\,\hat{}, \quad t \to \sigma_e(C_t)\,\hat{} \quad \text{and}$$

$$t \to \{\lambda : C_t - \lambda \text{ is not semi-Fredholm}\}\,\hat{}$$

are not continuous, in general.

The operator $L = e^{in\,A}$ ($A$ as above, $n$ "large") provides another example of the same kind, where the connectivity of $\sigma(L_t)$ increases from 2 to a very large number, as $t$ moves from 0 to 1.

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