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The Eigenvalue Problem for Grassmann Matrices

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We investigate the eigenvalue problem for Grassmann matrices. By considering two independent versions of this problem we define the left and right eigenvalues of a Grassmann matrix. It is shown that the numeric parts of left and right eigenvalues coincide.

1. Introduction

The supersymmetric theories of elementary particles, in which states are Grassmann algebra-valued functions on space-time, provide motivation for the study of Grassmannified mathematical structures [1]. By Grassmanification we mean the replacement of an underlying field of scalars or parameters, usually the reals or complex numbers, by a Grassmann algebra.

Linear transformations between particle states can be represented by matrices whose entries are homogeneous elements of a fixed Grassmann algebra $B_p(F)$ on p generators, where F is the field of reals or complex numbers. These matrices are called supermatrices and have the special partitioned form

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

where the entries in the square matrices A, D (the rectangular matrices B, C) belong to the even (odd) part, $B_{p,e}(B_{p,u})$, of $B_p(F)$, considered as a Z_2-graded algebra.

Although physicists are mainly interested in supermatrices, they really are special cases of the general Grassmann matrices for which there is no restriction on the matrix elements.

Let now M be an $n\times n$ Grassmann matrix over a fixed Grassmann algebra $B_p(F)$. Then we can write

$$M=M_0+M_n, \tag{2}$$

where the matrix elements of M_0 (resp., M_n) contain only the numeric (resp., nilpotent) part of the matrix elements of M. In these terms, M is invertible if and only if the ordinary matrix M_0 is invertible, and

$$M^{-1}=M_0^{-1}(I+M_n M_0^{-1})^{-1}. \tag{3}$$

Also of interest are eigenvalue problems for Grassmann matrices. F. Berezin in his article [2] on representations of the supergroup $U(p,q)$ made use of the eigenvalues of a diagonal supermatrix, as sums of a certain numeric part plus an arbitrary nilpotent part. In this article we study in detail the eigenvalue problem for a Grassmann matrix.

2. Eigenvalues of a Grassmann matrix

Let M be an $n\times n$ Grassmann matrix, $\mathbf{x}=(x_1,\dots,x_n)^t$ be a column vector with entries in the Grassmann algebra $B_p(F)$. We consider the system

$$M\mathbf{x}=\lambda\mathbf{x}, \tag{4}$$

where λ is a Grassmann number. Then analogously to the case of numeric matrices we can give the following definition of the eigenvalue of Grassmann matrix:

Definition 1. Any Grassmann number λ for which there exists a nonzero solution $\mathbf{x}$ of the system (4) is said to be a left eigenvalue of the Grassmann matrix M.

Bearing in mind the non-commutativity of the Grassmann algebra $B_p(F)$, we consider the system

$$M\mathbf{x}=\mathbf{x}\lambda, \tag{5}$$

which is inequivalent to the system (4). Therefore it is natural to define the right eigenvalue of a Grassmann matrix M as follows:

Definition 2. Any Grassmann number λ for which there exists a nonzero solution of the system (5) is said to be a right eigenvalue of the Grassmann matrix M.

Let us now write the Grassmann matrix M as a sum of its homogeneous parts, that is,, $M=M_e+M_u$. In a similar fashion we can write $\mathbf{x}=\mathbf{x}_e+\mathbf{x}_u$, $\lambda=\lambda_e+\lambda_u$. The system (4) can be written as

$$M_e\mathbf{x}_e+M_u\mathbf{x}_u=\lambda_e\mathbf{x}_e+\lambda_u\mathbf{x}_u,$$

$$M_u\mathbf{x}_e+M_e\mathbf{x}_u=\lambda_u\mathbf{x}_e+\lambda_e\mathbf{x}_u,$$

or equivalently, using the unit matrix I_n or simply I,

$$\begin{pmatrix} M_e-\lambda_e I & M_u-\lambda_u I \\ M_u-\lambda_u I & M_e-\lambda_e I\end{pmatrix}\begin{pmatrix}x_e\\ x_u\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}. \tag{6}$$

Similarly the system (5) can be written equivalently in the form

$$\begin{pmatrix} M_e-\lambda_e I & M_u+\lambda_u I \\ M_u-\lambda_u I & M_e-\lambda_e I\end{pmatrix}\begin{pmatrix}\mathbf{x}_e\\ \mathbf{x}_u\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}. \tag{7}$$

Consequently the system (4) (resp., (5)) has a nonzero solution if and only if the system (6) (resp., (7)) has a nonzero solution. However, it is necessary now to interject a number of preliminary lemmas.

Lemma 1. *Let A be an $n \times n$ Grassmann matrix over $B_{p,e}(F)$ and $\mathbf{x}=(x_1,\ldots,x_n)^t$ be a column vector with entries in $B_{p,e}(F)$ or in $B_{p,u}(F)$. Then the system*

$$(8) \qquad A\mathbf{x}=\mathbf{0}$$

has a nonzero solution if and only if A is not invertible.

Proof. Let the system (8) has a nonzero solution. Then, if A is invertible, multiplying by A^{-1} both sides of (8), we find $\mathbf{x}=\mathbf{0}$ (contradiction). Hence, A is not invertible. For the converse, when $\mathbf{x}$ is over $B_{p,e}(F)$, we consider

$$(9) \qquad \mathbf{x}=(k_1,\ldots,k_n)^t\,\theta_1\,\theta_2\ldots\theta_{2e}$$

as a candidate solution of the system (8), where $2e=p$ or $p-1$ and $k_i \in F$, $i=1, 2,\ldots,n$. Writing $A=A_0+A_n$, where A_0 (resp., A_n) contains the numeric (resp., the nilpotent) part of the entries of A, the column vector $\mathbf{k}=(k_1,\ldots,k_n)$ is forced to satisfy the numerical system

$$(10) \qquad A_0\,\mathbf{k}=\mathbf{0}.$$

Since A_0 is not invertible, it is a standard result that the homogeneous system of linear equations (10) has an infinity of nontrivial solutions.
When $\mathbf{x}$ is over $B_{p,u}(F)$, we may consider

$$(11) \qquad \mathbf{x}=(k_1,\ldots,k_n)^t\,\theta_1\,\theta_2\ldots\theta_q,$$

where $q=p$ if p is odd and $q=p-1$ if p is even.

Lemma 2. *Let M be a supermatrix of the form* (1) *and $\mathbf{x}=(\mathbf{x}_e,\mathbf{x}_u)^t$ be a column vector, where $\mathbf{x}_e$* (resp., $\mathbf{x}_u$) *is an $m \times 1$* (resp., $n \times 1$) *column vector over $B_{p,e}(F)$ (resp., $B_{p,u}(F)$).*
Then the system

$$(12) \qquad M\mathbf{x}=\mathbf{0}$$

has a nonzero solution if and only if the supermatrix M is not invertible.

Proof. If the system (12) has a nonzero solution, then M must be not invertible (the proof as in Lemma 1).
Let us now suppose that the supermatrix M is not invertible.
The system (12) is written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x}_e \\ \mathbf{x}_u \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

or equivalently,

$$Ax_e + Bx_u = \mathbf{0}, \qquad (13)$$
$$Cx_e + Dx_u = \mathbf{0}.$$

We distinguish three cases dependent on the invertibility of the even Grassmann matrices A and D.

I. A is invertible, D is not invertible

Then system (13) is equivalent to

$$\mathbf{x}_e = -A^{-1}B\mathbf{x}_u \qquad (14)$$
$$(D - CA^{-1}B)\mathbf{x}_u = \mathbf{0}.$$

Since the matrix $D - CA^{-1}B$ is not invertible, using Lemma 1, we get an infinity of nonzero solutions for the second equation of the system.

II. A is not invertible, D is invertible

We work as in case I.

III. A and D are not invertible

For $p = 2l$ we choose $\mathbf{x}_u = \mathbf{0}$. Then system (13) becomes

$$A\mathbf{x}_e = \mathbf{0}, \quad C\mathbf{x}_e = \mathbf{0},$$

and because of Lemma 1, the first equation has an infinity of nonzero solutions $\mathbf{x}_p = (k_1, \ldots, k_n)^t \theta_1 \theta_2 \ldots \theta_p$, which also satisfy the second equation.

For $p = 2l + 1$, we choose $\mathbf{x}_e = \mathbf{0}$. Then system (13) becomes

$$B\mathbf{x}_u = \mathbf{0}, \quad D\mathbf{x}_u = \mathbf{0},$$

and because of Lemma 1, the second equation has an infinity of nonzero solutions.

Coming again to the systems (6) and (7), from Lemma 2 we conclude that they have a nonzero solution if and only if the supermatrices

$$\begin{pmatrix} M_e - \lambda_e I & M_u \mp \lambda_u I \\ M_u - \lambda_u I & M_e - \lambda_e I \end{pmatrix}$$

are not invertible, or equivalently, the numeric matrix $M_0 - \lambda_0 I$ is not invertible. Therefore the numeric parts of the eigenvalues of the Grassmann matrix M are the roots of the equation

$$\det(M_0 - \lambda_0 I) = 0. \qquad (15)$$

However, the nilpotent part of the eigenvalues of the Grassmann matrix M can be taken arbitrarily.

From the above discussion we can state the following:

Theorem 1. *Let M be an $n\times n$ Grassmann matrix over $B_p(F)$ and $\mathbf{x}=(x_1,\dots,x_n)^t$ be a column vector with entries in $B_p(F)$. For λ in $B_p(F)$ we consider the systems*

$$M\mathbf{x}=\lambda\mathbf{x},$$

$$M\mathbf{x}=\mathbf{x}\lambda.$$

Then both systems have a nonzero solution if and only if the numeric matrix $M_0-\lambda_0 I$ is not invertible.

The numeric parts of the left and right eigenvalues of M are the roots of the polynomial equation

$$\det(M_0-\lambda_0 I)=0,$$

where M_0 (resp., λ_0) is the numeric part of M (resp., λ), whereas the nilpotent parts of the eigenvalues of M can be taken arbitrarily.

Comments

1. Considering supermatrices as special cases of Grassmann matrices, from Theorem 1, we have that the real parts of their eigenvalues are the roots of the polynomial equation

$$\det(A_0-\lambda_0 I)\det(D_0-\lambda_0 I)=0. \tag{16}$$

2. Also from Theorem 1, we have that the left and right eigenvalues of a Grassmann matrix coincide.

References

1. P. Van Nieuwenhuizen. Supergravity. *Phys. Rep.*, **68**, 1981 (for the physical background).
D. Ebner. The supergroups of supersymmetry and supergravity as ordinary real Lie groups. *Gen. Relativity Gravitation*, **14**, 1982, 1001-1006.
N. B. Backhouse, A. G. Fellouris. Grassmann analogues of the classical matrix groups. *Journ. of Math. Physics*, **26** (6), 1985, 1146-1151.
2. F. A. Berezin. Representations of the supergroup U(p, q). *Funct. Analysis*, **10** (3), 1976, 70-71.

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