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Markushevich Bases and Vector Measures

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Some properties of vector measures are studied and some criteria for regularity, absolute continuity and extension of vector measures are shown.

1. Introduction

J. Diestel [2] originally applied the theory of Schauder bases to the study of vector measures. His ideas have been developed subsequently by R. A. Alo and A. De Korvin [1], and Z. Lipecki and K. Musial [11].

This note is placed in the framework of the above ideas. In particular, we are dealing with vector measures taking values in locally convex topological vector spaces with a Markushevich basis.

Let S be a non empty set, Q an algebra and R a σ-ring of subsets of S. We say that a set function μ from Q to a locally convex space (abbreviated l.c.s) X is a finitely additive vector measure or simply vector measure if, for every A_1, $A_2 \in Q$ with $A_1 \cap A_2 = \emptyset$, $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$. If in addition $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, in the topology τ of X, for all sequences (A_n) of pairwise disjoint members of Q such that $\bigcup_{n=1}^{\infty} A_n \in Q$, then μ is called a countably additive vector measure or simply σ-additive vector measure. The vector measure μ is called bounded if its range $\mu[Q]$ is a bounded subset of X. Moreover, μ is said to be strongly bounded (abbreviated s-bounded) if, for every sequence (A_n) of mutually disjoint sets from Q, $\lim_{n\to\infty} \mu(A_n) = 0$.

For a σ-additive vector measure $\mu : Q \to X$ and a seminorm $P : X \to \mathrm{R}$ on X, the P-semivariation of μ is defined by $p(\mu) = \sup\{U(x^* \mu, A) : x^* \in U_p^0\}$.

A biorthogonal collection $\{x_i, f_i\}$ in (X, X^*) is a Markushevich basis (abbreviated M-basis) for X if and only if $\{x_i\}_{i\in I}$ is fundamental in (X, τ) and $\{f_i\}_{i\in I}$ is total over X. We set $\mu_n = f_n \circ \mu$, $n \in \mathrm{N}$. (For further details we refer to [2], [7], [8].)

2. The case of S-bounded and absolutely continuous vector measures

Proposition 2.1. *Let X be a l.c.s. with a M-basis (x_n, f_n) and $\mu : Q \to X$ be a vector measure. The following conditions are equivalent:*
(i) μ is σ-additive;
(ii) μ_n is σ-additive, $\forall n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii): For every sequence (A_k) of mutually disjoint sets from Q, with $A_k \searrow \emptyset$, we have $\lim_{k\to\infty} \mu(A_k)=0$. Therefore, for all $n\in\mathbb{N}$, $\lim_{k\to\infty} \mu_n(A_k) = \lim_{k\to\infty} (f_n \circ \mu)(A_k)=0$.

(ii) $\Rightarrow$ (i): By the assumption $\lim_{k\to\infty} \mu_n(A_k)= \lim_{k\to\infty} (f_n \circ \mu)(A_k)=0$, for every sequence (A_k) of mutually disjoint sets from (Q) with $A_k \searrow \emptyset$ for all $n\in\mathbb{N}$. Since (x_n, f_n) is a M-basis it follows that $\lim_{k\to\infty} \mu(A_k)=0$.

Proposition 2.2. *With the assumptions of the preceding Proposition the following conditions are equivalent:*
(i) μ is s-bounded;
(ii) μ_n is s-bounded, $\forall n \in \mathbb{N}$.

Proof. By the hypothesis, $\lim_{k\to\infty} \mu(A_k)=0$ for every sequence (A_k) of mutually disjoint sets from Q. Thus, $\lim_{k\to\infty} \mu_n(A_k)=0$ for all $n\in\mathbb{N}$, which shows that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): As the analogue of Proposition 2.1.

Proposition 2.3. *With the same assumptions as above, the following are equivalent:*
(i) μ is bounded;
(ii) μ_n is bounded, $\forall n \in \mathbb{N}$.

Proof. $P(\mu(Q))$ is a bounded subset of $\mathbb{R}$, for every continuous seminorm P (from the family Γ of seminorms defining the topology of X). Therefore $|\mu_n(Q)| = |f_n \mu(Q)| \leqslant \lambda_n P(\mu(Q))$ for $n\in\mathbb{N}$, thus proving the implication (i) $\Rightarrow$ (ii).
Conversely, because $\mathbb{R}$ has finite-dimension, μ_n is s-bounded. Hence, μ is bounded, since, by Proposition 2.2, μ is s-bounded.
The previous Propositions are summarized in the following result, extending to the case of M-bases Proposition 1.5 of [2].

Theorem 2.4. *Let X be a l.c.s. with a M-basis (x_n, f_n) and $\mu : Q \to X$ a vector measure. Then the following conditions are equivalent:*
(i) μ is bounded;
(ii) μ_n is bounded, $\forall n \in \mathbb{N}$;
(iii) μ_n is s-bounded, $\forall n \in \mathbb{N}$;
(iv) μ is bounded.

Proof. (i) $\leftrightarrow$ (ii) and (iii) $\leftrightarrow$ (iv) follow from Propositions 2.3 and 2.2, respectively. The remaining equivalence (ii) $\leftrightarrow$ (iii) holds because the notion of boundedness coincides with the one of s-boundedness in finite dimensional normed spaces.

Definition 2.5. Let X be a l. c. s, $\mu : Q \to X$ be a σ-additive vector measure and $\lambda : Q \to [0, \infty)$ a σ-additive non-negative measure.

(i) μ is called absolutely continuous (or continuous) with respect to λ (notation: $\mu \ll \lambda$) if, for every $A \in Q$ such that $\lambda(A) = 0$, $\mu(A) = 0$.

(ii) μ is topologically λ-continuous (notation: $\mu \underset{\tau}{\ll} \lambda$) if, for every τ-neighborhood U of zero in X, there exists $\delta > 0$ such that $\mu(A) \in U$, whenever $A \in Q$ and $\lambda(A) < \delta$.

An immediate consequence of the above is

Proposition 2.6. *Let X be a l. c. s. with a M-basis (x_n, f_n), $\mu : Q \to X$ a σ-additive vector measure and $\lambda : Q \to [0, \infty)$ a non-negative σ-additive measure. The following conditions are equivalent:*

(i) $\mu \ll \lambda$;

(ii) $\mu_n \ll \lambda$, $\forall n \in \mathrm{N}$.

Proof. By Definition 2.5 (i) $\mu(A) = 0$, whenever $\lambda(A) = 0$, $A \in Q$. Therefore, $\mu_n(A) = f_n(\mu(A)) = 0$ for all $n \in \mathrm{N}$, $A \in Q$, whenever $\lambda(A) = 0$. So $\mu_n \ll \lambda$ for all $n \in \mathrm{N}$, thus proving (i) $\Rightarrow$ (ii).

Conversely, by the hypothesis, $\mu_n(A) = 0$ for all $n \in \mathrm{N}$, whenever $\lambda(A) = 0$ and $A \in Q$. Therefore, by the definition of a M-basis $\mu(A) = 0$, whenever $\lambda(A) = 0$, $A \in Q$. Hence $\mu \ll \lambda$, thus proving the Proposition.

Lemma 2.7. *Let (X, τ) be a metrizable l.c.s, $\mu : R \to X$ a σ-additive vector measure and $\lambda : R \to [0, \infty)$ a non-negative σ-additive measure. The following conditions are equivalent:*

(i) $\mu \ll \lambda$;

(ii) $\mu \underset{\tau}{\ll} \lambda$;

(iii) $\lambda(A) \to 0 \Rightarrow P(\mu(A)) \to 0$, *for every seminorm* $P \in \Gamma$ *and* $A \in R$.

Proof. For (i) $\leftrightarrow$ (ii) cf. [13], Theorem 2.

(iii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii): Assume that the conclusion is false. Then there exists $\varepsilon > 0$ such that, for every $\delta > 0$, exists a set $A \in R$ such that $\lambda(A) < \delta$ and $P(\mu)(A) \geqq \varepsilon$, for every seminorm $P \in \Gamma$. Taking $\delta = \frac{1}{2^n}$, we have that $\lambda(A) < \frac{1}{2^n}$ and $P(\mu)(A_n) \geqq \varepsilon$, $P \in \Gamma$ for all $n \in \mathrm{N}$. If we set $B_n = \bigcup_{k=n}^{\infty} A_k$ and $B = \bigcap_{n=1}^{\infty} B_n$, then $\lambda(B_n) \leqq \sum_{k=n}^{\infty} \lambda(A_k) \leqq \frac{1}{2^{n+1}}$, for all $n \in \mathrm{N}$. Hence $\lambda(B) = 0$. As a result, for every $\Gamma \in R$ with $\Gamma \subset B$, $\lambda(\Gamma) = 0$ and since $\mu \ll \lambda$, $\mu(\Gamma) = 0$. For $\mu_P(A) = \sup\{P(\mu(F)) : F \subseteq A, A \in R\}$ we have $\mu_P(A) = 0$. On the other hand, $\mu_P(B_n) \geqq \mu_P(A_n) \geqq P(\mu)(A_n) \geqq \varepsilon$ for all $n \in \mathrm{N}$. Since $\{B_n\}_{n \in \mathrm{N}}$ is a decreasing sequence, we conclude $\mu_P(B) = \lim_{n \to \infty} \mu_P(\mathrm{B}_n) \geqq \varepsilon$, which is a contradiction.

If $(\mu_n)_{n \in \mathrm{N}}$ is a sequence of positive measures on Q, then a σ-additive vector measure $\mu : Q \to X$ is said to be uniformly absolutely continuous over Q, relative to $(\mu_n)_{n \in \mathrm{N}}$, if, for every τ-neighborhood U of zero in X, there exists $\delta > 0$ such that $\mu(A) \in U$, whenever $\mu_n(A) < \delta$ for all $n \in \mathrm{N}$ and $A \in Q$. In this case we write $\mu \underset{u}{\ll} \mu_n$ (cf. [1]).

The next theorem yields a necessary and sufficient condition for the measure μ to be extendable to $\sigma(Q)$ (: the σ-algebra generated by Q).

Theorem 2.8. *Let X be a sequentially complete metrizable l. c. s. with a M-basis (x_n, f_n) and let $\mu : Q \to X$ be a bounded σ-additive vector measure. Then the following conditions are equivalent:*

(i) μ is uniquely extendable (as σ-additive vector measure) to $\sigma(Q)$;

(ii) $\mu \underset{u}{\ll} U(\mu_n)$, where $U(\mu_n)$ is the total variation of μ_n.

Proof. (i)$\Rightarrow$(ii): We set $\lambda : \sigma(Q) \to \mathsf{R}$ with

$$\lambda(A) = \sum_{n=1}^{\infty} \frac{U(\mu_n, A)}{2^n[1+U(\mu_n, S)]}. \tag{2.1}$$

If $\lambda(A)=0$, then $\mu_n(A)=0$. Therefore $\mu_n \ll \lambda$ for all $n \in \mathsf{N}$ and $\mu \ll \lambda$ by the Proposition 2.6. In virtue of Lemma 2.7 for every τ-neighborhood U of zero in X there exists $\delta > 0$ such that

$$\mu(A) \in U, \tag{2.2}$$

whenever $\lambda(A) < \delta$, $A \in Q$.

Taking $U(\mu_n, A) < \delta$ for all $n \in \mathsf{N}$, (2.1) implies that $\lambda(A) \leqq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\delta}{1+\delta} < \delta$, which combined with (2.2) implies that $\mu(A) \in U$. Thus $\mu \underset{u}{\ll} U(\mu_n)$, for $n \in \mathsf{N}$.

(ii)$\Rightarrow$(i) : $\mu \underset{u}{\ll} U(\mu_n)$ yields $\mu \ll U(\mu_n)$. Then $\lim P(\mu(A)) = 0$, $P \in \Gamma$, whenever $U(\mu_n, A) \to 0$. Thus by [3], Corollary 2.1 there exists a unique extension of μ on $\sigma(Q)$.

3. The case of regular vector measures

In this section we are mainly dealing with the regularity of a vector valued measure.

Definition 3.1. Let S be a locally compact Hausdorff space, $\mathscr{B}(S)$ be the σ-ring generated by the compact sets of S and (X, τ) a l. c. s.

(i) Any σ-additive vector measure $\mu : \mathscr{B}(S) \to X$ is called Borel measure on S whereas a non-negative measure on $\mathscr{B}(S)$ is called a Borel measure if it is a Borel measure in the sense of Halmos [4], §52.

(ii) A σ-additive vector measure $\mu : \mathscr{B}(S) \to X$ is regular (with respect to τ), if, for each $E \in \mathscr{B}(S)$, $\varepsilon > 0$ and continuous seminorm P on X, there exists a compact set $K \subset E$ and an open set G in $\mathscr{B}(S)$ with $G \supset E$ and such that $P(\mu)(G \setminus K) < \varepsilon$.

Theorem 3.2. *Let X be a metrizable l. c. s, $\mu : \mathscr{B}(S) \to X$ a Borel measure and $\lambda : \mathscr{B}(S) \to [0, \infty)$ a non-negative, σ-additive regular Borel measure. If $\mu \ll \lambda$, then μ is regular.*

Proof. Let $\varepsilon > 0$. Since $\mu \ll \lambda$, by Lemma 2.7, for every continuous seminorm $P \in \Gamma$ there exists $\delta > 0$ such that $P(\mu)(A) < \varepsilon$, whenever $\lambda(A) < \delta$, $A \in \mathscr{B}(S)$. By the regularity of λ, for any $E\in\mathscr{B}(S)$ and the above $\delta>0$, there exists a compact set K in $\mathscr{B}(S)$, $K\subset E$ and an open set $G\in\mathscr{B}(S)$ with $G\supset E$ such that $\lambda(G\setminus K)<\delta$. Thus $P(\mu)(G\setminus K)<\varepsilon$.

Lemma 3.3. *Let X be a l.c.s. and $\mu:R\to X$ a σ-additive vector measure. Then, for every seminorm $P\in\Gamma$, there exists a non-negative σ-additive measure $\lambda_P:R\to[0,\infty)$ such that*

$$\lim_{\lambda_P(A)\to 0} P(\mu(A))=0 \quad \text{and} \quad \lambda_P(E)\leqq\sup\{P(\mu(A)):A\subseteq E\}.$$

Proof. Refer to [3], Theorem 1.1.

Proposition 3.4. *Let X be a metrizable l.c.s. $\mu:\mathscr{B}(S)\to X$ a Borel σ-additive vector measure, and for every seminorm $P\in\Gamma$, λ_P be the non-negative σ-additive measure determined by the above Lemma. Then, the following conditions are equivalent:*

(i) μ is regular;

(ii) λ_P is regular, for every $P\in\Gamma$.

Proof. (i) ⇒ (ii) : Let $P\in\Gamma$. Since μ is regular, for any $E\in\mathscr{B}(S)$ and $\varepsilon>0$ there exists a compact set K in $\mathscr{B}(S)$, $K\subset E$ and an open set $G\in\mathscr{B}(S)$, $G\supset E$ such that $P(\mu)(G\setminus K)<\varepsilon$. From the inequality $P(\mu(G\setminus K))\leqq P(\mu)(G\setminus K)$ and Lemma 3.3. we obtain that $\lambda_P(G\setminus K)<\varepsilon$.

(ii) ⇒ (i) : From the inequality
$P(\mu(E))\leqq P(\mu)(E)\leqq 4\{\sup P(\mu(F)):F\subseteq E,\ E\in\mathscr{B}(S)\}$ and our hypothesis we have that $\lim_{\lambda_P(A)\to 0} P(\mu)(A)=0$. By Lemma 2.7 and Theorem 3.2 μ is regular.

Theorem 3.5. *Let X be a metrizable l.c.s. and $\mu:\mathscr{B}(S)\to X$ a σ-additive vector measure. Then the following conditions are valid:*

(i) There exists a non-negative σ-additive measure $\lambda:\mathscr{B}(S)\to[0,\infty)$ such that $\mu\ll\lambda$;

(ii) μ is regular if and only if λ is regular.

Proof. (i). Since X is metrizable l.c.s. there exists an increasing sequence of continuous seminorms $(P_n)_{n\in\mathrm{N}}$ generating the topology τ of X. By Lemma 3.3, for every seminorm P_n, there exists a non-negative σ-additive measure $\lambda_{P_n}:\mathscr{B}(S)\to[0,\infty)$ with $P_n(\mu)\ll\lambda_{P_n}$. We set

$$\lambda(A)=\sum_{n=1}^{\infty}\frac{\lambda_{P_n}(A)}{2^n\lambda_{P_n}(S)},\quad A\in R. \tag{3.1}$$

Then $P_n(\mu)\ll\lambda$, $n\in\mathrm{N}$. By [5] Proposition 4.3, $P(\mu)\ll\lambda$ for every continuous seminorm P on X. Thus Lemma 2.7 implies that $\mu\ll\lambda$.

(ii) If λ is regular, Theorem 3.2 gives the regularity of μ.

Conversely, by the regularity of μ and Proposition 3.4 λ_{P_n} is regular, for all $n\in\mathrm{N}$. Hence, by equality (3.1), Lemma 2.7 and Theorem 3.2 λ is regular.

In the case of a locally convex space X with shrinking M-basis (cf. [7]), the following Theorem describes the regularity of μ in terms of the μ_n's (recall that $\mu_n=f_n\circ\mu$).

Theorem 3.6. *Let X be a l.c.s. with a shrinking M-basis (x_n,f_n) and $\mu:\mathscr{B}(S)\to X$ a σ-additive vector measure. Assume that there exists a Wells class*

W such that the sequence $((f_n \circ \mu)(E))_{n\in N}$ *is convergent for each* $E\in W$. *Then, the following conditions are equivalent*:

(*i*) μ *is regular*;

(*ii*) μ_n *is regular*, $\forall n\in N$.

Proof. (i)$\Rightarrow$(ii): It is clear.

(ii)$\Rightarrow$(i): By the definition of shrinking M-basis, the sequence $(f_n, Jx_n)_{n\in N}$ (where $J: X\to X^{**}$ is the canonical embedding) is an M-basis for X^*, when X^* is endowed with the strong topology. Therefore, for each $x^*\in X^*$, there exists a sequence $(y_n^*)_{n\in N}$ in X^*, each term of which is a finite linear combination of some f_n's such that $y_n^* \underset{s}{\to} x^*$. By our hypothesis and [9], Theorem 5.2, $x^*\mu$ is a regular Borel measure for each $x^*\in X^*$, and so μ is regular for the weak topology of X. Therefore, by [10], Theorem 1.6, μ is regular for the τ-topology of X.

Lemma 3.7. ([8], 2.1, Corollary 2.) *If* X *is a metrizable l.c.s. and* $\mu: R\to X$ *a* σ*-additive vector measure, then there exists a non-negative* σ*-additive measure* $\lambda: R\to[0,\infty)$ *equivalent to* μ.

Theorem 3.8. *Let* X *be a matrizable l.c.s. and* $\mu: \mathscr{B}(S)\to X$ *a Borel* σ*-additive vector measure. The following conditions are equivalent*:

(*i*) μ *is regular*;

(*ii*) *For every compact set* K *there exists a compact* G_δ *set* B *such that* $K\subset B$ *and* $\mu(B\setminus K)=0$.

Proof. Let $\lambda: \mathscr{B}(S)\to[0,\infty)$ be the non-negative measure determined by Lemma 3.7. By Theorem 3.2 and the equivalence of λ and μ it follows that λ is regular. Thus there exists a sequence (A_n) of open Borel sets such that $K\subset A_n$ and $\lambda(K)=\inf_n \lambda(A_n)$. By [4], Theorem 50.D, there exists a compact G_δ set B_n such that $K\subset B_n\subset A_n$. Then $B=\bigcap_{n=1}^{\infty} B_n$ is a compact G_δ set, $B\supset K$ and $\lambda(K)\leqq\lambda(B)\leqq\lambda(B_n)\leqq\lambda(A_n)$, for all $n\in N$. Hence, $\lambda(K)=\lambda(B)$ implies that $\lambda(B\setminus K)=0$. Since $\mu\ll\lambda$, we get $\mu(B\setminus K)=0$. Thus showing the first part.

Conversely, since $\lambda\ll\mu$, $\lambda(B\setminus K)=0$ holds. Let $(A_n)_{n\in N}$ be a sequence of open sets such that $A=\bigcap_{n=1}^{\infty} A_n$. By [4], Theorem 60.D, there exists, for each n, an open Baire set U_n such that $A\subset U_n\subset A_n$. Then we have $A=\bigcap_{n=1}^{\infty} U_n$ and therefore $\lim_{n\to\infty}\lambda(\bigcap_{i=1}^{n} U_i)=\lambda(A)=\lambda(K)$. By [4], Theorem 52.H, λ is regular. Since $\mu\ll\lambda$, Theorem 3.2 shows the regularity of μ and completes the proof.

It is known that a set $A\subset S$ is called locally measurable if $A\cap B\in R$, for every $B\in R$. If $\mu: R\to X$ is a σ-additive vector measure, we say (cf. [6]) that μ is singular with respect to λ (notation: $\mu\perp\lambda$) if there exists a locally measurable set A such that $\mu(A\cap B)=0$ and $\lambda(B\setminus A)=0$, for every $B\in R$.

Proposition 3.9. *Let* X *be a l.c.s. with a* M*-basis* (x_n, f_n), $\mu: R\to X$ *a* σ*-additive vector measure and* $\lambda: R\to[0,\infty)$ *a non-negative measure. Then the following conditions are equivalent*:

(*i*) $\mu\perp\lambda$;

(*ii*) $\mu_n\perp\lambda$, $\forall n\in N$.

Proof. (i)$\Rightarrow$(ii): There exists a locally measurable set A such that $\mu(E\cap A)=0$ and $\lambda(E\setminus A)=0$, for every $E\in R$. Therefore, $\mu_n(E\cap A)=0$, for all $n\in\mathbb{N}$, and $\lambda(E\setminus A)=0$.

(ii)$\Rightarrow$(i): Since $\mu_n\perp\lambda$ for all $n\in\mathbb{N}$, there exists a locally measurable set such that $\mu_n(E\cap A)=0$ and $\lambda(E\setminus A)=0$. From the definition of an M-basis, $\mu(E\cap A)=0$ and $\lambda(E\setminus A)=0$, thus $\mu\perp\lambda$.

Proposition 3.10. *Let X be a metrizable l.c.s., $\mu:\mathscr{B}(S)\to X$ a σ-additive vector measure and $\lambda:\mathscr{B}(S)\to[0,\infty)$ a non-negative σ-additive measure. Then there exist uniquely σ-additive vector measures $\mu_1,\mu_2:\mathscr{B}(S)\to X$ such that $\mu=\mu_1+\mu_2$, where $\mu\ll\lambda$ and $\mu_2\perp\lambda$.*

Proof. This is a direct concequence of Lemma 2.7 Theorem 3.5 and [12], Theorem 2.1.

Proposition 3.11. *Let X be a metrizable l.c.s. with a Schauder basis (x_n,f_n), $\mu:R\to X$ a σ-additive vector measure and $\lambda:R\to[0,\infty)$ a non-negative σ-additive measure. If μ_i, (i=1, 2) are the vector measures determined by proposition 3.10, $\mu_n=f_n\circ\mu$ $\forall n\in\mathbb{N}$, $\mu_{i,n}=f_n\circ\mu_i$ $(i=1,2)$ and*

$$\mu_n=\mu_{n,1}+\mu_{n,2} \tag{3.2}$$

with $\mu_{n,1}\ll\lambda$ and $\mu_{n,2}\perp\lambda$, $n\in\mathbb{N}$, then $\mu_{n,i}=\mu_{i,n}$, for all $n\in\mathbb{N}$ and $i=1,2$.

Proof. By Propositions 2.6 and 3.9 we have that $\mu_{1,n}\ll\lambda$ and $\mu_{2,n}\perp\lambda$. From the equality $\mu=\mu_1+\mu_2$ it follows that $\mu(A)=\sum_{n=1}^{\infty}f_n(\mu(A))x_n$, $A\in R$; hence,

$$\mu(A)=\sum_{n=1}^{\infty}f_n((\mu_1+\mu_2)(A))x_n=\sum_{n=1}^{\infty}(\mu_{1,n}+\mu_{2,n})(A)x_n.$$

By the uniqueness of the last expression we have

$$\mu_n=\mu_{1,n}+\mu_{2,n} \tag{3.3}$$

with $\mu_{1,n}\ll\lambda$ and $\mu_{2,n}\perp\lambda$.
Now (3.2) and (3.3) imply that $\mu_{i,n}=\mu_{n,i}$, for all $n\in\mathbb{N}$ and $i=1,2$.

For the sake of completeness, we state the following result, which can be thought of as a variation of [11] Corollary 3, using [7], Theorem 13.

Proposition 3.12. *Let X be a Banach space with a uniformly bounded, unconditional Schauder basis (x_n,f_n). The following conditions are equivalent:*

(i) (x_n) is a boundedly complete M-basis;

(ii) X has the Radon–Nikodym property;

(iii) X contains no subspace isomorphic to c_0.

References

1. R. A. Alo, A. De Korvin. Some approximation theorems for vector measures. *Rev. Roum. Math.*, **23** (9), 1978, 1289-1295.
2. J. Diestel. Applications of weak compactness and bases for vector measures and vectorial integration. *Rev. Roum. Math.*, **17** (2), 1973, 211-224.

3. N. Dinculeanu, I. Kluvanek. On vector measures. *Proc. London Math. Soc.*, **17**(3), 1967, 505-512.
4. P. Halmos. Measure theory. New York, 1950.
5. J. Hofman-Jørgensen. Vector measures. *Math. Scand.*, **28**, 1971, 5-32.
6. R. A. Johnson. On the Lebesgue decomposition theorems. *Proc. Amer. Math. Soc.*, **18**, 1967, 628-632.
7. W. B. Johnson. Markushevich bases and duality theory. *Tran. Amer. Soc.*, **149**, 1970, 171-177.
8. I. Kluvanek, G. Knowles. Vector measures and Control systems. New York, 1976.
9. J. Kupka. Uniform boundedness principles for regular Borel vector measures. *J. Austr. Math. Soc.* (series A), **29**, 1980, 206-218.
10. D. R. Lewis. Integration with respect to vector measures. *Pacific. Journ. Math.*, **33** (1), 1970, 157-165.
11. Z. Lipecki, K. Musial. On the Radon-Nikodym derivative of a measure taking values in a Banach space with basis. *Rev. Roum. Math.*, **23** (6), 1978, 911-915.
12. S. Ohba. The decomposition theorems for vector measures. *Sûgaku*, **25** (2), 1973, 23-28.
13. T. Traynor. Absolute continuity for group valued measures. *Canad. Math. Bull.*, **16** (4), 1973, 577-579.

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