Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

Now Series Vol. 3, 1989, Fasc. 3-4

Extension of a Valuative Order Ranging over a P. O. Abelian Group

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§1. Introduction. G-Valuative Orders

It is the main purpose of this paper to generalize, in the case of the partial order, the next theorem referred to Krull valuations: (see. f.e.[6], p. 31) *Given a Krull valuation defined on a commutative field and ranged over a linear abelian group, there exists an extension of the valuation which is defined on an extension of the field and which has as range the given group.* It is a simple consequence of this theorem to define a Krull valuation with domain an extension of a given field and range a given group.

The problem to define a valuation ranging over a partially ordered structure has been succeeded in several ways. The crucial point of such a definition is the meaning we intend to give to the hypermetric property. Especially if this range is a p.o. abelian group the s e m i - v a l u a t i o n has been defined as a multiplicative homomorphism v with $v(1)=0$ and which fulfils the property (x, y are elements of the field):

$$v(x+y)\geqq inf_G\{v(x), v(y)\},$$

where the last relation means that: $v(x+y)\geqq c$, for all c such that $v(x)\geqq c$, $v(y)\geqq c$. (See for example [5], p. 577, 578.)

We have faced the problem considering that $v(x+y)$ is not strictly smaller than $v(x)$ and $v(y)$. In §2 we prove that the symbol $\inf_G\{v(x), v(y)\}$ as well as the analogous one of our definition are the infimum of $v(x)$ and $v(y)$ in two different complements to which the p.o. group is embedded.

If the p.o group is torsion-free, we can formulate an analogous theorem to the above one extending the group (via Lorenzen – Simbireva – Everett's theorem, [3], p. 39) to a linear one. Paragraph 3 is devoted to establish an analogous statement in the case of a mixed group.

1.1 In the next $(K, +, .)$ and $(G, +, \leqq)$ are the structures of a commutative field and a partially ordered abelian group respectively (we use the terms "the field K" or "the group G"). As usual, we symbolize by $G^+=\{x\in G: x\geqq 0\}$, $K^*=K\backslash\{0\}$ and G^* the maximal torsion subgroup of G. Finally let it be $\hat{G}=G\cup\{\infty\}$, where ∞ is a new element such that $a+\infty=\infty+a=\infty+\infty=\infty$ and $a<\infty$, for each a in G.

1.2. **Definition.** A G-valuative order v is a function of $(K, +, .)$ into $(\hat{G}, +, \leqq)$ with the following properties:

(v_1) $v(x)=\infty \leftrightarrow x=0$;

(v_2) $v(x)=v(-x)$, for each x in K;

(v_3) $v(x.y)=v(x)+v(y)$ for every element x, y of K;

(v_4) If $v(x)>\gamma$ and $v(y)>\gamma$, then $v(x+y)>\gamma$ for x, y in K and γ in G. If $v(0)=\infty$, $v(x)=0$ for every $x \in K^*$, then v is called trivial.

1.3 Examples.

1. Every Krull valuation v of a commutative field K into a totally ordered abelian group G, is a G-valuative order.

2. We consider the set K of the elements u of the form:

$$(1) \qquad u=a_1 x^{\gamma_1}+a_2 x^{\gamma_2}+\ldots+a_n x^{\gamma_n}+\ldots,$$

where a_i belongs to a field F and $(\gamma_i)_{i\in N}$ is a strictly increasing sequence of integers converging to $+\infty$. The x is a generic symbol used as a variable. So $x^0=1$, $0.x=0$, e.t.c. K is a field (see [1], §5) and the function $v: K \to Z\cup\{\infty\}$ (∞ as in 1.1) with $v(0)=\infty$ and for $u\neq 0$ as in (1) and $v(u)=\gamma_1$, is a Krull valuation.

Let now $(G, +, \leqq)$, instead of Z, be a p.o. abelian group, G^+ its positive cone. We extend G to a totally ordered group (see [3], p. 39) and in the next we consider it ordered by this new order. Consider a totally well-ordered index-set I and an isotone function γ of I in G (symbolize γ_i, instead of $\gamma(i)$). Define the set K_1 of the elements u (generalized formal series)

$u=\sum_{i\in J\subset I} a_i x^{\gamma_i}$, where $a_i \in F$, $\gamma_i \in G^+$, x as above and the family $(\gamma_i)_{i\in J}$ is increasing and converges to ∞ (in the sense that for every $a\in G$, there exists $j \in J$, with $\gamma_j \geqq a$).

Put ord u-minimum $\{\gamma_i;\, i\in J\}$. K_1 is a ring with identity; on the other hand, if f, g are polynomials with variable the above x and exponents from G^+ and if x will be substituted by a generalized formal series u, there holds $(f.g)(u)=f(u).g(u)$ (which is proved as in [1], prop. 3, §5.5).

(1) Consider now

$$(1) \qquad u=a_0+a_1 x^{i_1}+a_2 x^{i_2}+\ldots,$$

where $a_0\neq 0$, $0<i_1<i_2<\ldots$ It is $u=a_0(1-a_0^{-1} v)$ and because of $\dfrac{1}{1-t}=\sum_{i=0}^{\infty} t^i$, the inverse of u is $\sum_{i=0}^{\infty} a_0^{-(i+1)} v^i$.

Obviously v^i is a formal series, because of the exponents of x in v^i being elements of the group.

Next, let $\tilde{K}$ be the set of the elements of the form:

$$u^*=\sum_{i\in J\subset I}^{\infty} a_i x^{\gamma_i},$$

where $a_i \in F$ and $(\gamma_i)_{i\in J}$ is an increasing and converging to ∞ family of elements of G.

Symbolize ord $u^*=\gamma_{i^*}$, where γ_{i^*} is the minimum of γ_i, $i\in J$. Then u^* can be written in the form

$$u^* = x^{\gamma_{i^*}}(a_{i^*} + \sum_{i \in J \setminus \{i^*\}} a_i x^{\gamma_i}),$$

where $\gamma_i > 0$ and the element in parenthesis is a formal series of the form (1). So if $u^* \neq 0$, there is the converse of u in $\tilde{K}$ and $\tilde{K}$ is a field (c.f [1], §§5.5, 5.6, 5.7).

We define on $\tilde{K}$ a G-valuative order v_1 as follows:

$$v_1(0) = \infty \quad \text{and for} \quad u \neq 0, \quad v_1(u) = \operatorname{ord} u.$$

3. If all the elements of a p. o. abelian group G are parallel one to another, then every homomorphism v of the multiplicative group of a commutative field, into G with order of $v(-1) \neq 2$ becomes a G-valuative order putting $v(0) = \infty$.

4. We consider in the Cartesian plane two non-zero vectors $\vec{c_1}, \vec{c_2}$ adapted at the origin of the axes and the module $\mathscr{M}$ which is generated from these vectors over the ring Z of integers. In $\mathscr{M}$ we introduce the well-known order: $(a, b) \precsim (a', b') \leftrightarrow a < a'$ and $b < b'$, the last inequalities refer to the natural orders (over the axes).

The $(\mathscr{M}, \tilde{\leqq})$ can be considered as a p. o. abelian group (in fact it is a module lattice). Putting $\hat{\mathscr{M}} = \mathscr{M} \cup \{\infty\}$, ∞ as above, we define — for given prime numbers p_1, p_2 — on Q a G-valuative order v, as follows:

$$v(0) = \infty \quad \text{and} \quad v(k) = n_1 \vec{c_1} + n_2 \vec{c_2},$$

where $k = p_1^{n_1} p_2^{n_2} \frac{a}{b}$, a, b integers $\neq 0$ prime to p_1, p_2 and η_1, η_2 integers.

The definition can be extended to a finite number of prime integers.

1.4. Remark. In the previous examples the group G was not mixed. The construction of §3 gives us examples of a different kind.

§2. The hypermetric property given by an order completion

In this paragraph we give an axiom, equivalent to (v_4) of definition 1.2. So this axiom gets the form — and in many points the meaning — of the hypermetric property. Analogous results we get for the case of a semi-valuation. We will use the well-known Mac Neille's complement ([4]) as well as a variation of it.

2.1. Given an order structure $(E, \leqq)$ put (for any A, B subsets of E) $A^+ = \{y \in E : a < y, \text{ for every } a \in A\}$,

$$B^- = \{x \in E : x < \beta, \text{ for every } \beta \in B\}.$$

The couple (A, B) is called a cut (A the lower and B the upper class of the cut). Each cut whose classes are without ends is called a gap and the set of gaps is symbolized by L(E).

The Mac Neille's complement could be defined as the set $E' = E \cup L(E)$ and a cut $(A, B) \in L(E)$ defines a new element lying between the elements of A and B. (The order between two new elements (A, B), (A', B') of E' is defined by the relation

$(A, B) < (A', B') \leftrightarrow A \subset A'$.)

The complement E' is a complete lattice. In fact, if the non-void subset S of E has not an infimum (resp., supremum), then the cut $(S^-,(S^-)^+)$ (resp., $((S^+)^-,S^+)$.) is the infimum (resp., supremum) of S in E'.

In the present paper we will exclusively refer to the infimum case and the symbol $\inf_{E'} S$ (the infimum of S in Mac Neille's complement of E) means the infimum of S in E (if it exists), otherwise it means the cut $(S^-, (S^-)^+)$.

2.2. Proposition. *The relation $a_0 \geqq \inf_{E'} \{a_1, a_2\}$ is equivalent to the relation "$a_1 \geqq \gamma$, $a_2 \geqq \gamma \Rightarrow a_0 \geqq \gamma$".*

Proof. If inf $\{a_1, a_2\}$ in E exists, the proof is trivial. Now let $a_1 \geqq \gamma$, $a_2 \geqq \gamma$. Put $S = \{a_1, a_2\}$. Then $\inf_{E'} S = (S^-, (S^-)^+)$ and $\gamma \in S^-$. The relation $a_0 \geqq \inf_{E'} S$ means $a_0 \geqq \gamma$.

Conversely, if $a_1 \geqq \gamma$, $a_2 \geqq \gamma$ and $a_0 \geqq \gamma$, then $\gamma \in S^-$, hence in all the cases $a_0 \geqq \inf_{E'} S$.

2.3. Remark. In accordance with the above 2.2. Proposition the hypermetric property of a semi-valuation v ranging over a partially ordered group G can be formulated as follows:

$$v(x+y) \geqq \inf_{G'} \{v(x), v(y)\}.$$

2.4. Consider again the order structure $(E, \leqq)$ and the set $L^*_+(E)$ (resp., $L^*_-(E)$) of the upper (resp., lower) classes without ends of cuts of E; L. Dokas defined (in [2]) an order structure on the set $\tilde{E} = E \cup L^*_+(E) \cup L^*_-(E)$ which could be described as follows: to each cut $(A, B) \in L^*_-(E) \times L^*_+(E)$ are corresponded two elements $x^- = A$, $x^+ = B$, such that $a < x^- < x^+ < b$, for every $a \in A, b \in B$. If one of the classes, say B, has an end, then $x^+ = B$. This complement is a complete lattice.

Any subset of the form $[x, \rightarrow$ [or] $\leftarrow$, $x]$ could be a class of a cut, hence x could be identified with this class, but it is possible none class to be identified with x.

2.5. Remark. To each subset S is corresponded the cut $(S^-, (S^-)^+)$. In the case where S hasn't infimum, this cut is the infimum of S in E' and the class $(S^-)^+$ is the infimum of S in $\tilde{E}$. But if S has infimum $e \in E$, then, either $e \in S$ or $e \notin S$ (at the first case $e \in S^-$, at the second $e \in (S^-)^+$), it is not sure, that e is an end of any of these classes. Even in the case where S is a singleton, say $S = \{e\}$, e is not necessarily the infimum of the class $(S^-)^+$. This case appears whenever there is not a cut with upper class the set $[e, \rightarrow [$.

2.6. Notation. The symbol $\inf_{\tilde{E}} S$ means always the class $(S^-)^+$. We call it the infimum of the classes of S (if it is not confused) or the infimum of S in the complement of the classes.

We can prove the following:

2.7. Proposition. *The relation $a_0 \geqq \inf_{\tilde{E}} \{a_1, a_2\}$ is equivalent to the relation "$a_1 > \gamma$, $a_2 > \gamma \Rightarrow a_0 > \gamma$".*

2.8. Remark. In accordance with the above 2.7. Proposition the hypermetric property of a G-valuative order ranging over a p. o. group G can be formulated as follows:

$$v(x+y) \geqq \inf_{\tilde{G}} \{v(x), v(y)\}.$$

2.9. Proposition. *If A is a subset of an ordered group G, then*

$$(\inf_{\tilde{G}} A)+\gamma=\inf_{\tilde{G}}(A+\gamma),$$

for any $\gamma\in G$.

Proof. Write $A_1<A_2$, if $a_1<a_2$ for any $a_1\in A_1$ and $a_2\in A_2$. (If one of the subsets is singleton, say $A_1=\{x\}$, write $x<A_2$.)

First we prove:

$$A^-+\gamma=(A+\gamma)^-. \tag{1}$$

If $x\in A^-+\gamma$, then $x=y+\gamma\,(y\in A^-)$, $y+\gamma<A+\gamma$ and $x\in(A+\gamma)^-$. On the other hand, if $x\in(A+\gamma)^-$, then $x-\gamma<A$, hence $x\in A^-+\gamma$.

Let now be $x\in(A^-)^++\gamma$; then $x-\gamma>A^-$, $x>A^-+\gamma$ and, because of (1), $x>(A+\gamma)^-$ and $x\in((A+\gamma)^-)^+$. Hence

$$(A^-)^++\gamma\subset((A+\gamma)^-)^+. \tag{2}$$

Similarly, if $x\in((A+\gamma)^-)^+$, then $x>(A+\gamma)^-=A^-+\gamma$, hence $x-\gamma\in(A^-)^+$ and

$$((A+\gamma)^-)^+\subset(A^-)^++\gamma. \tag{3}$$

Relations (2) and (3) imply the statement.

§3. An extension theorem of G-valuated fields

The main result of the present paper is described in this paragraph. Beginning from a G-valuated commutative field with values into a partially ordered abelian group, we extend the field, the valuation and the order of the group, to a new G-valuated field whose values cover entirely the group G.

3.1. Consider a p. o. abelian group $(G,\ +,\ \leqq)$ and G^* the maximal torsion group of G. For $a\in G$, $\bar{a}$ is the class mod G^*. We symbolize $a/\!/b$, if a and b are not comparable.

There have been proved the following:

Proposition. (i). *Every class of ${}^G/_{G^*}$ contains elements parallel one to another, hence the elements of G^* are parallel to zero.*
(ii). *If $a<b$, then for every $a'\in\bar{a}$, $b'\in\bar{b}$, $a'<b'$ or $a'/\!/b'$.*

The $\leqq$ induces in ${}^G/_{G^*}$ an order relation R as follows: $\bar{a}R\bar{b}\leftrightarrow\bar{a}=\bar{b}$, or there exist $a_0\in\bar{a}, b_0\in\bar{b}$ with $a_0<b_0$.

On the other hand, ${}^G/_{G^*}$ is torsion free and so R can be extended to a total order $\tilde{\leqq}$ in ${}^G/_{G^*}$.

Finally, $\tilde{\leqq}$ defines in G an order $\leqq_1$, extension of the initial $\leqq$, as follows:

$$a\leqq_1 b\leftrightarrow a=b,\ \text{or}\ \bar{a}\tilde{<}\bar{b}.$$

In the next, for simplicity, we use the symbol $\leqq_1$, instead of $\tilde{\leqq}$ or $\leqq_1$ as long as it is clear that the inequality is referred to classes or to elements.

3.2. If G is a mixed abelian group, it is said to be split, if $G=G^* \oplus \Gamma$, where Γ is a torsion-free group isomorphic to ${}^{G}/_{G^*}$. Γ is called thereafter system of representatives.

Among the groups that can be split are all the direct sums of cyclic groups, in particular, all the finitely generated abelian groups, all divisible groups and a lot of others.

In the proof of the basic theorem there will be used the following proposition:

Proposition. *Every abelian group is an homomorphic image of a free abelian group.*

3.3. Finish this introduction by inducing a total order to an arbitrary free abelian group $\mathscr{F}$ considered multiplicatively. This ordered free group plays a basic role to the proof of the theorem.

Let X be a free system of generators of $\mathscr{F}$. We order it by a total order $\leqq^*$ and consider the set $\mathscr{M}_1$ of monomials of the type

$$M=x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_n}^{a_n}, \tag{1}$$

where n is finite, a_k integer and $x_{i_1} <^* x_{i_2} <^* \dots <^* x_{i_n}$. Consider these monomials in "reduced" form, that is, if $a_k=0$, the corresponding factor is omitted except if all the exponents are zero, in which case we put $M=1$. Define in the known way the product in $\mathscr{M}_1$ and the inverse of a monomial and write $M_1 . M_2$ for its reduced form. So the monomials are written in a constant form and the structures $(\mathscr{M}_1,.)$ and $(\mathscr{F},.)$ coincide. We order $\mathscr{M}_1$ as following: put $\mathscr{M}^* > 1$, if $a_1>0$ and $\mathscr{M}^* < 1$, if $a_1<0$ (M as in (1)). For M_1, M_2 in $\mathscr{M}_1$, if $M_1 . M_2^{-1} {}^* > 1$. $\leqq^*$is compatible with the multiplication in $\mathscr{M}_1$.

3.4. Theorem. Given a commutative field $(K, +, .)$ and a G-valuative order v, ranging over a splitting partially ordered abelian group $(G, +, \leqq)$, there exist an extension $\tilde{K}$ of K, an extension $\tilde{\leqq}$ of $\leqq$ and an extension $\tilde{v}$ of v, such that $\tilde{v}(\tilde{K}^*)=G\,(\tilde{K}^*=\tilde{K}\setminus\{0\})$.

Proof. Let be $G=G^*\oplus\Gamma$, $\Gamma=(\gamma_i)_{i\in I}$ a system of representatives $(\mathscr{F},.)$ a free abelian group whose homomorphic image by an homomorphism ∂ is G^* and which is totally ordered by $\leqq^*$ (3.3). Let also $\leqq_1$ be the extension of $\leqq$, as well as the total order of ${}^{G}/_{G^*}$ (see 3.1).

The proof is realized in two steps; in the first (non-torsion step) we construct an extension K_1 of K and an extension v_1 of v, to"cover"by v_1 the group Γ. The second step (torsion step) refers to the extension $\tilde{K}$ of K_1 and $\tilde{v}$ of v_1 in order to cover G^* and after that the total group G.

Non-torsion step

Consider the ring $K[x]^{\Gamma^+}$ of the generalized polynomials S,

$$S=a_1 x^{\gamma_1}+a_2 x^{\gamma_2}+\dots+a_n x^{\gamma_n},$$

$$\gamma_i\in\Gamma^+,\ a_i\in K \text{ and } \Gamma^+=\{\gamma\in\Gamma: 0\ \leqq_1 \gamma\}.$$

We correspond to each polynomial S the set

$$\Lambda(S)=\{v(a_i)+\gamma_i : i\in\{1, 2,\dots,n\}\}\subset G.$$

We symbolize by $\Lambda_0(S)$ the set of the elements of $\Lambda(S)$ which belong to the smallest – according to $\leqq_1$ – class (mod G^*).

We define $v_1 : K[x]^{\Gamma^+} \to \hat{G} = G \cup \{\infty\}$ as follows:

$v_1(0) = \infty$,

$v_1(S) = v(a_k) + \gamma_k$, where $S \neq 0$, γ_k is the minimum of γ_i with $v(a_i) + \gamma_i \in \Lambda_0(S)$.

For every $i \neq k$ we have that $v(a_k) + \gamma_k <_1 v(a_i) + \gamma_i$, or $v(a_k) + \gamma_k$, $v(a_i) + \gamma_i$ belong to $\Lambda_0(S)$ and $\gamma_k <_1 \gamma_i$.

We prove that v_1 is a G-valuative order.

Let S be as above, $t = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + \ldots + b_m x^{\lambda_m}$, $v_1(S) = v(a_k) + \gamma_k$ and $v_1(t) = v(b_p)^+ \lambda_p$.

We have the following:

(i) $\boldsymbol{v_1(S.t) = v_1(S) + v_1(t)}$

The polynomial $S.t$ contains the summand $a_k b_p x^{\gamma_k + \lambda_p}$ and

$$v_1(a_k b_p x^{\gamma_k + \lambda_p}) = v(a_k) + v(b_p) + \gamma_k + \lambda_p. \tag{1}$$

For every $i \neq k$ and $j \neq p$, we have

$$A = v(a_k) + \gamma_k + v(b_p) + \lambda_p <_1 v(a_i) + \gamma_i + v(b_j) + \lambda_j = B, \tag{2}$$

or

A and B belong to $\Lambda_0(S.t)$ with $\gamma_k + \lambda_p <_1 \gamma_i + \lambda_j$. Indeed, if $\Lambda_0(S)$, $\Lambda_0(t)$ are singletons, relation (2) is obvious. In the other cases, B either will belong to a class (mod G^*) larger than the class (mod G^*) of A, (in which case (2) is satisfied) or A and B are equivalent (mod G^*). In this last case

$$\gamma_k <_1 \gamma_i \text{ and } \lambda_p <_1 \lambda_j, \text{ so}$$

$$v_1(S.t) = v_1(a_k b_p x^{\gamma_k + \lambda_p}) = v(a_k) + \gamma_k + v(b_p) + \lambda_p = v_1(S) + v_1(t).$$

(ii) $\boldsymbol{v_1(S+t) \geqq \inf_{\tilde{G}} \{v_1(S), v_1(t)\}}$

Let $v_1(S)$, $v_1(t)$ be as above.

$$S + t = \sum_{\substack{i \in \{1, \ldots, n\} \\ j \in \{1, \ldots, n\}}} (a_i + b_j) x^\sigma \text{ (where } \sigma = \gamma_i \text{ or } \sigma = \lambda_j).$$

We have $v_1[(a_i + b_j) x^\sigma] = v(a_i + b_j) + \sigma \geqq_1 \inf_{\tilde{G}} \{v(a_i), v(b_j)\} + \sigma = \inf_{\tilde{G}} \{v(a_i) + \sigma, v(b_j) + \sigma\}$ (2.9 Prop.).

So, $v_1[(a_i + b_j) x^\sigma] \geqq_1 \inf_{\tilde{G}} \{v(a_k) + \gamma_k, v(b_p) + \lambda_p\} = \inf_{\tilde{G}} \{v_1(S), v_1(t)\}$.

So $v_1(S+t) \geqq_1 \inf_{\tilde{G}} \{v_1(S), v_1(t)\}$.

(iii) **There is a field $\boldsymbol{K_1 \supset K}$ and an extension $\boldsymbol{v_1}$ of $\boldsymbol{v}$ covering $\boldsymbol{\Gamma^+}$**

For $a \in K$, $v_1(a) = v(a)$ and for $\gamma \in \Gamma^+$, $v_1(x^\gamma) = \gamma$.

If K_1 is the set of the fractions having terms from K, define

$$v_1\left(\frac{S}{t}\right) = v_1(S) - v_1(t), \quad \frac{S}{t} \in K_1^* = K_1 \setminus \{0\}.$$

(We use the same symbol v_1 for the new G-valuative order and we consider the usual equivalence between them.) Evidently

$$v_1\left(\frac{S}{t}\frac{S^*}{t^*}\right)=v_1(SS^*)-v_1(tt^*)=v_1\left(\frac{S}{t}\right)+v_1\left(\frac{S^*}{t^*}\right)$$

and

$$v_1\left(\frac{S}{t}+\frac{S^*}{t^*}\right)=v_1(St^*+S^*t)-v_1(tt^*)\geqq 1$$

$$\inf_{\tilde{G}}\{v_1(St^*), v_1(S^*t)\}-v_1(tt^*)=\inf_{\tilde{G}}\{v_1\left(\frac{S}{t}\right), v_1\left(\frac{S^*}{t^*}\right)\}.$$

Torsion step

We consider now the ring of the polynomials $K_1[X]$ of the form $S=u_1M_1+u_2M_2+\ldots+u_nM_n$, where X is a base of the free abelian group $\mathscr{F}$, $M_i\in\mathscr{F}$ and $u_1\in K_1$.

As in the previous step, define the set

$$\Lambda^*(S)=\{v_1(u_1)+\partial(M_i) : i\in\{1,\ldots,n\}\}.$$

The set of the elements of $\Lambda^*(S)$ which belongs to the smallest — according to $\leqq_1$ — class (mod G^*) symbolize by $\Lambda_0^*(S)$. We define $\tilde{v}: K_1[X]\to\hat{G}$, as follows: $\tilde{v}(0)=\infty$ $\tilde{v}(S)=v_1(u_k)+\partial(M_k)$, $S\neq 0$, M_k is the minimum (according to $\leqq^*$) of M_i, $i\in\{1, 2,\ldots,n\}$ and $v_1(u_i)+\partial(M_i)\in\Lambda_0^*(S)$.

We prove that $\tilde{v}$ is a G-valuative order:

Let S be as above, $t=p_1N_1+p_2N_2+\ldots+p_mN_m$, $\tilde{v}(S)=v_1(u_k)+\partial(M_k)$ and $\tilde{v}(t)=v_1(p_\lambda)+\partial(N_\lambda)$.

(i^*) $\tilde{v}(S.t)=\tilde{v}(S)+\tilde{v}(t)$

The polynomial $S.t$ contains the summand $u_kp_\lambda M_kN_\lambda$ and there holds

$$(1^*)\qquad \tilde{v}(u_kp_\lambda M_kN_\lambda)=v_1(u_k)+v_1(p_\lambda)+\partial(M_k)+\partial(N_\lambda).$$

For every $i\neq K$ and every $j\neq\lambda$, we have

$$(2^*)\qquad A^*=v_1(u_k)+\partial(M_k)+v_1(p_\lambda)+\partial(N_\lambda)<_1 v_1(u_i)+\partial(M_i)+v_1(p_j)+\partial(N_j)=B^*,$$

or

the elements A and B belong to $\Lambda_0^*(S.t)$ with $M_k.N_\lambda<^*M_iN_j$, because of the compatibility of $<^*$. Indeed, if $\Lambda_0^*(S)$ and $\Lambda_0^*(t)$ are singletons, relation (2*) is obvious. In the other cases, the element B^* of relation (2*) either belongs to the class of A^* (mod G^*) (in which case (2*) holds), or A^* and B^* are equivalent (mod G^*).

But $M_k<^*M_i$ and $N_\lambda<^*N_j$. So

$\tilde{v}(S.t)=\tilde{v}(u_kp_\lambda M_kN_\lambda)=v_1(u_k)+\partial(M_k)v_1(p_\lambda)+\partial(N_\lambda)=\tilde{v}(S)+\tilde{v}(t)$.

(ii*) $\tilde{v}(S+t)\geqq_1\inf_{\tilde{G}}\{\tilde{v}(S), \tilde{v}(t)\}$

Suppose that $\tilde{v}(S)$, $\tilde{v}(t)$ are as above.

$$S+t=\sum_{\substack{i\in\{1,\dots,n\}\\ j\in\{1,\dots,n\}}}(u_i+p_j)M_\sigma \text{ (where } M_\sigma=M_i \text{ or } M_\sigma=N_j).$$

We have $\tilde{v}[(u_i+p_j)M_\sigma]=v_1(u_i+p_j)+\partial(M_\sigma)\geqq_1 \inf_{\tilde{G}}\{v_1(u_i), v_1(p_j)\}+\partial(M_\sigma)$ $=\inf_{\tilde{G}}\{v_1(u_i)+\partial(M_i), v_1(p_j)+\partial(N_j)\}$.
Hence, $\tilde{v}[(u_i+p_j)M_\sigma]\geqq_1 \inf_{\tilde{G}}\{v_1(u_k)+\partial(M_k), v_1(p_\lambda)+\partial(M_\lambda)\}=\inf_{\tilde{G}}\{\tilde{v}(S),\tilde{v}(t)\}$.

As in the non-torsion step case we can define extension $\tilde{K}$ of K_1, and an extension $\tilde{v}$ of v_1 which covers $\hat{G}$.

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Received 01. 03. 1989