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# Sufficient Conditions for Certain Multivalent Functions

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The object of the present paper is to derive some interesting sufficient conditions for $p$-valently convex, starlike, and close-to-convex functions in the unit disk.

## I. Introduction and Preliminaries

Let $\mathscr{A}(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \qquad (p\in\mathscr{N}=\{1,2,3,\dots\})$$

which are analytic in the unit disk $\mathscr{D}=\{z:|z|<1\}$.

A function $f(z)$ belonging to the class $\mathscr{A}(p)$ is said to be $p$-valently convex iff

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathscr{D}).$$

A function $f(z)$ in $\mathscr{A}(p)$ is said to be $p$-valently starlike iff

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in\mathscr{D}).$$

Further, a function $f(z)$ belonging to $\mathscr{A}(p)$ is said to be p-valently close-to-convex iff there exists a $p$-valently starlike function $g(z)$ such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>0 \qquad (z\in\mathscr{D}).$$

Some interesting results for certain multivalent functions were proved by D. A. Patil and N. K. Thakare [3], S. Owa [2], and by M. Nunokawa [1]. In order to derive our results, we need the following lemmas:

**Lemma 1** (Nunokawa [1, Lemma 1]). *Let* $f(z)\in\mathscr{A}(p)$ *and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>k \qquad (z\in\mathscr{D}),$$

*where* $k$ *is a real bounded constant, then we have* $f(z)\neq 0$ *for* $0<|z|<1$.

**Lemma 2.** *Let $f(z) \in \mathscr{A}(p)$ and there exist a $(p-k+1)$-valently starlike function*

$$g(z) = z^{p-k+1} + \sum_{n=p-k+2}^{\infty} b_n z^n$$

*which satisfies*

$$\mathrm{Re}\left\{\frac{zf^{(k)}(z)}{g(z)}\right\} > 0 \qquad (z \in \mathscr{D}),$$

*then $f(z)$ is $p$-valently close to-convex in the unit disk $\mathscr{D}$.*

Proof. Using the result by M. Nunokawa [1, Theorem 8], we see that

$$\mathrm{Re}\left\{\frac{zf'(z)}{G(z)}\right\} > 0 \qquad (z \in \mathscr{D})$$

for a $p$-valently starlike function $G(z)$ in $\mathscr{D}$. This implies that $f(z)$ is $p$-valently close-to-convex in the unit disk $\mathscr{D}$.

**Lemma 3** (Nunokawa [1, Theorem 5]). *Let $f(z) \in \mathscr{A}(p)$ and suppose that*

$$\mathrm{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \qquad (z \in \mathscr{D}).$$

*Then we have*

$$\mathrm{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0 \qquad (z \in \mathscr{D})$$

*for $k = 1, 2, 3, \ldots, p-1$.*

**Lemma 4** (Nunokawa [1, Theorem 1]). *Let $f(z) \in \mathscr{A}(p)$ and suppose*

$$p + \mathrm{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > 0 \qquad (z \in \mathscr{D}).$$

*Then $f(z)$ is $p$-valently convex in $\mathscr{D}$ and*

$$k + \mathrm{Re}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0 \qquad (z \in \mathscr{D})$$

*for $k = 1, 2, 3, \ldots, p-1$.*

## 2. Sufficient conditions for certain multivalent functions

We begin with the statement and the proof of the following result.

**Theorem 1.** *Let $f(z) \in \mathscr{A}(p)$ and suppose that there exists a positive integer $k$ and a $(p-k+1)$-valently starlike function*

$$g(z)=z^{p-k+1}+\sum_{n=p-k+2}^{\infty} b_n z^n$$

*which satisfies*

$$\left|\operatorname{Im}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}-\frac{zg'(z)}{g(z)}\right\}\right| \leqq \frac{\pi}{2}|z| \qquad (z\in\mathscr{D}), \tag{1}$$

*where* $1\leqq k\leqq p$. *Then* $f(z)$ *is p-valently close-to-convex in the unit disk* $\mathscr{D}$.

Proof. Since $g(z)$ is $(p-k+1)$-valently starlike in $\mathscr{D}$, we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\}>0 \qquad (z\in\mathscr{D}).$$

Therefore, using Lemma 1, we see that $g(z)\neq 0$ for $0<|z|<1$. On the other hand, it is trivial that $zg'(z)/g(z)$ is analytic at the origin. After this, we need to prove that $f^{(k)}(z)$ has no zero for $0<|z|<1$.

Suppose that $f^{(k)}(z)$ has a zero of order $m\,(m\geqq 1)$ at a point $\alpha$ that satisfies $0<|\alpha|<1$. Then $f^{(k)}(z)$ can be written as $f^{(k)}(z)=(z-\alpha)^m q(z)$, where $q(z)$ is analytic in $\mathscr{D}$ and $q(\alpha)\neq 0$.

Then it follows that

$$\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}=\frac{mz}{z-\alpha}+\frac{zq'(z)}{q(z)}.$$

Therefore, by a simple calculation, we have

$$\lim_{z\to\alpha}(z-\alpha)\left(\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}-\frac{zg'(z)}{g(z)}\right)$$
$$=\lim_{z\to\alpha}\left(mz+(z-\alpha)\frac{zq'(z)}{q(z)}-(z-\alpha)\frac{zg'(z)}{g(z)}\right)=m\alpha\neq 0.$$

This contradicts to the condition (1), because the condition (1) implies that $zf^{(k+1)}(z)/f^{(k)}(z)-zg'(z)/g(z)$ has no pole for $0<|z|<1$. Thus we know that $f^{(k)}(z)$ does not have any zero for $0<|z|<1$. A simple computation gives that

$$\log\left(\frac{zf^{(k)}(z)}{p(p-1)(p-2)\ldots(p-k+1)g(z)}\right)$$
$$=\int_0^z\left(\frac{1}{t}+\frac{f^{(k+1)}(t)}{f^{(k)}(t)}-\frac{g'(t)}{g(t)}\right)dt$$
$$=\int_0^r\left(1+\frac{tf^{(k+1)}(t)}{f^{(k)}(t)}-\frac{tg'(t)}{g(t)}\right)\frac{ds}{s},$$

where $z=re^{i\theta}$, $t=se^{i\theta}$, and $0\leqq s\leqq r<1$. Therefore, it follows that

$$\arg\left(\frac{zf^{(k)}(z)}{g(z)}\right)=\operatorname{Im}\left\{\log\left(\frac{zf^{(k)}(z)}{p(p-1)(p-2)\ldots(p-k+1)g(z)}\right)\right\}$$

$$=\int_0^r \mathrm{Im}\left(1+\frac{tf^{(k+1)}(t)}{f^{(k)}(t)}-\frac{tg'(t)}{g(t)}\right)\frac{ds}{s}$$

$$=\int_0^r \mathrm{Im}\left(\frac{tf^{(k+1)}(t)}{f^{(k)}(t)}-\frac{tg'(t)}{g(t)}\right)\frac{ds}{s}.$$

From the assumption (1), we obtain

$$\left|\arg\left(\frac{zf^{(k)}(z)}{g(z)}\right)\right|\leqq\int_0^r\left|\mathrm{Im}\left(\frac{tf^{(k+1)}(t)}{f^{(k)}(t)}-\frac{tg'(t)}{g(t)}\right)\right|\frac{ds}{s}$$

$$=\int_0^r \frac{\pi}{2}ds=\frac{\pi}{2}r<\frac{\pi}{2}.$$

This shows that

$$\mathrm{Re}\left\{\frac{zf^{(k)}(z)}{g(z)}\right\}>0 \qquad (z\in\mathscr{D}).$$

Then, using Lemma 2, we have that $f(z)$ is $p$-valently close-to-convex in the unit disk $\mathscr{D}$. This completes the assertion of Theorem 1.

**Corollary 1.** *Let $f(z)\in\mathscr{A}(p)$ and suppose that there exists a positive integer $k$ such that*

$$\left|\mathrm{Im}\left(\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}-\frac{2(p-k+1)r\sin\theta}{1-2r\cos\theta+r^2}\right)\right|\leqq\frac{\pi}{2}r$$

*for $z\in\mathscr{D}$, where $1\leqq k\leqq p$, $z=re^{i\theta}$, and $0\leqq\theta<2\pi$. Then $f(z)$ is $p$-valently close-to-convex in the unit disk $\mathscr{D}$.*

Proof. Noting that the function $g(z)=z^{p-k+1}/(1-z)^{2(p-k+1)}$ is $(p-k+1)$-valently starlike in $\mathscr{D}$, and that

$$\mathrm{Im}\left(\frac{zg'(z)}{g(z)}\right)=\frac{2(p-k+1)r\sin\theta}{1-2r\cos\theta+r^2}.$$

Thus, our conclusion follows from Theorem 1.

Next we prove

**Theorem 2.** *Let $f(z)\in\mathscr{A}(p)$ and suppose that*

$$\left|\mathrm{Im}\left(\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)\right|\leqq\frac{\pi}{2}|z| \qquad (z\in\mathscr{D}). \tag{2}$$

*Then $f(z)$ is $p$-valently starlike in the unit disk $\mathscr{D}$.*

Proof. We first prove that $f^{(p-1)}(z)\neq0$ for $0<|z|<1$. Suppose that $f^{(p-1)}(z)$ has a zero of order $m\,(m\geqq1)$ at a point $\alpha$ $(0<|\alpha|<1)$. Then $f^{(p-1)}(z)$ can be written as $f^{(p-1)}(z)=(z-\alpha)^m q(z)$, where $q(z)$ is analytic in $\mathscr{D}$ and $q(\alpha)\neq 0$. Then, we see that

$$\lim_{z\to\alpha}(z-\alpha)\left(\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)$$

$$=\lim_{z\to\alpha}\left\{z\frac{m(m-1)q(z)+2m(z-\alpha)q'(z)+(z-\alpha)^2q''(z)}{mq(z)+(z-\alpha)q'(z)}-mz-\frac{zq'(z)}{q(z)}(z-\alpha)\right\}$$

$$=(m-1)\alpha-m\alpha$$

$$=-\alpha\neq 0.$$

This contradicts to the condition (2), because the condition (2) shows that

$$\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$$

has no pole for $0<|z|<1$. Thus $f^{(p-1)}(z)$ can not have any zero for $0<|z|<1$.

Next, we need to prove that $f^{(p)}(z)$ has no zero for $0<|z|<1$. Suppose that $f^{(p)}(z)$ has a zero of order $m$ $(m\geqq 1)$ at a point $\beta$ $(0<|\beta|<1)$. Then $f^{(p)}(z)$ can be written as

(3) $$f^{(p)}(z)=(z-\beta)^m g(z),$$

where $g(z)$ is analytic in $\mathscr{D}$ and $g(\beta)\neq 0$. In this case, if $f^{(p-1)}(z)$ does not become zero at a point $\beta$, then we have

$$\lim_{z\to\beta}(z-\beta)\left(\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)$$

$$=\lim_{z\to\beta}\left(mz+(z-\beta)\frac{zg'(z)}{g(z)}-(z-\beta)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)=m\beta\neq 0.$$

Also, this contradicts to the condition (2), because the condition (2) implies that

$$\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$$

has no zero for $0<|z|<1$. Therefore, in this case, $f^{(p)}(z)$ has no zero for $0<|z|<1$.

On the other hand, if $f^{(p)}(z)$ has a zero of order $m$ $(m\geqq 1)$ at a point $z=\beta$, and, at the same time, $f^{(p-1)}(z)$ has a zero of order $n$ $(n\geqq 1)$ at a point $z=\beta$, then $f^{(p-1)}(z)$ can be written as

(4) $$f^{(p-1)}(z)=(z-\beta)^n h(z),$$

where $h(z)$ is analytic in $\mathscr{D}$ and $h(\beta)\neq 0$. With the aid of (3) and (4), we have

(5) $$f^{(p)}(z)=(z-\beta)^{n-1}\{nh(z)+(z-\beta)h'(z)\}$$

$$=(z-\beta)^m g(z).$$

It follows from (5) that $nh(\beta)\neq 0$ and $g(\beta)\neq 0$. This shows that $m=n-1$ or $n=m+1$. Therefore, by a simple calculation, we obtain

$$\lim_{z\to\beta}(z-\beta)\left(\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)$$

$$=\lim_{z\to\beta}\left(mz+(z-\beta)\frac{zg'(z)}{g(z)}-(m+1)z-(z-\beta)\frac{zh'(z)}{h(z)}\right)-\beta\neq 0,$$

which contradicts to the condition (2). Consequently, in any case, $f^{(p)}(z)$ has no zero for $0<|z|<1$.

Noting that

$$\log\left(\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)=\int_0^z\left(\frac{1}{t}+\frac{f^{(p+1)}(t)}{f^{(p)}(t)}-\frac{f^{(p)}(t)}{f^{(p-1)}(t)}\right)dt$$

$$=\int_0^r\left(1+\frac{tf^{(p+1)}(t)}{f^{(p)}(t)}-\frac{tf^{(p)}(t)}{f^{(p-1)}(t)}\right)\frac{ds}{s},$$

where $z=re^{i\theta}$, $t=se^{i\theta}$, and $0\leqq s\leqq r<1$, we have

$$\arg\left(\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)=\operatorname{Im}\left\{\log\left(\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)\right\}$$

$$=\int_0^r\operatorname{Im}\left(1+\frac{tf^{(p+1)}(t)}{f^{(p)}(t)}-\frac{tf^{(p)}(t)}{f^{(p-1)}(t)}\right)\frac{ds}{s}$$

$$=\int_0^r\operatorname{Im}\left(\frac{tf^{(p+1)}(t)}{f^{(p)}(t)}-\frac{tf^{(p)}(t)}{f^{(p-1)}(t)}\right)\frac{ds}{s}.$$

Hence, with the help of the assumption (2), we obtain

$$\left|\arg\left(\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)\right|\leqq\int_0^r\left|\operatorname{Im}\left(\frac{tf^{(p+1)}(t)}{f^{(p)}(t)}-\frac{tf^{(p)}(t)}{f^{(p-1)}(t)}\right)\right|\frac{ds}{s}$$

$$\leqq\int_0^r\frac{\pi}{2}ds<\frac{\pi}{2}.$$

This proves that

$$\operatorname{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}>0\qquad(z\in\mathscr{D}).$$

Finally, applying Lemma 3, we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0\qquad(z\in\mathscr{D}),$$

which shows that $f(z)$ is $p$-valently starlike in $\mathscr{D}$.

**Theorem 3.** *Let $f(z)\in\mathcal{A}(p)$ and suppose that*

$$\left|\operatorname{Im}\left(\frac{(p+1)zf^{(p+1)}(z)+z^2f^{(p+1)}(z)}{pf^{(p)}(z)+zf^{(p+1)}(z)}-\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right|\leqq\frac{\pi}{2}|z| \tag{6}$$

*for $z\in\mathcal{D}$. Then $f(z)$ is $p$-valently convex in $\mathcal{D}$.*

Proof. Applying the same method as in the proof of Theorem 2, and from the assumption (6), we obtain

$$\left|\arg\left(\frac{pf^{(p)}(z)+zf^{(p+1)}(z)}{pf^{(p)}(z)}\right)\right|=\left|\arg\left(p+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right|$$
$$<\frac{\pi}{2}$$

for $z\in\mathcal{D}$. Therefore, it follows that

$$p+\operatorname{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}>0 \qquad (z\in\mathcal{D}).$$

Thus, using Lemma 4, we have

$$1+\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathcal{D}),$$

which completes the proof of Theorem 3.

Finally, applying the same method as in the proof of Theorem 2, and using Lemma 3, we have

**Theorem 4.** *Let $f(z)\in\mathcal{A}(p)$ and suppose that*

$$\left|\operatorname{Im}\left(\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)\right|\leqq\frac{\pi}{2}|z| \qquad (z\in\mathcal{D}).$$

*Then $f(z)$ is $p$-valently starlike in the unit disk $\mathcal{D}$.*

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