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Topologies on Classes

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1. Introduction

In this paper we shall study the problem of introducing topologies on proper classes. Similar problem was considered by P. Vopěnka [4] and Chudachek in alternative set theory. Namely, there were introduced such notions as "a closure of a class", etc. However, we remind the reader that the alternative set theory is not an extension of Cantorian set theory (i.e. ZF set theory). In the following, we shall assume NBG system, or, occasionally, Morse theory of classes which are in fact extensions of ZF. There are natural examples of topological class-spaces, as we shall show. Secondly, there might be some applications of such constructions in the foundation of set theory and topology.

In building notions related to class-spaces, it is not possible to transfer in the straightforward way definitions and constructions from classical topology. The reason is that many notions and constructions in classical topology are complementary, i.e. one has to use the set-theoretical complement operation. This problem can be avoided in many cases defining separately classes of open subsets and classes of closed subsets of a space, as it is indicated for standard spaces in the following theorem.

Theorem 1.1. *Assume topological spaces $\mathscr{X}_1$ and $\mathscr{X}_2$ are given by $\mathscr{X}_1=(X, \tau)$, $\mathscr{X}_2=(X, \sigma)$, where τ is the set of open subsets of $\mathscr{X}_1$, and σ is the set of all closed subsets of $\mathscr{X}_2$. If for all $x\in\tau$, $y\in\sigma$ we have $x-y\in\tau$ and $y-x\in\sigma$, then τ is the set of all open subsets of $\mathscr{X}_2$ and σ is the set of all closed subsets of $\mathscr{X}_1$.*

Proof. In the following, for any $Y\subseteq X$, Y^c stands for the complement of Y in respect of X. Let τ_i, $i=1, 2$ be the set of all open sets in $\mathscr{X}_i$. Then

$$U\in\tau_1,\ V\in\tau_2 \Rightarrow U\in\tau,\ V^c\in\sigma \Rightarrow U-V^c\in\tau \Rightarrow U\cap V\in\tau.$$

Taking $U=X$ we find $V\in\tau_2\Rightarrow V\in\tau$, i.e.

$$(1) \qquad \tau_2\subseteq\tau_1.$$

Further, assume $U\in\tau_1$ and $V\in\tau_2$, and choose $A, B\subseteq X$ such that $U=A^c$, $V=B^c$. Then $A^c\in\tau$, and $B\in\sigma$, so $B-A^c\in\sigma$. Therefore, $A\cap B\in\sigma$, i.e. $(U^c\cap V^c)^c\in\tau_2$. Thus we proved the implication

$$U\in\tau_1,\ V\in\tau_2 \Rightarrow U\cup V\in\tau_2.$$

Taking $V=\emptyset$, we find that $U\in\tau_1$ implies $U\in\tau_2$, i.e.

(2) $$\tau_1\subseteq\tau_2.$$

Therefore, by (1) and (2) we have $\tau_1=\tau_2$. ■

According to this theorem we may define the notion of a topological space as a triple (X, τ, σ), where τ satisfies the usual axioms for the set of open subset, and σ satisfies the axioms for closed subsets with the additional condition:

(C) $$U\in\tau,\ V\in\sigma \Rightarrow U-V\in\tau,\ V-U\in\sigma.$$

We remark that this approach may be useful for studying topology on the basis of intuitionism, first because the notions of open and closed subsets are independently defined (i.e. they are not complementary notions), and secondly, for differences in (C) we may take the relative complement.

2. Topological class-spaces

In this section we shall define topologies on classes. First, we introduce some notion and terminology. By capital letters $X, Y, Z, \ldots$ we denote classes, and by x, y, z sets. Greek letters may stand both for classes and for sets. For our metatheory we shall take NBG class theory if not otherwise stated. Further, we shall assume the usual constructions and definitions from set theory and class theory, e.g. $(x, y)=\{\{x\}, \{x, y\}\}$, $(X, Y)=X\times\{0\}\cup Y\times\{1\}$, V is the class of all sets, ORD is the class of all ordinals, CARD is the class of all cardinal numbers, etc.

Now we shall state the axioms for topologies on classes. Let K be a class, and τ, σ two classes of subsets of K. We call the triple (K, τ, σ) a topological class-space iff the following are satisfied:

1. $x,\ y\in\tau \Rightarrow x\cap y\in\tau$.
2. For any i, and $\langle x_j | j\in i\rangle$, $(\forall j\in i\ x_j\in\tau) \Rightarrow \cup_j x_j\in\tau$.
3. For any $a\in K$ there is $x\in\tau$ such that $a\in x$.
4. $\forall x\in\tau\forall y\in\sigma\ x-y\in\sigma$.

1'. $x,\ y\in\sigma \Rightarrow x\cup y\in\sigma$.
2'. For any i, and $\langle x_j | j\in i\rangle$, $(\forall j\in i\ x_j\in\sigma) \Rightarrow \cap_j x_j\in\sigma$.
3'. For any subset x of K there is $y\in\sigma$ such that $x\subseteq y$.
4'. $\forall x\in\tau\forall y\in\sigma\ y-x\in\tau$.

Example 2.1. 1. Discrete class-space on V, (V, V, V).
2. Let for any $\alpha\in\mathrm{ORD}$, τ_α be the set of all open subsets, and σ_α the set of all closed subsets of α in respect to the order topology of α. As $\alpha\leqq\beta$ implies $\tau_\alpha\subseteq\tau_\beta$, and $\sigma_\alpha\subseteq\sigma_\beta$, it is easy to see that for $\tau=\cup_{\alpha\in\mathrm{ORD}}\tau_\alpha$, $\sigma=\cup_{\alpha\in\mathrm{ORD}}\sigma_\alpha$, $(\mathrm{ORD}, \tau, \sigma)$ is a class-space.
3. Same as 2, but taking CARD instead of ORD.
4. Let u be any set-topological space, τ_u be the set of all open and σ_u the set of all closed subsets of u. For any class K let us define

$$\tau=\{x\subseteq K|x\cap u\in\tau_u\},\quad \sigma=\{x\subseteq K|x\cap u\in\sigma_u\}.$$

Then (K,τ,σ) is a topological class-space.

5. While the Axiom 3 is obvious (as we need that every point has a neighborhood), in order to justify 3′ consider the following example. Let a be any set and

$$\tau=\{\{a\}\cup x|x\in V\},\quad \sigma=\{x|a\notin x\}.$$

Then it is easy to see that all the axioms 1-4 and 1′, 2′, 4′ are satisfied on V but not the Axiom 3′. If we assume the usual definition of the closure of a set, in so defined space the closure of $\{a\}$ is V, i.e. the closure of $\{a\}$, is not a set.

Let (K,τ,σ) be a class-space and $x\subseteq K$. Then we define restrictions of τ and σ to x as follows:

$$\tau|x=\{y\cap x|y\in\tau\},\quad \sigma|x=\{y\cap x|y\in\sigma\}.$$

Now we shall see that class-space induce a topology on every $x\subseteq K$.

Proposition 2.2. *Let $K=(K,\tau,\sigma)$ be any class-space. Then*
1. *For any $x\subseteq K$, $(x,\tau|x,\sigma|x)$ is a set-space.*
2. $\tau=\cup_{x\subseteq K}\tau|x$, $\sigma=\cup_{x\subseteq K}\sigma|x$.

Proof. It is easy to see that $(x,\tau|x)$ and $(x,\sigma|x)$ are topological spaces under usual axioms for open (closed) subsets. Further, let $u\in\tau|x$, and $v\in\sigma|x$. Then there are $a\in\tau$ and $b\in\sigma$ such that $u=x\cap a$ and $v=x\cap b$. Therefore,

$$u-v=(x\cap a)-(x\cap b)=x\cap(a-b),$$

but $a-b\in\tau$, hence $u-v\in\tau|x$. In a similar way we can prove that $v-u\in\sigma|x$, so by Theorem 1.1. the proof is finished. ■

As in the case of topological set-spaces, we can define most of usual notions, and perform topological constructions as well. Objects introduced in this way satisfy in general the expected properties. For example, for a topological class-space $\mathscr{K}=(K,\tau,\sigma)$ we have:

(1) *Every point of $\mathscr{K}$ has a neighborhood.*

(2) *$u\in\tau$ iff u is a neighborhood of every $a\in u$.*

(3) *For every $x\in K$ there is $u\in\tau$ such that $x\subseteq u$.*

For example, let us prove (3). We have $\forall y\in x\,\exists\, v\in\tau\ y\in v$. So for each $y\in x$ there is $v_y\in\tau$ such that $y\in v_y$. Then for $u=\cup_{y\in x}v_y$ we have $u\in\tau$ and $x\subseteq u$.

For $x\subseteq K$ let us define $\bar{x}=\cap\{y\in\sigma|x\subseteq y\}$. The set $\bar{x}$ is well defined as for every $x\subseteq K$ there is $y\in\sigma$ such that $x\subseteq y$. Observe that $\bar{x}\in\sigma$.

Proposition 2.3. $\bar{x}=\{y|$ *every neighborhood of y intersects x*$\}$.

Proof. Suppose $y\in\bar{x}$, i.e. y belongs to every closed u such that $x\subseteq u$. Further, suppose that there is $v\in\tau$ such that $y\in v$ and $v\cap x=\emptyset$. But then $\bar{x}-v\in\sigma$ and $x\subseteq\bar{x}-v$, so $y\in\bar{x}-v$, a contradiction. Therefore, we proved

(1) $\qquad \bar{x}\subseteq\{y|$ every neighborhood of y intersects $x\}$.

Now, suppose that every neighborhood of y intersects x, and assume there is a closed set c such that $x \subset c$ and $y \notin c$. Taking any open neighborhood v of y, we have that $v-c$ is still a neighborhood of y, but $v-c$ does not intersect x, and this is a contradiction. Therefore, we proved

(2) $\qquad \{y \mid \text{ every neighborhood of } y \text{ intersects } x\} \subseteq \bar{x},$

and this finishes the proof. ■

We shall use later the following proposition.

Proposition 2.4. *Let $\mathscr{X}=(X,\tau,\sigma)$ be a class-space, and $f \subseteq X$. Then*
1. *f is closed iff $\forall y \notin f \exists v \in \tau\ (y \in v \wedge v \cap f = \emptyset)$.*
2. *If f is closed and $a \cap f = \emptyset$, $a \subseteq X$, then there is an open v such that $a \subseteq v$ and $v \cap f = \emptyset$.*

Proof. 1. ($\Rightarrow$) Suppose f is closed and assume $y \notin f$. Then there is $u \in \tau$ such that $y \in u$. For $v=u-f$ it follows that $v \in \tau$, $y \in v$, and $v \cap f = \emptyset$.
($\Leftarrow$) If f is not closed then $\bar{f} \neq f$, so there is $y \in \bar{f}-f$. But by the previous proposition y violates the righthand side of 1.
2. Suppose f is closed and assume $a \subseteq X$, $a \cap f = \emptyset$. For each $x \in a$ there is an open v_x such that $x \in v_x$ and $v_x \cap f = \emptyset$. Then for $v = \cup_{x \in a} v_x$, $a \subseteq v$ and $v \cap f = \emptyset$. ■

If X and Y are classes, recall that a map from X into Y is every class $F \subseteq X \times Y$ such that $\forall x \in X \exists_1 y \in Y\ (x,y) \in F$. If $\mathscr{X}$ and $\mathscr{Y}$ are class-spaces, we can define in a natural way the notion of continuity of maps from $\mathscr{X}$ into $\mathscr{Y}$.

Definition. *Let $\mathscr{X}$ and $\mathscr{Y}$ be class-spaces. A map $F: X \to Y$ is continuous iff the restriction $F|u$ to every $u \in \tau_X$ is a continuous function.*

It is easy to see that F is continuous if for all $u \in \tau_X$, $v \in \tau_Y$, we have $\{x \in u \mid F(x) \in v\} \in \tau_X$. For example, we shall see that projection maps $\pi_1: X \times Y \to X$, $\pi_2: X \times Y \to Y$, are continuous in respect to product topology $\mathscr{X} \times \mathscr{Y}$.

We can transfer many constructions from classical topology to class-spaces. For example, the product of two class-spaces $\mathscr{K}_i=(K_i,\tau_i,\sigma_i)$, $1 \leqq i \leqq 2$, is the class-space $\mathscr{K}=(K,\tau,\sigma)$, where $K=K_1 \times K_2$, and the basis for τ and σ are the classes

$$\tau_0=\{\pi_1^{-1}(u) \cap \pi_2^{-1}(v) \mid\ u \in \tau_1,\quad v \in \tau_2\},$$

$$\sigma_0=\{\pi_1^{-1}(u) \cap \pi_2^{-1}(v) \mid\ u \in \sigma_1,\quad v \in \sigma_2\}.$$

Here, π_1 and π_2 are projection maps from K to K_1 and K_2 respectively.

3. Compact class-spaces

The following proposition will enable us to introduce the notion of compact class-space.

Proposition 3.1. *Let $\mathscr{X}$ be a Hausdorff class-space. Then the following are equivalent:*
1. *Every set of subsets of $\mathscr{X}$ which has the finite intersection property has the empty intersection.*

2. *Every closed subspace of* $\mathscr{X}$ *is compact.*

Proof. ($1 \Rightarrow 2$) Suppose 1 and let be $y \in \sigma$. If f_i, $i \in s$ is a family of open subsets of a subspace y such that $y = \cup_{i \in s} f_i$, then there are $g_i \in \tau$ so that $f_i = y \cap g_i$. Therefore, $y \subseteq \cup_{i \in s} g_i$, i.e. $\cap_{i \in s}(y - g_i) = \emptyset$. As for all $i \in s$ we have $y - g_i \in \sigma$, by our assumption there is a finite $t \subseteq s$ such that $\cap_{i \in t}(y - g_i) = \emptyset$. Then $\{f_i | \ i \in t\}$ is a finite cover of y.
($2 \Rightarrow 1$) Suppose 2, and let $f = \{f_i | \ i \in s\}$ be a set of closed subsets of $\mathscr{X}$ with finite intersection property. Assume $\cap_{i \in s} f_i = \emptyset$, and let $x \in \sigma$ be such that $\cup_{i \in s} f_i \subseteq x$. Then for $u_i = x - f_i$, u_i is an open set in x (since $f_i = f_i \cap x$ is a closed subset of subspace x), so u_i, $i \in s$, is an open cover of x, thus by the compactness of x, there is a finite $t \in s$ such that $x = \cup_{i \in t} u_i$. Hence

$$x = \cup_{i \in t}(x - f_i) = x - \cap_{i \in t} f_i,$$

i.e. $\cap_{i \in t} f_i = \emptyset$, contradicting our assumption that f has the finite intersection property. ■

Therefore we can define a class-space $\mathscr{X}$ to be compact iff $\mathscr{X}$ has either of the listed properties 1 and 2 in the last proposition. An example of a compact class-space is the order topology on ORD. Really, every closed subset x of ORD is a closed subset of some $\alpha \in \mathrm{ORD}$ in respect to the order topology of α, but it is well-known that this topology makes α a compact space, thus x is compact as well.

Let us remark that if $\mathscr{X}$ is compact, and $Y \subseteq X$ is a class of closed subsets with finite intersection property, then $\cap Y \neq \emptyset$. To see this, let $Y = \{v_\alpha | \ \alpha \in \mathrm{ORD}\}$. We can list elements of Y in this way if we assume the Global Choice. Recall that the Global Choice holds in the constructible universe, therefore it is consistent with NBG system. Further, let us define the sequence s_α by $s_\alpha = \cap_{\beta \in \alpha} v_\beta$, $\alpha \in \mathrm{ORD}$, $\alpha > 0$, $s_0 = v_0$. Then for each $\alpha \in \mathrm{ORD}$, $s_\alpha \neq \emptyset$, and $s_0 \supseteq s_1 \supseteq \cdots \supseteq s_\alpha \supseteq \cdots$. If $\cap Y = \emptyset$, then we would have a subsequence s_{α_ξ}, $\xi \in \mathrm{ORD}$, which is strictly decreasing, and this is a contradiction, as otherwise s_0 would contain a proper class.

As in the case of set-spaces it is possible to establish a characterization of compact spaces in terms of cartesian products. For this we need the following lemma, a variation of Lemma 3.1.15 in [2].

Lemma 3.2. *If a is a compact subspace of a class-space $\mathscr{X}$, and y is a point of a class-space $\mathscr{Y}$, then for every open $w \subseteq X \times Y$ containing $a \times \{y\}$ there exist open sets $u \subseteq X$ and $v \subseteq Y$ such that*

$$a \times \{y\} \subseteq u \times v \subseteq w.$$

The proof of this lemma is same as in [2], so we omit it. Now we state and prove a variant of Kuratowski theorem for compact class-spaces, cf. 3.1.16. in [2].

The Kuratowski theorem. *For a Hausdorff class-space $\mathscr{X}$ the following are equivalent*:
1. $\mathscr{X}$ *is compact.*
2. *For every class-space $\mathscr{Y}$, the projection $\pi_Y : X \times Y \to Y$ is closed.*
3. *For every normal class-space $\mathscr{Y}$ the projection $\pi_Y : X \times Y \to Y$ is closed.*

P r o o f. ($1 \Rightarrow 2$) Let X be compact and $f \subseteq X \times Y$ be closed. Further, let v be a closed subset of X such that $\pi_X(f) \subseteq v$, and suppose $y \notin \pi_Y(f)$. As $v \times \{y\} \cap f = \emptyset$ by Proposition 2.4. there is an open subset u of $X \times Y$ such that $u \cap f = \emptyset$ and $v \times \{y\} \subseteq u$. Then by Lemma 3.2. there are $u_1 \in \tau_X$, $u_2 \in \tau_Y$ such that $v \times \{y\} \subseteq u_1 \times u_2 \subseteq u$, thus $u_1 \times u_2 \cap f = \emptyset$. If there is $b \in \pi_Y(f) \cap u_2$ then there is an element a such that $(a, b) \in f$. Then we would have $a \notin u_1$ and $a \in \pi_X(f)$, so $a \in v$, and therefore $a \in u_1$, but this is a contradiction. Hence $\pi_Y(f) \cap u_2 = \emptyset$, i.e. u_2 is a neighborhood of y which does not intersects $\pi_Y(f)$, so by Proposition 2.4. $\pi_Y(f)$ is a closed subset of Y.
($2 \Rightarrow 3$) This part is obvious.
($3 \Rightarrow 1$) The proof of this assertion is an adaptation of the proof of the Theorem 3.1.16. part ($3 \Rightarrow 1$) in [2]. Namely, if $\{f_i | \ i \in s\}$ is a set of closed subsets with finite intersection property, then we choose a closed subset $x \supseteq \cup_i f_i$ and the subspace topology on x. Taking x for the ground space in the mentioned proof in [2], $y_0 \in X - x$, $Y = x \cup \{y_0\}$, and restriction of p_Y to $x \times y$, one can obtain the proof of this part. ■

Corollary. *If $\mathscr{X}$ and $\mathscr{Y}$ are compact spaces, then $\mathscr{X} \times \mathscr{Y}$ is a compact space*

P r o o f. Projections $\pi_X : X \times Y \times Z \to Y \times Z$, $\pi_Y : Y \times Z \to Y$ are closed mappings in respect to appropriate product topologies, so the composition π of π_X and π_Y, $\pi : X \times Y \times Z \to Y$ is closed, too, therefore by the previous theorem $\mathscr{X} \times \mathscr{Y}$ is compact. Iterating the construction finitely many times, one can obtain that the product of finitely many compact spaces is a compact class-space. ■

Finally, it would be interesting to see which other properties and notions of standard topological spaces can be transferred to class-spaces. For example it is easy to code a family of classes Y_i where i runs through a set s by a class Y, and then to define a class-product ΠY of such a family. However there is no obvious way for defining the product topology on ΠY if s is infinite.

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