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Lebesgue-Type Decompositions of Pairs of Commuting Contractions

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1. Introduction

Let H be a complex separable Hilbert space. By $B(H)$ we denote the algebra of all bounded linear operators on H. If $T \in B(H)$ is a contraction (i.e. $\|T\| \leqq 1$), then H can be decomposed into an orthogonal sum $H = H_0 \oplus H_1$ of two subspaces reducing T such that $T|H_0$ is unitary and $T|H_1$ is completely non-unitary (c.n.u.) ([NF], Thm. I.1.1.).

Let us denote by Alg T the smallest weakly closed subalgebra of $B(H)$ containing T and the identity operator I. The disadvantage of Wold-type decomposition is that Alg T does not split into the orthogonal sum Alg $T_0 \oplus$ Alg T_1.

To see this, it suffices to consider $T = U \oplus S$, where U is the bilateral shift and S is the unilateral shift, both of multiplicity 1.

If $p_n(U \oplus S)$ converged weakly to $0 \oplus S$ for a net of polynomials p_n, then $p_n(S)$ would converge to S, and $p_n(U)$ would converge to 0. But it is impossible because S is the restriction of U to an invariant subspace.

There is another natural decomposition of a contraction $T \in B(H)$ which does not have the above mentioned disadvantage:

1.1. Definition (cf. [Z2]). A contraction $T \in B(H)$ is called absolutely continuous (a.c.) or singular unitary (s.u.) if the spectral measure of its minimal unitary dilation is a.c. or singular, respectively, with respect to the Lebesgue measure (on the unit circle).

All c.n.u. contractions are a.c. ([NF], Thm. II.6.4.). Decomposing the unitary part of the contraction $T \in B(H)$ we can easily obtain another decomposition (see e.g. [Z1], [Z2] and references quoted there):

1.2. Definition. Let $T \in B(H)$ be a contraction. Then $H = H_a \oplus H_s$, where H_a, H_s are subspaces reducing T. $T|H_a$ is a.c., $T|H_s$ is s.u. This decomposition is uniquely determined and it will be called the Lebesgue-type decomposition of T.

2. Decomposition I (of pairs of commuting contractions)

Several authors (e.g. [HM1], [HM2], [K2], [S1], [S2]) studied pairs of commuting contractions. The Wold-type decomposition for pairs T_1, T_2 of doubly commuting (i.e. $T_1T_2=T_2T_1$, $T_1T_2^*=T_2^*T_1$) contractions was due to M. Słocinski [S2]. A Lebesgue-type decomposition is constructed in [K2] for any pair of commuting contractions. We shall present in this section another Lebesgue-type decomposition:

Theorem 1. *Let T_1, $T_2 \in B(H)$ be contractions and let $T_1T_2=T_2T_1$. Then H decomposes into an orthogonal sum of four subspaces reducing both T_1 and T_2:*

$$H=H_1 \oplus H_2 \oplus H_3 \oplus H_4 \tag{2.1}$$

and such that

$$T_1|H_1,\ T_1|H_2,\ T_2|H_1,\ T_2|H_3 \text{ are a.c.} \tag{2.2}$$

$$T_1|H_3,\ T_1|H_4,\ T_2|H_2,\ T_2|H_4 \text{ are s.u.} \tag{2.3}$$

The decomposition (2.1) *is uniquely determined by* (2.2), (2.3).

2.4. Remark. Theorem 1 can be deduced from a more general theorem (M2, Thm. 2.1), but we give below an elementary proof omitting the technique of representations of function algebras.

Proof. Let us consider the Lebesgue-type decomposition of T_1: $H=H_a \oplus H_s$, $T_1=T_1|H_a \oplus T_1|H_s$. It was proved in [Z1] that the commutant $\{T_1\}$ of T_2 splits into $\{T_1|H_a\} \oplus \{T_1|H_s\}$. This means that H_a, H_s are invariant under T_2 as well. Now we consider the Lebesgue-type decomposition for operators $T_2|H_a$, $T_2|H_s$:

$$H_a=H_{aa} \oplus H_{as},\quad H_s=H_{sa} \oplus H_{ss}.$$

Again by [Z1] the subspaces

$$H_1=H_{aa},\quad H_2=H_{as},\quad H_3=H_{sa},\quad H_4=H_{ss}$$

are reducing for both T_1 and T_2 and we obtain the desired decomposition (2.1) satisfying (2.1), (2.3).

If there was another decomposition $H=H'_1 \oplus H'_2 \oplus H'_3 \oplus H'_4$ satisfying (2.2), (2.3), then $T_1|(H'_1 \oplus H'_2)$, $T_2|(H'_1 \oplus H'_3)$, would be the a.c. parts of T_1 and T_2 respectively, and $T_1|(H'_3 \oplus H'_4)$, $T_2|(H'_2 \oplus H'_4)$ would be s.u. parts of T_1 and T_2 respectively. Because the Lebesgue-type decomposition of a single contraction is uniquely determined, $H_1 \oplus H_2=H'_1 \oplus H'_2$, $H_3 \oplus H_4=H'_3 \oplus H'_4$.

To finish the proof it suffices to observe that $H'_1 \oplus H'_2$, $H'_3 \oplus H'_4$ are Lebesgue-type decompositions of $T_2|(H'_1 \oplus H'_2)$, $T_2|(H'_3 \oplus H'_4)$ respectively. The uniqueness of the Lebesgue-type decompositions of $T_2|(H_1 \oplus H_2) = T_2|(H'_1 \oplus H'_2)$ and $T_2|(H_3 \oplus H_4)=T_2|(H'_3 \oplus H'_4)$ implies $H_i=H'_i$ $(i=1,2,3,4)$.

Theorem 2. *Let T_1, T_2 be a pair of commuting contractions. Let*

(2.5) $H=H_1\oplus H_2\oplus H_3\oplus H_4,\ T_{1i}=T_1|H_i,\ T_{2i}=T_2|H_i,\ i=1,2,3,4.$

be its Lebesgue-type decomposition given by theorem 1. *Then*

$$\mathrm{Alg}(T_1,T_2)=\mathrm{Alg}(T_{11},T_{21})\oplus\mathrm{Alg}(T_{12},T_{22})\oplus\mathrm{Alg}(T_{13},T_{23})\oplus\mathrm{Alg}(T_{14},T_{24}).$$

Proof. The inclusion

$$\mathrm{Alg}(T_1,T_2)\subset\mathrm{Alg}(T_{11},T_{21})\oplus\mathrm{Alg}(T_{12},T_{22})\oplus\mathrm{Alg}(T_{13},T_{23})\oplus\mathrm{Alg}(T_{14},T_{24})$$

is obvious. To prove the opposite inclusion it suffices to show that the operators

$$P_1=I\oplus 0\oplus 0\oplus 0,\qquad P_2=0\oplus I\oplus 0\oplus 0,$$
$$P_3=0\oplus 0\oplus I\oplus 0,\qquad P_4=0\oplus 0\oplus 0\oplus I$$

and

$$A_{11}=T_{11}\oplus 0\oplus 0\oplus 0,\qquad A_{21}=T_{21}\oplus 0\oplus 0\oplus 0,$$
$$A_{12}=0\oplus T_{12}\oplus 0\oplus 0,\qquad A_{22}=0\oplus T_{22}\oplus 0\oplus 0,$$
$$A_{13}=0\oplus 0\oplus T_{13}\oplus 0,\qquad A_{23}=0\oplus 0\oplus T_{23}\oplus 0,$$
$$A_{14}=0\oplus 0\oplus 0\oplus T_{14},\qquad A_{24}=0\oplus 0\oplus 0\oplus T_{24}$$

belong to $\mathrm{Alg}(T_1,\ T_2)$.

Consider again the Lebesgue-type decompositions of T_1 and T_2:

$$T_1=T_{1a}\oplus T_{1s},\qquad T_2=T_{2a}\oplus T_{2s}.$$

Note that $T_{1a}=T_{11}\oplus T_{12}$, $T_2=T_{21}\oplus T_{23}$.
By [Z1] we have $\mathrm{Alg}(T_i)=\mathrm{Alg}(T_{ia})\oplus\mathrm{Alg}(T_{is})$ for $i=1,2$ which implies

$$P_1\oplus P_2\in\mathrm{Alg}(T_1)\subset\mathrm{Alg}(T_1,T_2)$$
$$P_1\oplus P_3\in\mathrm{Alg}(T_2)\subset\mathrm{Alg}(T_1,T_2)$$

and so $P_1=(P_1\oplus P_2)(P_1\oplus P_3)\in\mathrm{Alg}(T_1,T_2)$. Analogously we obtain

$$P_i\in\mathrm{Alg}(T_1,T_2),\qquad i=2,3,4.$$

To finish the proof note that $A_{ij}=T_iP_j$ $(i=1,2;\ j=1,2,3,4)$ also belong to $\mathrm{Alg}(T_1,T_2)$.

3. **Decomposition II.** In this section we recall the Lebesgue-type decomposition from [K2] and show that it implies also a decomposition of weakly closed algebras generated by a pair of commuting contractions. We will also compare this decomposition with that described in section 2.

Let us introduce now the concept of absolute continuity and singularity of pairs of commuting contractions. Denote by T^2 the unit torus in C^2, i.e. $\mathsf{T}^2=\{(z_1,z_2)\in\mathsf{C}^2:|z_1|=|z_2|=1\}$. It is well-known (see [M1]) that for commuting contractions T_1, T_2 there exists a collection of complex Borel measures $\{\mu_{x,y}:x,y\in H\}$ on T^2 such that for every analytic polynomial w we have

(3.1) $$(w(T_1,T_2)x,y)=\int w\,d\mu_{x,y},\qquad x,y\in H.$$

These measures are called elementary measures of T_1, T_2.

3.2. Definition. A pair (T_1, T_2) of commuting contractions on a separable Hilbert space is said to be absolutely continuous (a. c.) if it has a collection of elementary measures $\{\mu_{x,y}: x, y \in H\}$ absolutely continuous with respect to some positive Borel measure ν on the torus T^2, which satisfies the equality $\int w\, d\nu = w(0,0)$ for every analytic polynomial w (cf. [K1], [K2]).

Recall that F is a common spectral measure of a pair (T_1, T_2) of commuting contractions if $w(T_1, T_2) = \int w\, dF$ for all analytic polynomials w.

3.3. Definition. A pair (T_1, T_2) is said to be completely singular (c. s.) if it has a unique common spectral measure on T^2 singular to all measures annihilating analytic polynomials (i. e. to all measures satisfying the condition $\int w\, d\eta = 0$ for each analytic polynomial w), (cf. [K1], [K2]).

It is obvious that if (T_1, T_2) is a c. s. pair then contractions T_1, T_2 are unitary.

It is possible to construct (see [K1], [K2]) a decomposition of pairs of commuting contractions different from that in theorem 1:

3.4. Proposition. *If (T_1, T_2) is a pair of commuting contractions on a Hilbert space H then H decomposes into an orthogonal sum of subspaces reducing T_1 and T_2:*

$$H = K_1 \oplus K_2 \oplus K_3 \oplus K_4,$$

where

$$T_1|K_1,\ T_1|K_2,\ T_2|K_1,\ T_2|K_3 \text{ are a.c.} \tag{3.5}$$

$$T_1|K_3,\ T_2|K_2 \text{ are s.u.} \tag{3.6.}$$

$$(T_1, T_2)|K_1 \text{ is a.c.}, \quad (T_1, T_2)|K_4 \text{ c.s.} \tag{3.7}$$

Comparing the above decomposition with theorem 1 we obtain

3.8 Corollary. $K_1 \subset H_1$, $K_2 = H_2$, $K_3 = H_3$, $H_4 \subset K_4$.

3.9. Remark. It follows from the example in [K3] that there may exist a pair of commuting contractions for which $H = K_4 = H_1$. In the case when both contractions T_1, T_2 are without unitary a. c. parts the decompositions given in theorem 1 and in proposition 3.4 are the same. It can be easily seen that in that case any pair of commuting contractions has a Wold-type decomposition due to M. Słocinski [S1].

Theorem 3. *Let $T_1 . T_2 \in B(H)$ be a pair of commuting contractions. Let $H = K_1 \oplus K_2 \oplus K_3 \oplus K_4$ be its Lebesgue-type decomposition given by proposition 3.4. Then*

$$\mathrm{Alg}(T_1, T_2) = \mathrm{Alg}((T_1, T_2)|K_1) \oplus \mathrm{Alg}((T_1, T_2)|K_2) \oplus$$
$$\oplus \mathrm{Alg}((T_1, T_2)|K_3) \oplus \mathrm{Alg}((T_1, T_2)|K_4).$$

Proof. By theorem 2, proposition 3.4, and corollary 3.8, it is sufficient to consider only the case when $H = H_1$, i. e. when both contractions $T_1 . T_2$ are a. c.

(separately). In this case $H=K_1 \oplus K_4$ and $P_1=I \oplus 0$, $P_4=0 \oplus I$, where P_i denotes the orthogonal projection onto the space K_i $(i=1,4)$. By the same argument as in the proof of theorem 2, it suffices to show that $P_1 \in \mathrm{Alg}(T_1, T_2)$.

Denote by A the uniform algebra of all continuous complex functions on the closed unit bidisc which are analytic on its interior. By Oka-Weil theorem (see [G], III.5.1), analytic polynomials are uniformly dense in A. Then the absolute continuity and complete singularity of the appropriate parts of (T_1, T_2) can be expressed in the following way (see [K1], [K2]):

Denote by M_0 the set of all positive Borel measures ν on T^2 such that $\int u\,d\nu=u(0,0)$ $\forall u \in A$. Then

(3.10) *There exists a measure $\nu \in M_0$ such that $(T_1, T_2)|K_1$ has a system of elementary measures absolutely continuous with respect to ν.*

(3.11) *$(T_1, T_2)|K_4$ has a unique common spectral measure which is singular to all measures in M_0.*

The condition (3.11) follows from the fact that each measure in M_0 is absolutely continuous with respect to some measure anihilating A.

Since M_0 is a weak-star compact convex set of positive measures, by [(G], II.7.5) we have the following

3.12. Lemma. *Every regular Borel measure singular to M_0 (i.e. singular to each measure in M_0) is supported on an F_σ set E such that $\nu(E)=0 \,\forall \nu \in M_0$.* We will need also the following Forelli lemma ([G], II.7.3.):

3.13. Lemma. *Let E be an F_σ set such that $\nu(E)=0 \,\forall \nu \in M_0$. Then there is a bounded sequence $f_n \in A$ such that $f_n(z) \to 0$ for all $z \in E$, and $f_n \to I[\nu]$ almost everywhere $\forall \nu \in M_0$.*

Let $\{x_n : n \in \mathsf{N}\}$ be an orthogonal basis in H (we consider only separable Hilbert spaces). Let F be the spectral measure of $(T_1, T_2)|K_4$. The set $\{(F(\bullet)x_n, x_m) : n, m \in \mathsf{N}\}$ is a countable collection of elementary measures, and (hence by lemma 3.12, all these measures are supported on an F_σ set E such that $\nu(E)=0 \,\forall \nu \in M_0$. By an elementary calculation we check that each elementary measure $(F(\bullet)x, y)$, $(x, y \in K_4)$ must be also supported on E. Let f_n be a sequence as in lemma 3.13 which tends to 0 on E. Then for $x, y \in K_4$ we have

$$(3.14) \qquad (f_n(T_1, T_2)x, y)=\int f_n \,d(F(\bullet)x, y)=\int_E f_n \,d(F(\bullet)x, y) \to 0.$$

If $\{\mu_{x,y} : x, y \in K_1\}$ is a system of elementary measures of $(T_1, T_2)|K_1$ then by (3.10) and lemma 3.13, we get for each $x, y \in K_1$

$$(3.15) \qquad (f_n(T_1, T_2)x, y)=\int f_n \,d\mu_{x,y} \to \int 1 \,d\mu_{x,y}=(x, y).$$

As a consequence of (3.14) and (3.15) we get

$$f_n(T_1, T_2) \to P_1=I \oplus 0$$

which completes the proof.

3.16. Remark. Obviously theorems 1.2, and 3 can be extended to the case of n-tuples of mutually commuting contractions (for any positive integer n). In that case we have 2^n direct orthogonal summands in the Lebesgue-type decomposition. (Some results in that direction are included in [K3].)

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