Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New series Vol. 4, 1990, Fasc. 1

Majorization, Factorization and Systems of Linear Operator Equations

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Presented by M. Putinar

1. Introduction

This note concerns the solvability of systems of linear operator equations. Let $\mathcal{H}$ denote a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. The following theorem of R. G. Douglas [9] is fundamental in the theory of operator ranges [12, 15] and in the study of similarity and quasisimilarity [5].

Theorem 1.1. [9] *For* A, $B\in\mathcal{L}(\mathcal{H})$, *the following are equivalent*:
1) (*Majorization*) $AA^*\leqq\lambda^2BB^*$ *for some* $\lambda\geqq 0$.
2) (*Factorization*) $A=BC$ *for some* $C\in\mathcal{L}(\mathcal{H})$.
3) (*Range Inclusion*) $\operatorname{Ran} A\subset\operatorname{Ran} B$.

Analogues of Theorem 1.1 for Banach space operators are given in [3, 11]. Note that majorization and factorization are C*-theoretic concepts, so we may consider the following possible property of a C*-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$: If A, $B\in\mathcal{A}$ satisfy $AA^*\leqq\lambda^2BB^*$, then there exists $C\in\mathcal{A}$ such that $A=BC$; it is natural also to consider the additional requirement that $\|C\|\leqq\lambda$.

In section 2 we present basic properties and examples of such "majorization-factorization" algebras. Our main results (section 3) completely characterize these algebras under a mild separability hypothesis, which includes the important case when the underlying space $\mathcal{H}$ is separable. For the general case, our results lead to a conjecture concerning the SAW*-algebras of G. Pedersen [19]. In section 4, as an application, we study the "uniformity" of right projections in AW*-algebras. Throughout, our motivating concern is the solvability of systems of linear operator equations, so we introduce and study versions of majorization and factorization appropriate for systems. We are also interested in identifying "minimal" environments in which systems of equations can be solved. If $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ does not have the majorization-factorization property, can one describe in concrete terms the minimal majorization-

[1] Research supported by a National Science Foundation Research Grant.

factorization subalgebras $\mathscr{B}$ such that $\mathscr{A}\subset\mathscr{B}\subset\mathscr{L}(\mathscr{H})$? This basic problem appears quite difficult, but we illustrate a special case at the conclusion of this section.

We next introduce our terminology and notation. Let $\mathscr{A}$ be a unital C*-algebra and let $M_n(\mathscr{A})$ denote the C*-algebra of all $n\times n$ matrices over $\mathscr{A}$[17, page 2]; for $1\leqq m, n\leqq k$, we regard $M_{mn}(\mathscr{A})$, the $m\times n$ matrices over $\mathscr{A}$, as a closed subspace of $M_k(\mathscr{A})$. Let

$$a\equiv\begin{pmatrix}a_1\\ \vdots\\ a_m\end{pmatrix}\in M_{m1}(\mathscr{A}),\quad B\equiv(b_{ij})\in M_{mn}(\mathscr{A}),\quad\text{and}\quad x\equiv\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in M_{n1}(\mathscr{A})$$

and consider the system of linear equations represented by the matrix equation

$$a=Bx. \tag{1.1}$$

If a admits such a "factorization" through B with $x=c\in M_{n1}(\mathscr{A})$, then for $\lambda=\|c\|$ we have the "majorization"

$$aa^*\leqq\lambda^2BB^*. \tag{1.2}$$

We are interested in algebras $\mathscr{A}$ for which every majorization as in (1.2) (with $\lambda>0$) implies a factorization $a=Bc$, and we call such an algebra a *majorization-factorization* (or MF) *algebra for $m\times n$ systems*. If we may always choose c with the additional property that $\|c\|\leqq\lambda$, then we say that $\mathscr{A}$ is a *uniform majorization-factorization* (or UMF) *algebra for $m\times n$ systems*. For $m=n=1$, we refer to such algebras simply as MF or UMF as the case may be; MF algebras were introduced by D. Handelman [14] under the name "$\aleph_0$-injective" algebras. Note that if $M_k(\mathscr{A})$ is MF (resp., UMF), then $\mathscr{A}$ is MF (resp., UMF) for $m\times n$ systems for all $m, n\leqq k$. Moreover, $\mathscr{A}$ is MF for $m\times n$ systems if and only if whenever $A\in M_{mp}(\mathscr{A})$ and $B\in M_{mn}(\mathscr{A})$ satisfy $AA^*\leqq BB^*$, then there exists $C\in M_{np}(\mathscr{A})$ with $A=BC$.

We will consider the following questions.

Question 1.2. For m, $n\geqq 1$, which C*-algebras are MF (resp., UMF) for $m\times n$ systems;

Question 1.3. If $\mathscr{A}$ is MF for $m\times n$ systems, is $\mathscr{A}$ UMF for $m\times n$ systems?

Question 1.4. If $\mathscr{A}$ is MF (resp., UMF) and $n>0$, is $M_n(\mathscr{A})$ MF (resp., UMF)?

Question 1.5. If $\mathscr{A}$ is UMF for $n\times n$ systems, is $M_n(\mathscr{A})$ UMF?

In section 3 we resolve these questions under the following separability hypothesis considered by D. Handelman [14]:

(H) $\mathscr{A}$ contains no uncountable set of mutually orthogonal self-adjoint elements.

Let $\mathscr{A}$ denote a C*-subalgebra of $\mathscr{L}(\mathscr{H})$ and assume that $\mathscr{A}$ is *not* UMF. Any UMF subalgebra $\mathscr{B}$ of $\mathscr{L}(\mathscr{H})$ satisfying $\mathscr{A}\subset\mathscr{B}$ is a *factorization algebra* for $\mathscr{A}$; W*($\mathscr{A}$), the von Neumann algebra generated by $\mathscr{A}$, is always a factorization algebra (see Corollary 2.2. below). Since we are interested in solving systems of linear equations from $\mathscr{A}$ (with majorization) as "efficiently" as possible, we consider the following questions: Is W*($\mathscr{A}$) the smallest factorization algebra for $\mathscr{A}$? Is $W^*(\mathscr{A})$ a minimal factorization algebra for $\mathscr{A}$ (with respect to set inclusion)? Does there exist a minimal factorization algebra for $\mathscr{A}$?

These questions seem to be very difficult in general, but we conclude this section with an example for which we can answer them. Let $\mathscr{H}=\Sigma_{n=1}^{\infty}\oplus\mathscr{H}_n$ denote an orthogonal decomposition of the Hilbert space $\mathscr{H}$. For $b=\{b_n\}\in\ell^\infty$, let $\tilde{b}=\Sigma_{n=1}^{\infty}\oplus b_n 1_{\mathscr{H}_n}$. We regard ℓ^∞ as a commutative C*-algebra and we let $\mathbf{c}$ denote the C*-subalgebra of all convergent sequences. The maximal ideal space of ℓ^∞ is identified with $\beta\mathsf{N}$, the Stone-Cech compactification of N (see [10, page 58]); $n\in\mathsf{N}$ is identified with point evaluation at n. Thus N is identified as an open, dense subset of $\beta\mathsf{N}$, and it follows that a functional $f\in\beta(\mathsf{N})$ belongs to the corona set $\beta\mathsf{N}\backslash\mathsf{N}$ if and only if $f(b)=\lim b_n$ for every $b\in\mathbf{c}$. For a Hilbert space $\mathscr{K}$ and $\lambda\in\mathsf{C}$, let $[\tilde{b},\lambda]=\tilde{b}\oplus\lambda 1_{\mathscr{K}}$, acting on $\mathscr{H}\oplus\mathscr{K}$.

In the sequel let $m=\{m_n\}\in\mathbf{c}$, let $\lambda=\lim m_n$ and suppose $m_n\neq m_k$, for $n\neq k$, $\lambda\neq m_n$ for $n\geqq 1$. Let $M=\tilde{m}$ and let $N=[\tilde{m},\lambda]$. Routine calculations show that $C^*(M)=\{\tilde{a}:a\in\mathbf{c}\}$, $W^*(M)=\{\tilde{b}:b\in\ell^\infty\}$, and $W^*(M)$ is the smallest MF subalgebra of $\mathscr{L}(\mathscr{H})$ containing $C^*(M)$. By way of contrast, we have the following result:

Proposition 1.6. i) $C^*(N)=\{[\tilde{a},\alpha]:a\in\mathbf{c},\ \alpha=\lim a_n\}$.
ii) $W^*(N)=\{[\tilde{b},\beta]:b\in\ell^\infty,\ \beta\in\mathsf{C}\}$.
iii) *For* $f\in\beta\mathsf{N}\backslash\mathsf{N}$, *let* $\mathscr{A}_f=\{[\tilde{b},f(b)]:b\in\ell^\infty\}$. $\mathscr{A}_f$ *is a minimal factorization algebra for* $C^*(N)$ *and* $C^*(N)\subsetneqq\mathscr{A}_f\subsetneqq W^*(N)$. *Every MF algebra* $\mathscr{B}$ *with* $C^*(N)\subset\mathscr{B}\subsetneqq W^*(N)$ *is of the form* $\mathscr{B}=\mathscr{A}_f$ *for a unique* $f\in\beta\mathsf{N}\backslash\mathsf{N}$. $\mathscr{A}_f\cap\mathscr{A}_g$ *is not MF for* $f\neq g$, *and* $C^*(N)=\cap_{f\in\beta\mathsf{N}\backslash\mathsf{N}}\mathscr{A}_f$.

Proof. In the sequel we may assume $\lambda=0$. i) and ii) are routine calculations. iii) Since $f\in\beta\mathsf{N}\backslash\mathsf{N}$ is positive, linear, multiplicative, and norm continuous, $\mathscr{A}_f$ is a C*-subalgebra of $\mathscr{L}(\mathscr{H})$ and (by ii)) a proper subalgebra of $W^*(N)$; since $f\notin\mathsf{N}$, i) implies $C^*(N)\subsetneqq\mathscr{A}_f$. Suppose $[\tilde{a},f(a)]$, $[\tilde{b},f(b)]\in\mathscr{A}_f$ with $[\tilde{a},f(a)]\,[\tilde{a},f(a)]^*\leqq[\tilde{b},f(b)]\,[\tilde{b},f(b)]^*$. Then for $n\geqq 1$, $|a_n|^2\leqq|b_n|^2$, and we define

$$c_n=\begin{cases}a_n/b_n & \text{if } b_n\neq 0\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $c\equiv\{c_n\}\in\ell^\infty$, $[\tilde{c},f(c)]\in\mathscr{A}_f$, $[\tilde{a},f(a)]=[\tilde{b},f(b)]\,[\tilde{c},f(c)]$, and $\|[\tilde{c},f(c)]\|\leqq 1$ since $\|f\|=1$. Thus $\mathscr{A}_f$ is a factorization algebra for $C^*(N)$.

Suppose $\mathscr{B}\subsetneqq W^*(N)$ is a factorization algebra for $C^*(N)$. For $b\in\ell^\infty$, $[\tilde{m}\tilde{b},0]\in C^*(N)$, and $[\tilde{m}\tilde{b},0]\,[\tilde{m}\tilde{b},0]^*\leqq\|b\|_\infty^2\,[\tilde{m},0]\,[\tilde{m},0]^*$. Thus there exists $B\in\mathscr{B}$ $(\subset W^*(N))$, $B=[\tilde{d},\beta]$ for some $d\in\ell^\infty$ and $\beta\in\mathsf{C}$, such that $[\tilde{m}\tilde{b},0]=[\tilde{m},0]\,B$. It follows that $\tilde{d}=\tilde{b}$; thus for $b\in\ell^\infty$, there exists $\beta\in\mathsf{C}$ such that $[\tilde{b},\beta]\in\mathscr{B}$. Since $\mathscr{B}\neq W^*(N)$, β is unique for b. Since $\mathscr{B}$ is a unital C*-algebra, it follows that the map $f\colon\ell^\infty\to\mathsf{C}$, given by $f(b)=\beta$, defines an element $f\in\beta\mathsf{N}$. If $c\in\mathbf{c}$, then $[\tilde{c},\lim c_n]\in C^*(N)\subset\mathscr{B}$, so $f(c)=\lim c_n$, and it follows that $f\in\beta\mathsf{N}\backslash\mathsf{N}$ and that $\mathscr{B}=\mathscr{A}_f$.

Let $f,g,h\in\beta\mathsf{N}\backslash\mathsf{N}$. Since $\mathscr{A}_g\subset\mathscr{A}_f$ implies $f=g$, $\mathscr{A}_f$ is a minimal factorization algebra for $C^*(N)$. Similarly, if $\mathscr{A}_h=\mathscr{A}_g\cap\mathscr{A}_f$, then $f=g=h$; thus if $f\neq g$, $\mathscr{A}_f\cap\mathscr{A}_g$

is not MF. Note also that if $b \in \ell^\infty$ is nonconvergent, then there exist f, $g \in \beta\mathrm{N}\backslash\mathrm{N}$ such that $f(b) \neq g(b)$, and it follows that

$$\bigcap_{f \in \beta\mathrm{N}\backslash\mathrm{N}} \mathscr{A}_f \subset C^*(N).$$

Question 1.7. What are the minimal factorization algebras for $C^*(T)$ when T is a normal operator whose spectrum is an interval?

Acknowledgement. The authors are very grateful to Professors J. Bunce, D. Hadwin, and V. Paulsen for helpful suggestions concerning this work. Prof. Bunce referred us to the papers of D. Handelman [14] and S. Berberian [1]; he conjectured that every AW*-algebra is MF and that a proof could be based upon [1]. He also proved special cases of Proposition 4.3, extending our original proof for $M_n(C(X))$ [4].

2. Basic properties and examples

We first observe that every C*-algebra is UMF in an approximate sense. In the sequel, $\mathscr{B}_1(\mathscr{A})$ denotes the unit ball of the C*-algebra $\mathscr{A}$. The following result appears in [14], where the proof is attributed to J. Cuntz.

Proposition 2.1. [14] *Let $\mathscr{A}$ be a C*-algebra and let $a, b \in \mathscr{A}$ with $aa^* \leqq bb^*$; then there exists $\{c_n\} \subset \mathscr{B}_1(\mathscr{A})$ such that $\lim \|a - bc_n\| = 0$.*

Corollary 2.2. *If $\mathscr{A}$ is a W*-algebra, then $\mathscr{A}$ is UMF.*

Proof. We may assume that $\mathscr{A}$ is a weakly closed subalgebra of $\mathscr{L}(\mathscr{H})$. If a, $b \in \mathscr{A}$, $aa^* \leqq bb^*$, then from Proposition 2.1, there exists $\{c_n\} \subset \mathscr{B}_1(\mathscr{A})$ with $\lim \|a - bc_n\| = 0$. By the weak operator topology compactness of $\mathscr{B}_1(\mathscr{A})$ [7], we may assume $\{c_n\}$ converges weakly to some $c \in \mathscr{B}_1(\mathscr{A})$, whence $a = bc$.

The following alternate proof of Corollary 2.2 suggests the role of projections in establishing the UMF property. For $\mathscr{S} \subset \mathscr{L}(\mathscr{H})$, let $(\mathscr{S})'$ denote the commutant of $\mathscr{S}$ and let $(\mathscr{S})'' \equiv ((\mathscr{S})')'$ denote the double commutant. Suppose $\mathscr{A}$ is a weakly closed *-subalgebra of $\mathscr{L}(\mathscr{H})$ and let A, $B \in \mathscr{A}$ with $AA^* \leqq BB^*$. Define a map D: Ran $B^* \to$ Ran A^* by $D(B^*h) = A^*h$; D is a well-defined linear contraction, which thus extends to a contraction D: (Rn $B^*)^- \to$(Rn $A^*)^-$. Extend D to all of $\mathscr{H}$ by $DP = 0$, where P is the projection of $\mathscr{H}$ onto $\ker B$. Then $C = D^*$ satisfies $A = BC$ and $\|C\| \leqq 1$. The Double Commutant Theorem [8] implies $(\mathscr{A})'' = \mathrm{A}$, so to show $C \in \mathscr{A}$, it suffices to verify that $C^* \in (\mathscr{A})''$. Let $X \in \mathscr{A}'$; then $C^*XB^* = DXB^* = DB^*X = A^*X$ and $XC^*B^* = XA^* = A^*X$, whence $C^*X(1-P) = XC^*(1-P)$. To complete the proof, it suffices to show that $C^*XP = XC^*P$: since $P \in W^*(|B|) \subset \mathscr{A}$, then $C^*XP = C^*PX = DPX = 0 = XDP = XC^*P$.

The preceding argument suggest that the UMF property holds in algebras with sufficiency many projections, the results of section 3 further support this conjecture.

We present two applications of the UMF property for W*-algebras.

A result in the theory of operator ranges is that if A, B_1, $B_2 \in \mathscr{L}(\mathscr{H})$ satisfy

$AA^* \leqq B_1B_1^* + B_2B_2^*$, then there exist operaors C_1, $C_2 \in \mathscr{L}(\mathscr{H})$ such that $A = B_1C_1 + B_2C_2$ [12, Corollary 1, p. 260]. We may extend this result as follows.

Proposition 2.3. *Let $\mathscr{A}$ be a von Neumann algebra on $\mathscr{H}$. If $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ are sequences in $\mathscr{A}$ such that $\Sigma B_iB_i^*$ is strongly convergent and $\Sigma A_iA_i^* \leqq \Sigma B_iB_i^*$, then there exists $X \equiv (X_{ij}) \in \mathscr{L}(\mathscr{H}^{(\infty)})$, $\|X\| \leqq 1$, $X_{ij} \in \mathscr{A}$ for all i and j, such that for $j \geqq 1$,*

$$A_j = \sum_{i=1}^{\infty} B_iX_{ij}$$

(strong convergence).

Proof. Let $\mathscr{B}$ denote the von Neumann algebra on $\mathscr{H}^{(\infty)}$ consisting of all operators with matrices of the form (T_{ij}), where each $T_{ij} \in \mathscr{A}$. Then

$$A \equiv \begin{pmatrix} A_1 & A_2 & A_3 \dots \\ & 0 & \end{pmatrix} \quad \text{and} \quad B \equiv \begin{pmatrix} B_1 & B_2 & B_3 \dots \\ & 0 & \end{pmatrix}$$

belong to $\mathscr{B}$ and $AA^* \leqq BB^*$. Corollary 2.2 implies that there exists $X \equiv (X_{ij}) \in \mathscr{B}$, $\|X\| \leqq 1$, such that $A = BX$, and the result follows.

We next use the UMF property to characterize 2-positive maps between W*-algebras. In the sequel, A, B and C denote operators in $\mathscr{L}(\mathscr{H})$ with $A \geqq 0$ and $C \geqq 0$. We denote by $T \equiv T(A, B, C)$ the self-adjoint operator on $\mathscr{H} \oplus \mathscr{H}$ with operator matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. The following result is well-known.

Lemma 2.4. (cf. [6]) *If A is invertible, then $T \geqq 0$ if and only if $B^*A^{-1}B \leqq C$.*

The positivity of T may be characterized as follows:

Proposition 2.5. *$T \equiv T(A, B, C) \geqq 0$ if and only if there exists a contraction $E \in \mathscr{L}(\mathscr{H})$ such that $B = A^{1/2}EC^{1/2}$; in this case, E may be chosen from $W^*(A, B, C)$, and if A is invertible, E may be chosen from any prescribed UMF subalgebra of $\mathscr{L}(\mathscr{H})$ containing A, B and C.*

Proof. Assume that $T \geqq 0$ and consider first the case when A is invertible. From Lemma 2.4, $(B^*A^{-1/2})(B^*A^{-1/2})^* \leqq C^{1/2}C^{1/2}$. If $\mathscr{B}$ is any UMF subalgebra containing $B^*A^{-1/2}$ and C, then there exists $D \in \mathscr{B}$, $\|D\| \leqq 1$, such that $B^*A^{-1/2} = C^{1/2}D$, whence $B = A^{1/2}D^*C^{1/2}$; let $E = D^*$ in this case. In the general case when $T \geqq 0$, let

$$T_n = \begin{pmatrix} A + 1/n & B \\ B^* & C \end{pmatrix}, \quad n > 0.$$

The preceding argument and Corollary 2.2 imply that there exists $E_n \in W^*(A, B, C)$, $\|E_n\| \leqq 1$, such that $B = (A + 1/n)^{1/2}E_nC^{1/2}$. By weak compactness of $\mathscr{B}_1(\mathscr{L}(\mathscr{H}))$, we may assume $\{E_n\}$ is weakly convergent to some contraction $E \in W^*(A, B, C)$, whence $B = A^{1/2}EC^{1/2}$. The converse is a direct calculation.

Remark D. Xia [25] proves a result like Proposition 2.5, except the result in [25] does not locate E in any subalgebra associated with A, B, C, and it does not appear that the proof in [25] could be adapted to such a purpose.

In the sequel we employ the terminology of [17]. Let $\mathcal{A}$ be a unital C*-subalgebra of $\mathcal{L}(\mathcal{H})$. A subspace $\mathcal{S}$ of $\mathcal{A}$ is an operator system if $\mathcal{S}$ is self-adjoint and contains 1. If $\mathcal{B}$ is a C*-subalgebra of $\mathcal{L}(\mathcal{H})$, then a linear map $\Phi:\mathcal{S}\to\mathcal{B}$ is positive if whenever $a\in\mathcal{S}$ and $a\geqq 0$, then $\Phi(a)\geqq 0$; in this case $\Phi(a^*)=\Phi(a)^*$ $(a\in\mathcal{S})$ (see [17, page 9]). For $n>0$, the map $\Phi_n: M_n(\&)\to M_n(\mathcal{B})$ is defined by $\Phi_n((a_{ij}))=(\Phi(a_{ij}))$; Φ is n-positive if Φ_n is positive, in which case Φ is m-positive for $m<n$. Φ is completely positive if Φ_n is positive for every $n>0$. We characterize 2-positive maps as follows.

Proposition 2.6. *Let $\Phi:\mathcal{S}\to\mathcal{B}$ be a positive map. Φ is 2-positive if and only if whenever $A, C\in\mathcal{S}$ and $E\in\mathcal{L}(\mathcal{H})$ satisfy $A\geqq 0$, $C\geqq 0$, and $A^{1/2}EC^{1/2}\in S$, then there exists $F=F(A,C,E)\in\mathcal{L}(\mathcal{K})$, $\|F\|\leqq\|E\|$, such that*

$$\Phi(A^{1/2}EC^{1/2})=\Phi(A)^{1/2}F\,\Phi(C)^{1/2}.$$

Proof. By Proposition 2.5, the positive elements of $M_2(\mathcal{S})$ are precisely those matrices of the form $T=T(A,B,C)$, where $A\geqq 0$, $C\geqq 0$, and $B=A^{1/2}EC^{1/2}$ for some contraction E in $\mathcal{L}(\mathcal{H})$. Since Φ is positive, $\Phi_2(T)=T(\Phi(A),\Phi(B),\Phi(C))$; thus $\Phi_2(T)\geqq 0$ if and only if $\Phi(B)=\Phi(A)^{1/2}F\,\Phi(C)^{1/2}$ for some contraction $F\in\mathcal{L}(\mathcal{K})$. For the general case when $E\in\mathcal{L}(\mathcal{H})$ and $A^{1/2}EC^{1/2}\in\mathcal{S}$, apply the preceding argument to $(1/\|E\|)E$.

Proposition 2.6 can be used to recover several known properties of 2-positive maps. For example, the Schwarz Inequality for 2 positive maps [17, page 39] states that if $\Phi:\mathcal{A}\to\mathcal{B}$ is 2-positive and A, $B\in\mathcal{A}$, then $\|\Phi(A^*B)\|^2\leqq\|\Phi(A^*A)\|\,\|\Phi(B^*B)\|$. To see this, let $A=V|A|$ and $B=W|B|$ denote the polar decompositions of A and B. Since $\|V^*W\|\leqq 1$, Proposition 2.6 implies that there exists $F\in\mathcal{L}(\mathcal{K})$, $\|F\|\leqq 1$, such that $\Phi(A^*B)=\Phi((A^*A)^{1/2}V^*W(B^*B)^{1/2})=\Phi(A^*A)^{1/2}F\,\Phi(B^*B)^{1/2}$, and the result follows.

Corollary 2.7. *Let $\Phi:\mathcal{A}\to\mathcal{B}$ be a positive map between W*-algebras. Then Φ is 2-positive if whenever A, C, $E\in\mathcal{A}$ with $A\geqq 0$, $C\geqq 0$, then there exists $F\in\mathcal{B}$, $\|F\|\leqq\|E\|$, such that $\Phi(A^{1/2}EC^{1/2})=\Phi(A)^{1/2}F\Phi(C)^{1/2}$.*

Recall that a C*-subalgebra $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ is injective [17, page 82] if and only if there is a unital (contractive) completely positive map $\Phi:\mathcal{L}(\mathcal{H})\to\mathcal{A}$ such that $\Phi(A_1CA_2)=A_1\Phi(C)A_2$ for all A_1, $A_2\in\mathcal{A}$, $C\in\mathcal{L}(\mathcal{H})$. The following result was pointed out to us by J. Froelich.

Proposition 2.8. *If $\mathcal{A}$ is injective, then $\mathcal{A}$ is UMF.*

Proof. If A, $B\in\mathcal{A}$ satisfy $AA^*\leqq BB^*$, then there exists $C\in\mathcal{L}(\mathcal{H})$, $\|C\|\leqq 1$, with $A=BC$. Then $A=\Phi(A)=\Phi(BC)=B\Phi(C)$ and $\|\Phi(C)\|\leqq\|C\|\leqq 1$.

Since not every injective algebra is a W*-algebra, there exist UMF algebras that are not W*-algebras. Additional examples of this sort arise when we consider quotient algebras.

Proposition 2.9. *Let $\mathcal{J}$ be a closed 2-sided ideal of the C*-algebra $\mathcal{A}$. If m,*

$n \geqq 1$ and $\mathscr{A}$ is MF (resp., UMF) for $m \times n$ systems, then $\mathscr{A}/\mathscr{J}$ is MF (resp., UMF) for $m \times n$ systems.

Remark. For the case $m=n=1$, the MF result is contained in [14]; we base our proof on the following result.

Proposition 2.10. [18] *Let $\rho: \mathscr{A} \to \mathscr{B}$ be a *-homomorphism of $\mathscr{A}$ onto $\mathscr{B}$. If $a \in \mathscr{A}$, $a \geqq 0$, and $x \in \mathscr{B}$ with $x^*x \leqq \rho(a)$, then $x=\rho(y)$ for some $y \in \mathscr{A}$ with $y^*y \leqq a$.*

Proof of Proposition 2.9. Let $a=\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in M_{m1}(\mathscr{A})$, $B=(b_{ij}) \in M_{mn}(\mathscr{A})$ and suppose $\tilde{a}\tilde{a}^* \leqq \tilde{B}\tilde{B}^*$ in $M_m(\mathscr{A}/\mathscr{J})$.Let $\hat{a}=(a \;\; 0_{m,m-1}) \in M_m(\mathscr{A})$, so that $\tilde{\hat{a}}\tilde{\hat{a}}^*=\tilde{a}\tilde{a}^* \leqq \tilde{B}\tilde{B}^*$. Proposition 2.10 implies that there exists $c=(c_{ij}) \in M_m(\mathscr{J})$ such that $(\hat{a}+c)\;(\hat{a}+c)^* \leqq BB^*$; if $y=\begin{pmatrix} a_1+c_{11} \\ \vdots \\ a_m+c_{m1} \end{pmatrix}$, it follows that $yy^* \leqq (\hat{a}+c)$ $(\hat{a}+c)^* \leqq BB^*$. Since $\mathscr{A}$ is MF for $m \times n$ systems, there exists $x \in M_{n1}(\mathscr{A})$ such that $y=Bx$ (and in the UMF case we may assume $\|x\| \leqq 1$). Thus $\tilde{a}=\tilde{y}=\tilde{B}\tilde{x}$ (and in the UMF case, $\|\tilde{x}\| \leqq \|x\| \leqq 1$).

Remark. If $\mathscr{A}$ is a type II_∞ factor on $\mathscr{H}$ and $\mathscr{J}$ is the ideal generated by a finite projection, then a result of H. Takemoto [23] implies that $\mathscr{A}/\mathscr{J}$ is not a W*-algebra, although $\mathscr{A}/\mathscr{J}$ is UMF from Proposition 2.9. In particular, if $\mathscr{K}(\mathscr{H})$ denotes the ideal of compact operators on $\mathscr{H}$, then the Calkin algebra $\mathscr{L}(\mathscr{H})/_{=}\mathscr{K}(\mathscr{H})$ is UMF but not W*; this answers a question of [5].

We say that a C*-algebra $\mathscr{A}$ enjoys property (PD) if for each $x \in \mathscr{A}$, there exists $w \in \mathscr{A}$ such that $x=w|x|$. Clearly, each MF algebra satisfies (PD). For $x \in \mathscr{A}$, $\sigma(x)$ denotes the spectrum.

Proposition 2.11. *If $\mathscr{A}$ satisfies (PD) and is separable, then* $\dim \mathscr{A} < \infty$. *If* $\dim \mathscr{A} < \infty$, *then $\mathscr{A}$ is UMF.*

Proof. Assume that $\mathscr{A}$ satisfies (PD), is infinite dimensional, and acts on $\mathscr{H}$. Since $\mathscr{A}$ is infinite dimensional, it follows from [24, page 54] that $\mathscr{A}$ contains a positive operator $\mathscr{A}$ with infinite spectrum. Let E denote the spectral measure of A. Since $\sigma(A)$ is infinite, we may assume 0 as a point of accumulation of $\sigma(A)$, and we may construct four sequences of positive numbers, (a_k), (b_k), (c_k), (d_k), such that $d_1=\|A\|$,

$$d_{k+1} < a_k < b_k < c_k < d_k, \quad k \in \mathbf{N},$$

$\lim d_k=0$, and $E([b_k, c_k])$ is a nonzero projection for all k.

Let $f_k: [0, \|A\|] \to [0, 1]$ be a piecewise linear continuous function that is equal to 1 on $[b_k, c_k]$ and 0 on $[0, a_k] \cup [d_k, d_1]$. For each nonempty subset S of $\mathbf{N}$, define $h_S: [0, \|A\|] \to \mathbf{C}$ by

$$h_S(x)=\begin{cases} 1, & \text{if } x=0 \\ [\sum_{k \in S} f_k(x)]+i(1-[\sum_{k \in S} f_k(x)]^2)^{1/2}, & \text{if } x \neq 0. \end{cases}$$

Since h_S is a Borel measurable function with values on the unit circle, we may define the unitary operator $T_S \equiv h_S(A)$. T_S need not be in $\mathscr{A}$ (although $T_S \in W^*(A)$), but $T_S A$ is in $\mathscr{A}$ since the function $g(x) = xh_S(x)$ is continuous. Since T_S is unitary, $|T_S A| = A$. Since $T_S A \in \mathscr{A}$ and $\mathscr{A}$ satisfies (PD), there exists $V_S \in \mathscr{A}$ with $T_S A = V_S A$, and thus V_S coincides with T_S on $E([0, \|A\|]) = (\text{Ran } A)^-$.

Since the power set of N is uncountable, to show that $\mathscr{A}$ is nonseparable, it suffices to prove that if S and S' are distinct subsets of N, then $\|V_S - V_{S'}\| \geqq 2^{1/2}$. Assume $k \in S \backslash S'$ and choose a unit vector u in $E([b_k, c_k])\mathscr{H}$. Then $T_S(u) = u$ while $T_{S'}(u) = iu$; thus $\|V_S - V_{S'}\| \geqq \|V_S(u) - V_{S'}(u)\| = \|T_S(u) - T_{S'}(u)\| = 2^{1/2}$.

Conversely, each finite dimensional C*-algebra is W*, hence UMF.

3. Majorization and factorization in AW*-algebras

Let $\mathscr{A}$ be a unital C*-algebra and let & be a nonempty subset of $\mathscr{A}$. Let $R(\mathscr{S}) = \{x \in \mathscr{A} : sx = 0 \text{ for all } s \in \mathscr{S}\}$, the right annihilator of $\mathscr{S}$. $\mathscr{A}$ is an AW*-algebra if for every nonempty subset $\mathscr{S}$ of $\mathscr{A}$, $R(\mathscr{S}) = g\mathscr{A}$ for a suitable (self-adjoint) projection g [2]. (Equivalently, every maximal abelian self-adjoint subalgebra of $\mathscr{A}$ is *-isomorphic to $C(X)$ for some Stonean space X [14].) In this case, for $x \in \mathscr{A}$, there exists a unique projection $e \equiv e_x$ such that (3.1) $xe = x$ and (3.2) $xy = 0$ if and only if $ey = 0$; moreover, $R(\{x\}) = (1-e)\mathscr{A}$ and e is minimal among projections satisfying (3.1) [2, page 13]. The projection e is called the right projection of x and is denoted by RP(x). As a basic example, every W*-algebra is AW* [2, page 24]; if $\mathscr{A}$ is a von Neumann algebra on the Hilbert space $\mathscr{H}$ and $T \in \mathscr{A}$, RP(T) is the projection onto $\mathscr{H} \ominus \ker(T)$.

Following G. K. Pedersen [19], we say that a C*-algebra $\mathscr{A}$ is SAW* if whenever x and y are positive orthogonal elements of $\mathscr{A}$ (i.e., $x \geqq 0$, $y \geqq 0$, $xy = yx = 0$), there exists a self-adjoint element $e \in \mathscr{A}$ such that $xe = x$ and $ey = 0$. Clearly, each AW*-algebra is SAW*. The commutative SAW* algebras are those algebras *-isomorphic to $C(X)$ for some substonean space X [19].

We next examine connections between the SAW* and UMF concepts. In [14], D. Handelman proved that if $\mathscr{A}$ is MF and satisfies (H), then $\mathscr{A}$ is AW*. The proof of this result motivates our analogue for UMF algebras in the general case (Proposition 3.2 below). We require the following lemma, which was supplied to us by Prof. J. Bunce in order to simplify our first proof of Proposition 3.2. This lemma is implicitly assumed in the proofs of [20, Props. 3.1-3.3].

Lemma 3.1. *Let $\mathscr{A}$ be a C*-algebra. If $x, g \in \mathscr{A}$ satisfy $x = x^*$, $x = xg$, $\|g\| \leqq 1$, then, $x = xgg^*$.*

Proof. Since $0 = x^2 - xgg^*x = (x(1-gg^*)^{1/2})(x(1-gg^*)^{1/2})^*$, then $0 = x(1-gg^*)^{1/2}$ and $0 = x(1-gg^*)$.

Proposition 3.2. *If $\mathscr{A}$ is UMF, then $\mathscr{A}$ is SAW*.*

Proof. Let x and y be orthogonal positive elements of $\mathscr{A}$. Since $x^2 \leqq x^2 + y^2 = (x+y)^2$ and $\mathscr{A}$ is UMF, there exists $g \in \mathscr{A}$, $\|g\| \leqq 1$, such that $x = (x+y)g$. Thus $x^2 = x(x+y)g = x^2 g$, whence $p(x^2) = p(x^2)g$ for every polynomial

p satisfying $p(0)=0$. Since $x \geqq 0$, it follows by polynomial approximation that $x=(x^2)^{1/2}$ satisfies $x=xg$, whence $yg=0$. Thus $ygg^*=0$ and it suffices to show that $x=xgg^*$. Since this follows from Lemma 3.1, we conclude that $\mathscr{A}$ is SAW*.

Following [20], we say that a C*-algebra $\mathscr{A}$ is n-SAW* if $M_n(\mathscr{A})$ is SAW*; in this case, $\mathscr{A}$ is m-SAW* for $m \leqq n$. Proposition 3.2 may thus be viewed as a weak converse of the following result of Pedersen [20].

Proposition 3.3. (G. Pedersen [20, Proposition 3.3]) *If $\mathscr{A}$ is 4-SAW*, then $\mathscr{A}$ is UMF.*

We next present the main results.

Theorem 3.4. *If $\mathscr{A}$ is an AW*-algebra, then $M_n(\mathscr{A})$ is* UMF *for every $n \geqq 1$.*

Proof. For $n \geqq 1$, we have $M_4(M_n(\mathscr{A})) \approx M_{4n}(\mathscr{A})$ (see [17, page 3]). Since $\mathscr{A}$ is AW*, Berberian's theorem [1] implies that $M_k(\mathscr{A})$ is AW* for every $k>1$. Thus $M_4(M_n(\mathscr{A}))$ is SAW*, and it follows from Proposition 3.3 that $M_n(\mathscr{A})$ is UMF.

The next result shows that among the algebras satisfying (H), the AW*-algebras are precisely the algebras in which Douglas' theorem is valid; in exactly these algebras is it possible to solve linear systems with majorization. Following [20], we say that $\mathscr{A}$ is SSAW* if $M_n(\mathscr{A})$ is SAW* for each $n \geqq 1$.

Theorem 3.5. *For $\mathscr{A}$ satisfying (H) the following are equivalent:*

1) *For some m, $n \geqq 1$, $\mathscr{A}$ is an MF algebra for $m \times n$ systems;*
2) *$\mathscr{A}$ is an MF algebra;*
3) *$\mathscr{A}$ is an AW* algebra;*
4) *For every $n \geqq 1$, $M_n(\mathscr{A})$ is UMF;*
5) *For every m, $n \geqq 1$, $\mathscr{A}$ is UMF for $m \times n$ systems;*
6) *$\mathscr{A}$ is an SSAW* algebra.*

Proof. 1)⇒2) follows by specialization. 2)⇒3) is Handelman's result [14]. 3)⇒4) is Theorem 3.4. 4)⇒5) follows by specialization and 5)⇒1) is trivial. The equivalence of 4) and 6) follows from Propositions 3.2 and 3.3.

There are important UMF algebras which are not AW*-algebras, e. g., the Calkin algebra. For the general case, it is tempting to conjecture that $\mathscr{A}$ is UMF if and only if it is SAW*, and that in this case $M_n(\mathscr{A})$ inherits both of these properties. There is some evidence in this direction. In the commutative case, $\mathscr{A}$ is SAW* if and only if $\mathscr{A} \approx C(X)$ for some compact substonean space X [19]. D. Hadwin [13] has proved that these algebras are precisely the commutative MF algebras. Moreover, R. Smith and D. Williams [22] proved that if X is substonean, then $C(X)$ is SSAW*. Thus, via Proposition 3.3, $M_n(C(X))$ is also UMF.

For another class of examples, consider the corona algebras introduced by Pedersen [19]. In [16], C. Olsen and G. Pedersen proved that if $\mathscr{A}$ is a corona algebra, then $\mathscr{A}$ is SAW* and UMF; moreover, as G. K. Pedersen notes in [20], $M_n(\mathscr{A})$ is also a corona algebra. Further evidence is provided by basic results of SAW* theory which parallel corresponding UMF results: In [19, Corollary 2] G. K. Pedersen proved that an infinite dimensional SAW* algebra is nonseparable (the analogue of Proposition 2.11), and in [19, Proposition 3] G. K. Pedersen proved that if $\mathscr{A}$ is SAW* and $\mathscr{J}$ is a closed 2-sided ideal, then $\mathscr{A}/\mathscr{J}$ is SAW* (the analogue of Proposition 2.10). The main obstacle to understanding the general case is the lack of a structure theory for SAW* algebras. The principal (and deep) ingredient in the proof of Theorem 3.4 is Berberian's theorem, which depends on the rich structure theory for AW* algebras (cf. [2]).

For completeness, we include the following result which relates several possible properties of a C*-algebra. The first two implications are due to R. Smith and D. Williams [21].

Corollary 3.6. *For a C*-algebra $\mathscr{A}$, each of the following properties implies the next*:

1) $\mathscr{A}$ *is injective*;
2) $\mathscr{A}$ *satisfies the "decomposition property"* [21];
3) $\mathscr{A}$ *is AW**;
4) $\mathscr{A}$ *is UMF*;
5) $\mathscr{A}$ *is SAW**.

In [21] R. Smith and D. Williams describe sufficient conditions for 2)⇒1). The Calkin algebra is UMF but not AW*; ℓ^∞/c_0 [21] has the same features. Whether 5) implies 4) is open.

4. Uniform projections in AW*-algebras

We next introduce a framework for studying whether MF implies UMF for a particular algebra. Let $\mathscr{A}$ be a unital C*-algebra and let $B\in\mathscr{A}$. Suppose $Q\in\mathscr{A}$ is a self-adjoint projection satisfying the following properties:
(4.1) $B(1-Q)=0$;
(4.2) for each $C\in\mathscr{A}$,

$$\|QC\|=\min\{\lambda:\lambda\geqq 0 \text{ and } BCC^*B^*\leqq\lambda^2BB^*\}.$$

(To establish (4.2) for a projection Q, it suffices to show that if $\lambda\geqq 0$ and $BCC^*B^*\leqq\lambda^2BB^*$, then $\|QC\|\leqq\lambda$; this is because $BCC^*B^*=BQCC^*QB^*\leqq\|QC\|^2BB^*$.)

Proposition 4.1. *If a projection Q satisfies* (4.1) *and* (4.2), *then it is the unique projection satisfying these properties, and Q is minimal among projections satisfying* (4.1). *Moreover*, $R(\{B\})=(1-Q)\mathscr{A}$.

Proof. Note that if $C\in\mathscr{A}$ and $BC=0$, then (4.2) implies $QC=0$, whence $C=(Q+(1-Q))C=(1-Q)C$. Thus $R(\{B\})\subset(1-Q)\mathscr{A}$, and the reverse inclusion follows from (4.1). The remaining implications follow exactly as in the proofs of [2, Prop. 1, page 4] and [2, Prop. 3, page 13].

We denote the projection of Proposition 4.1 by Q_B, the uniform right projection for B. If $\mathscr{A}$ is AW*, then $Q_B=RP(B)$ by the uniqueness of right projections. The next result indicates the relevance of uniform right projections to the study of MF algebras.

Proposition 4.2. *If $\mathscr{A}$ is an MF algebra and each element of $\mathscr{A}$ has a uniform right projection, then $\mathscr{A}$ is a UMF algebra.*

Proof. Suppose A, $B \in \mathscr{A}$ and $AA^* \leqq \lambda^2 BB^*$; there exists $C \in \mathscr{A}$ such that $A = BC$. By (4.1), $A = B(Q_B C)$. Now $BCC^*B^* = AA^* \leqq \lambda^2 BB^*$, so (4.2) implies $\|Q_B C\| \leqq \lambda$. Thus $X = Q_B C$ is a uniform solution to $A = BX$.

If each element of $\mathscr{A}$ has a uniform right projection, then $\mathscr{A}$ is a Rickart *-algebra (in the sense of [2]) and is clearly SAW*; in general, however, a SAW* algebra may be projectionless [19].

In the sequel, if $\mathscr{M}$ is a subspace of a Hilbert space $\mathscr{H}$, $P_{\mathscr{M}}$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. For $T \in \mathscr{L}(\mathscr{H})$, init (T) denotes the initial space of T, i. e., init $(T) = \mathscr{H} \ominus \ker T$. If $\mathscr{A}$ is a von Neumann algebra on $\mathscr{H}$, then it is easy to check that $P_{\text{init}(T)}$ is a uniform right projection for $T \in \mathscr{A}$. We can extend this observation to AW*-algebras as follows.

Proposition 4.3. *If $\mathscr{A}$ is an AW*-algebra and $T \in \mathscr{A}$, then $RP(T)$ is a uniform right projection for T.*

Proof. Let $Q = RP(T)$. Since $TQ = T$, it suffices to check that if $C \in \mathscr{A}$ and $TCC^*T^* \leqq \lambda^2 TT^*$, then $\|QC\| \leqq \lambda$. By Theorem 3.4, $\mathscr{A}$ is UMF, so there exists $X \in \mathscr{A}$, $\|X\| \leqq \lambda$, such that $TC = TX$. Thus $T(C - X) = 0$, whence $Q(C - X) = 0$ and $\|QC\| = \|QX\| \leqq \|Q\| \; \|X\| \leqq \lambda$.

In the sequel we show that for the AW*-algebras $M_n(C(X))$, X Stonean, the uniform right projections admit a particularly nice description. We require several preliminary results.

Proposition 4.4. [24, Cor. 1.8, p. 105] *If $\mathscr{U}$ is a nonempty open subset of the Stonean space X and $f: \mathscr{U} \to \mathrm{C}$ is continuous, then f admits a continuous extension to the clopen set $\mathscr{U}^-$ (and hence to all of X).*

The proofs of the next two lemmas are standard, so we omit the details.

Lemma 4.5. *Let $B \in M_n(C(X))$.*

i) $\sigma(B) = \cup_{x \in X} \sigma(B(x))$;

ii) $B \geqq 0$ *if and only if* $B(x) \geqq 0$ *for each* $x \in X$;

iii) $\|B\| = \sup_{x \in X} \|B(x)\|$.

Lemma 4.6. *Let $1 \leqq k < \infty$. Let $\{T_j\} \subset \mathscr{L}(\mathscr{H})$ be a sequence of rank k operators norm convergent to the rank k operator T. Then lim $\|P_{\ker(T_j)} - P_{\ker(T)}\| = 0$.*

Theorem 4.7. *Let $B \in M_n(C(X))$. There exists an open dense subset $\mathscr{U}_B$ of X such that $RP(B)\,(x) = RP(B(x)) = P_{\text{init}(B(x))}$ $(x \in \mathscr{U}_B)$.*

Proof. We will define $Q_B: X \to M_n(\mathrm{C})$ pointwise on an open dense subset of X that we now describe. Let $\mathscr{U}_n = \{x \in X : \text{rank}\,(B(x)) = n\}$. For $0 \leqq j \leqq n-1$, assuming that $\mathscr{U}_n, \ldots, \mathscr{U}_{j+1}$ have been defined, let $\mathscr{U}_j = \{x \in X \setminus (\mathscr{U}_{j+1}^- \cup \ldots \cup \mathscr{U}_n^-) : \text{rank}\,(B(x)) = j\}$. We claim that each $\mathscr{U}_j$ is an open subset of X. That $\mathscr{U}_n$ is open is clear, since the map $d: X \to \mathrm{R}$ defined by $d(x) = \det B(x)$ is continuous. Assume $0 < j < n$ and let $x \in \mathscr{U}_j$. Since rank $B(x) = j$, then dim init $(B(x)) = j$; since $B(x)$ has closed range, there exists $\delta > 0$ such that $\|B(x)t\| \geqq \delta \|t\|$ $(t \in \text{init}\;(B(x)))$. Let $\mathscr{O}_x$ be an open subset of $X\backslash(\mathcal{U}_{j+1}^{-}\cup\ldots\cup\mathcal{U}_n^{-})$ such that $x\in\mathcal{O}_x$ and such that if $y\in\mathcal{O}_x$, then $\|B(y)-B(x)\|<\delta$ (B is continuous). If $y\in\mathcal{O}_x$, then rank $B(y)\leqq j$ by the very definition of the $\mathcal{U}_j$'s. If rank $B(y)<j$, then rank $B(y)^*=$ rank $B(y)<j$ $=\dim$ init $B(x)$, so there exists a unit vector $t\in$ init $B(x)\ominus$ range $B(y)^*$ $=$ init $B(x)\cap\ker B(y)$. Thus $\delta>\|B(x)-B(y)\|\geqq\|B(x)t-B(y)t\|=\|B(x)t\|\geqq\delta$, a contradiction; we conclude that $\mathcal{U}_j$ is open, $0<j<n$. Since $\mathcal{U}_0=X\backslash[\mathcal{U}_n^{-}\cup\ldots\cup\mathcal{U}_n^{-}]$, $\mathcal{U}_0$ is also open. We set $\tilde{\mathcal{U}}=\mathcal{U}_0\cup\ldots\cup\mathcal{U}_n$; $\tilde{\mathcal{U}}$ is open and dense in X.

We define $\tilde{Q}:\tilde{\mathcal{U}}\to M_n(\mathrm{C})$ as follows. For $1\leqq j\leqq n$ and $x\in\mathcal{U}_j$, let $\tilde{Q}(x)=1-P_{\ker(B(x))}$. Since $B:X\to M_n(\mathrm{C})$ is continuous, Lemma 4.6 and the definition of $\mathcal{U}_j$ imply that $\tilde{Q}:\mathcal{U}_j\to M_n(\mathrm{C})$ is continuous. Note that since X is Stonean, $\mathcal{U}_j^{-}\cap\mathcal{U}_k^{-}=\emptyset$ for $j\neq k$. Thus by Proposition 4.4 (extended to matrix valued functions), $\tilde{Q}$ admits a unique continuous extension $Q_B:X\to M_n(\mathrm{C})$.

To complete the proof, we show that Q_B is a uniform right projecion for B. Note that for S, $T\in M_n(C(X))$, $S(x)^*=S^*(x)$ and $ST(x)=S(x)T(x)$; since $Q_B(x)$ is a projection for $x\in\tilde{\mathcal{U}}$, it follows by the density of $\tilde{\mathcal{U}}$ and continuity, that Q_B is a self-adjoint projection. Similarly, since $B(x)$ $(1-\tilde{Q}(x))=B(x)P_{\ker(B(x))}=0$ for $x\in\tilde{\mathcal{U}}$, it follows that $B(1-Q_B)=0$. Let $C\in M_n(C(X))$. Suppose $\lambda>0$ and $BCC^*B^*\leqq\lambda^2BB^*$; from Lemma 4.5,

$$B(x)C(x)C(x)^*B(x)^*\leqq\lambda^2B(x)B(x)^*\quad(x\in X).$$

The remarks preceding Proposition 4.3 (applied to the Hilbert space C^n) imply that for $x\in\tilde{\mathcal{U}}$, $\|Q_B(x)C(x)\|=\|(1-P_{\ker(B(x))})C(x)\|\leqq\lambda$; by the continuity of Q_B, the density of $\tilde{\mathcal{U}}$, and Lemma 4.5, it follows that $\|Q_BC\|\leqq\lambda$. Thus Q_B is a uniform right projection for B.

Since $M_n(C(X))$ is an AW*-algebra, $Q_B=RP(B)$.

Added in proof.

i) The earliest characterization of the positivity of $T(A, B, C)$ is apparently due to J. L. Smul'jan [Mat. Sb. 91 (1959), 381–430].

ii) In a recent preprint, "Positive completions of matrices over C*-algebras", V. Paulsen and L. Rodman introduce (PC) algebras and (QF) algebras, both closely related to UMF algebras.

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Received 08.04.1989