Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 1

Convergence in Distribution of Supercritical Bellman-Harris Branching Processes with State-Dependent Immigration

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A necessary and sufficient condition for convergence in distribution of supercritical Bellman-Harris branching processes with state-dependent immigration is obtained.

1. Introduction

K. Athreya [2] first showed an analog of classical Kesten and Stigum theorem [1] for supercritical Bellman-Harris branching processes, which refine and make more precise the estimates of the growth of processes on the set of non-extinction.

In the present paper we investigate supercritical Bellman-Harris branching processes which admit immigration of new particles only in the state zero.

A model with state-dependent immigration component was first investigated by J. H. Foster [6] and A. G. Pakes [8,9]. They considered a modification of the Galton-Watson processes allowing immigration whenever the number of particles is zero. The continuous-time analog of this process was studied by M. Yamazato [11].

Our paper is closely connected with [10] and [12] where the asymptotic behavior of the two factorial moments is obtained and limit theorems are also proved in non-critical cases.

In the present work it is shown that for these processes the asymptotic results have an analogy with those obtained by K. Athreya.

2. Model and equations

Now we shall briefly recall the definition of Bellman-Harris processes with state-dependent immigration which was given by K. V. Mitov and N. M. Yanev [7].

Research supported in part by the Ministry of Science and Higher Education, Contract No 991/1988.

Let us have on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ three independent sets of random variables where:

1) $X=\{X_i\}_{i\geqq 1}$ is a set of independent identically distributed (i. i. d.) random variables (r. v.) with distribution function (d. f.) $K(t)=\mathsf{P}\{X_i\leqq t\}$, $K(0)=0$;

2) $Y=\{Y_i\}_{i\geqq 1}$ is a set of positive, integer-valued i. i. d. random variables with a probability generating function (p. g. f.)

$$f(s)=\mathsf{E}s^{Y_i}=\sum_{k=1}^{\infty} f_k s^k, \quad |s|\leqq 1;$$

and

3) $Z=\{Z_{ij}(t),\ t\geqq 0,\ i,\ j\geqq 1,\ Z_{ij}(0)=1\}$

is a set of i. i. d. Bellman-Harris branching processes defined by a particle-life distribution function $G(t)$, $G(0)=0$ and an offspring p. g. f.

$$h(s)=\sum_{k=0}^{\infty} p_k s^k, \quad |s|\leqq 1.$$

Then

$$Z_i(t)=\sum_{j=1}^{Y_i} Z_{ij}(t), \quad t\geqq 0, \quad i\geqq 1. \tag{2.1}$$

are i. i. d. Bellman-Harris branching processes starting with a random number $Y_i>0$ of ancestors.

Let T_i be the life-period of $Z_i(t)$, i. e.

$$T_i=\inf\{t: Z_i(t)=0\}, \quad i=1,\ 2, \dots\ . \tag{2.2}$$

Observe that $U_i=T_i+X_i$, $i\geqq 1$ are i. i. d. random variables which form the renewal process

$$S_0=0,\ S_n=\sum_{i=1}^{n} U_i,\ n\geqq 1 \text{ and } N(t)=\max\{n\geqq 0: S_n\leqq t\},\ t\geqq 0. \tag{2.3}$$

Then Bellman-Harris branching processes with state-dependent immigration can be defined as follows:

$$Z(0)=0,\quad Z(t)=Z_{N(t)+1}(t-S_{N(t)}-X_{N(t)+1})\,\mathsf{I}_{\{S_{N(t)}+X_{N(t)+1}\leqq t\}}\,. \tag{2.4}$$

The Foster-Pakes model follows from (2.4) with

$$G(t)=\begin{cases}0, & t\leqq 1,\\ 1, & t>1,\end{cases} \quad \text{and} \quad K(t)=\begin{cases}0, & t\leqq 0,\\ 1, & t>0.\end{cases}$$

We also obtain the Yamazato process if we suppose in (2.4)

$$G(t)=\begin{cases}0, & t\leqq 0,\\ 1-e^{-\lambda t}, & t>0,\end{cases} \quad \text{and} \quad K(t)=\begin{cases}0, & t\leqq 0,\\ 1, & t>0.\end{cases}$$

Denote $L(t)=\mathsf{P}\{X_i+T_i\leqq t\}=\int_0^t V(t-u)\,\mathrm{d}K(u)$ and suppose that $L(t)$ is non-lattice with $L(0)=0$, where $V(t)=\mathsf{P}(T_i\leqq t)$, $V(0)=0$.

3. Basic results

From now on it will be assumed:

1°) $1<A=h'(1)<\infty$, $m=\mathsf{E}Y_i=f'(1)<\infty$,

2°) $G(t)$ and $K(t)$ are non-lattice.

Define the Malthusian parameter α, provided it exists, as the root of equation

$$A\int_0^\infty e^{-\alpha u}\,dG(u)=1. \tag{3.1}$$

By the monotonicity of the left side of (3.1) and 1°) such a root always exists and $\alpha>0$.

It is well-known (see [3], p. 172) that in supercritical case $Z_{ij}(t)/\mathsf{E}Z_{ij}(t)\xrightarrow{d}\tilde{W}$, $t\to\infty$, and $\Psi(u)=\mathsf{E}e^{-u\tilde{W}}$, $u\geqq 0$ is the unique solution of the equation

$$\Psi(u)=\int_0^\infty f(\Psi(ue^{-\alpha y}))dG(y) \tag{3.2}$$

in the class

$$C=\{\Psi:\Psi(u)=\int_0^\infty e^{-ut}\,dF(t),\ F(0^+)<1,\ \int_0^\infty t\,dF(t)=1\} \tag{3.3}$$

iff

$$\sum_{j=2}^\infty p_j j\log j<\infty. \tag{3.4}$$

Theorem 1. *Assume* 1°) *and* 2°).

(*i*) *If* $\Sigma p_j j\log j<\infty$ *then* $W(t)=Z(t)/\mathsf{E}Z(t)$ *converges in distribution to a non-negative random variable* W *having the following properties*:

a) $\mathsf{E}W<\infty$;

b) $\theta(u)=\mathsf{E}e^{-uW}$, $u\geqq 0$, is the unique solution of the equation

$$\theta(u)=\int_0^\infty \theta(ue^{-\alpha t})\,dL(t)+\int_0^\infty f(\Psi(ue^{-\alpha t}))\,dK(t)-f(q) \tag{3.5}$$

in the class

$$B=\{\varphi:\varphi(u)=\int_0^\infty e^{-ut}\,dF(t),\ \varphi(0)=1\}, \tag{3.6}$$

where $\Psi(u)$ *satisfies* (3.2) *and* $\lim_{t\to\infty}\mathsf{P}\{Z_{ij}(t)=0\}=q$;

c) *the distribution of* W *is absolutely continuous on* $[0,\infty)$.

(ii) *If* $\Sigma p_j j\log j=\infty$ *then* $\lim_{t\to\infty}W(t)=0$ *in probability*.

4. Preliminaries

For classical Bellman-Harris branching processes $Z_{ij}(t)$ it is known (see [3], p. 152, Theorem 3A) that in supercritical case we have

(4.1) $$m(t)=\mathsf{E}Z_{ij}(t)\sim c_1 e^{\alpha t},\quad t\to\infty,$$

where

(4.2) $$c_1=\frac{A-1}{\alpha A^2\int\limits_0^\infty te^{-\alpha t}\mathrm{d}G(t)}.$$

Also if (3.4) holds, then the process $\tilde{W}(t)=Z_{ij}(t)/c_1e^{\alpha t}$ converges in distribution to the non-negative random variable $\tilde{W}$ having the following properties:

1) $\mathsf{E}\tilde{W}=1$;
2) The distribution of $\tilde{W}$ is absolutely continuous on $(0,\infty)$;
3) $\mathsf{P}\{\tilde{W}=0\}=q=\lim_{t\to\infty}\mathsf{P}\{Z_{ij}(t)=0\}$.

On the other hand, from R. A. Doney [4] it is known that if $A>1$ and (3.4) holds, then the process $(Z_{ij}(t)/c_1e^{\alpha t},\ v(t)/c_2e^{\alpha t})\xrightarrow[t\to\infty]{d}(\tilde{W},\ \tilde{W})$, where $v(t)$ denotes the total progeny (the number of particles born up to time t) of the process $Z_{ij}(t)$ and $c_2=Ac_1/(A-1)$.

Let $\mu(t)$ be the number of particles died up to time t of the process $Z_{ij}(t)$. Denote $F(t,\ s_1,\ s_2)=\mathsf{E}s_1^{Z_{ij}(t)}s_2^{\mu(t)}$. It is not difficult to obtain the following equation

(4.3) $$F(t,s_1,s_2)=s_1(1-G(t))+s_2\int\limits_0^t h(F(t-y,\ s_1,\ s_2))\mathrm{d}G(y),$$
$$|s_i|\leqq 1,\quad i=1,\ 2,\quad t\geqq 0.$$

If $s_1=1$ from (4.3) for $\delta(t,s)=\mathsf{E}s^{\mu(t)}$ we have

(4.4) $$\delta(t,\ s)=1-G(t)+s\int\limits_0^t h(\delta(t-y,\ s))\,\mathrm{d}G(y),\quad |s|\leqq 1,\quad t\geqq 0,$$

and by differentiating and letting $s=1$ for $n(t)=\mathsf{E}\mu(t)$ we obtain

(4.5) $$n(t)=G(t)+A\int\limits_0^t n(t-y)\mathrm{d}G(y).$$

Then applying the renewal theorem ([3], p. 147, Theorem 2) it is easy to see that

(4.6) $$n(t)\sim c_3e^{\alpha t},\quad t\to\infty,$$

where $\alpha>0$ is the Malthusian parameter and

$$c_3=\frac{\int\limits_0^\infty e^{-\alpha t}G(t)\mathrm{d}t}{A\int\limits_0^\infty te^{\alpha t}\mathrm{d}G(t)}=\frac{1}{\alpha A^2\int\limits_0^\infty te^{-\alpha t}\mathrm{d}G(t)}.$$

Theorem 2. *Let* $1<A<\infty$. *If* (3.4) *holds, then*

$$(Z_{ij}(t)/c_1e^{\alpha t},\ \mu(t)/c_3e^{\alpha t}) \xrightarrow{d} (\tilde{W},\ \tilde{W}),\ t\to\infty.$$

The proof of the theorem follows from the next lemmas.
Denote

$$g(u_1,\ u_2,\ t)=F\{\exp\{-u_1/c_1e^{\alpha t}\},\ \exp\{-u_2/c_3e^{\alpha t}\},\ t\},$$

$$R(u_1,\ u_2,\ t)=(u_1+u_2)^{-1}\left[g(u_1,\ u_2,\ t)+\frac{u_1m(t)}{c_1e^{\alpha t}}+\frac{u_2n(t)}{c_3e^{\alpha t}}-1\right].$$

Lemma 1. *Under conditions of the Theorem 2 there exist*

$$\lim_{t\to\infty} g(u_1,\ u_2,\ t)=\Phi(u_1,\ u_2),\ u_i>0,\ i=1,\ 2$$

and

$$\lim_{u_i\downarrow 0}\ \sup_{t\geq 0} R(u_1,\ u_2,\ t)=0.$$

Proof. Let $\xi_1(t)=Z_{ij}(t)/c_1e^{\alpha t}$, $\xi_2(t)=\mu(t)/c_3e^{\alpha t}$. Note that

$$R(u_1,\ u_2,\ t)=\mathsf{E}\{[e^{-(u_1\xi_1(t)+u_2\xi_2(t))}+u_1\xi_1(t)+u_2\xi_2(t)-1]/(u_1+u_2)\}.$$

Using $Z_{ij}(t,\omega)\leqq\nu(t,\omega)$ and $\mu(t,\omega)\leqq\nu(t,\ \omega)$ for every $\omega\in\Omega$ we have

$$u_1\xi_1(t)+u_2\xi_2(t)\leqq\xi(t)\,[u_1d_1+u_2d_2],$$

where $d_1=c_2/c_1$, $d_2=c_2/c_3$, $c_2=Ac_1/(A-1)$ and $\xi(t)=\nu(t)/c_2e^{\alpha t}$.

On the other hand, since the function $e^{-x}+x-1$ is non-decreasing for $x\geqq 0$, then

$$0\leqq R(u_1,\ u_2,\ t)\leqq\frac{d_1u_1+d_2u_2}{u_1+u_2}H\,(d_1u_1+d_2u_2,t), \tag{4.7}$$

where $H(u,t)=\mathsf{E}\,\{u^{-1}\,(e^{-u\xi(t)}+u\xi(t)-1)\}$.

Now applying Theorem 1 in [4] we have $\lim_{u\downarrow 0}\sup_{t\geq 0}H(u,\ t)=0$, so that from (4.7) we obtain

$$\lim_{u_i\downarrow 0}\ \sup_{t\geq 0} R(u_1,\ u_2,\ t)=0, \tag{4.8}$$

which implies that there exists $\Phi(u_1,\ u_2)=\lim_{t\to\infty}g(u_1,\ u_2,\ t)$.

Therefore from (4.3) making the substitutions $s_1=\exp\{-u_1/c_1e^{\alpha t}\}$, $s_2=\exp\{-u_2/c_3e^{\alpha t}\}$ it follows that as $t\to\infty$

$$\Phi(u_1,\ u_2)=\int_0^\infty h(\Phi(e^{-\alpha y}u_1,\ e^{-\alpha y}u_2))\,dG(y),\ u_i\geqq 0,\ i=1,\ 2. \tag{4.9}$$

Lemma 2. *There exists a unique solution* $\Phi(u_1,\ u_2)$ *of the equaion* (4.9), *such that*

$$(4.10)\qquad \begin{cases} \Phi(0,0)=1,\ 0<\Phi(u_1,u_2)\leqq 1, & u_i\geqq 0,\ i=1,2,\\ \lim\limits_{u_i\downarrow 0}\dfrac{1-\Phi(u_1,u_2)}{u_1+u_2}=1, & i=1,2,\\ \Phi(u_1+u_2)=\Psi(u_1+u_2), \end{cases}$$

iff (3.4) *holds.*

Proof. Suppose $\Phi_1(u_1, u_2)$ and $\Phi_2(u_1, u_2)$ are solutions of (4.9) satisfying (4.10). Let

$$\gamma(u_1, u_2)=|\Phi_1(u_1, u_2)-\Phi_2(u_1, u_2)|/(u_1+u_2),\ u_i>0,\ i=1,2.$$

Using $|h(x_1)-h(x_2)|\leqq A|x_1-x_2|$, $0\leqq x_i\leqq 1$, $i=1, 2$, and equation (4.9) we have

$$(4.11)\qquad \gamma(u_1, u_2)\leqq(u_1+u_2)^{-1}\int_0^\infty |h(\Phi_1(u_1e^{-\alpha y}, u_2e^{-\alpha y}))$$

$$-h(\Phi_2(u_1e^{-\alpha y}, u_2e^{-\alpha y}))|dG(y)\leqq A\int_0^\infty e^{-\alpha y}\gamma(u_1e^{-\alpha y}, u_2e^{-\alpha y})\,dG(y)$$

$$\leqq \mathsf{E}\{\gamma(u_1e^{-\alpha\xi}, u_2e^{-\alpha\xi})\},$$

where ξ is the random variable with distribution

$$G_\alpha(x)=\mathsf{P}\{\xi\leqq x\}=A\int_0^x e^{-\alpha y}\,dG(y)$$

and

$$\mathsf{E}\xi=A\int_0^\infty xe^{-\alpha x}\,dG(x)<\infty.$$

From (4.10) we have

$$\lim_{u_i\downarrow 0}\gamma(u_1, u_2)=0,\ i=1,2.$$

Iterating (4.11) we obtain

$$\gamma(u_1, u_2)\leqq \mathsf{E}\{\gamma(u_1e^{-\alpha\xi}, u_2e^{-\alpha\xi})\}\leqq \mathsf{E}\{\gamma(u_1e^{-\alpha(\xi_1+\xi_2)}, u_2e^{-\alpha(\xi_1+\xi_2)})\}\dots$$

$$\leqq \mathsf{E}\{\gamma(u_1e^{-\alpha S_n}, u_2e^{-\alpha S_n})\},$$

where $S_n=\Sigma_{i=1}^n \xi_i$ and $\{\xi_i\}_{i=1}^n$ are i.i.d.r.v. with common distribution function $G_\alpha(x)$.

By the strong law of the large numbers (SLLN) we have $S_n\to\infty$, $n\to\infty$, a.s. and hence the bounded convergence theorem yields

$$\gamma(u_1, u_2)\leqq \lim_{n\to\infty}\mathsf{E}\{\gamma(u_1e^{-\alpha S_n}, u_2e^{-\alpha S_n})\}=0,$$

for $u_i>0$, $i=1, 2$, which proves the uniqueness.

K. Athreya [2] has shown that (3.4) is a necessary and sufficient condition for existence of a unique solution $\Psi(u)$ of the equation (3.2) such that

$$(4.12)\qquad \begin{cases} \Psi(0)=1,\ 0<\Psi(u)\leqq 1,\ \text{for } u\geqq 0,\\ \lim\limits_{u\downarrow 0}\dfrac{1-\Psi(u)}{u}=1. \end{cases}$$

Let $\Psi(u)$ is such a solution and denote $\tilde{\Phi}(u_1, u_2)=\Psi(u_1+u_2)$. From (3.4) and (4.12) it follows that $\tilde{\Phi}(u_1, u_2)$ satisfies (4.9) and (4.10).

On the other hand, $\Phi(u_1, u_2)$ satisfies (4.9) and (4.10) and $\Phi(u, 0)=\Psi(u)$, which implies the lemma.

5. Proof of the Theorem 1

In the case $p_0=0$ it follows that $Z(t)$ is a classical Bellman-Harris process and the assertion of the theorem follows by Theorem 2, p. 172 in [3].

Let now $p_0>0$ and $v_i(t)$ be the number of particles in Bellman-Harris process $Z_i(t)$ which are born up to time t and $\mu_i(t)$ be the number of particles which are died up to time t. Denote

$$S_1(t)=\sum_{i=1}^{N(t)} v_i(t) \quad \text{and} \quad S_2(t)=\sum_{i=1}^{N(t)} \mu_i(t),$$

where $N(t)$ is defined in (2.3) and the process $N(t)$ is independent of $\{\mu_i(t)\}$ and $\{v_i(t)\}$.

We have the representation

$$(5.1)\qquad Z(t)=S_1(t)-S_2(t).$$

Under conditions of the theorem $N(t)\to v$, $t\to\infty$ a. s. and $\mathrm{E}v=1/(1-f(q))<\infty$ (see [5], Ch. XI, § 6).

On the other hand, we have for $i\geqq 1$

$$v_i(t)/e^{\alpha t} \xrightarrow{d} H_i,$$

(see [4]) and from Theorem 2 we obtain

$$\mu_i(t)/e^{\alpha t} \xrightarrow{d} \tilde{H}_i, \quad \text{as} \quad t\to\infty.$$

Therefore as $t\to\infty$

$$(5.2)\qquad \frac{S_1(t)}{e^{\alpha t}}=\sum_{i=1}^{N(t)} v_i(t)/e^{\alpha t} \xrightarrow{d} \sum_{i=1}^{v} H_i,$$

$$(5.3)\qquad \frac{S_2(t)}{e^{\alpha t}}=\sum_{i=1}^{N(t)} \mu_i(t)/e^{\alpha t} \xrightarrow{d} \sum_{i=1}^{v} \tilde{H}_i.$$

From (5.1)–(5.3) it follows that the process $W(t)$ converges in law.

The rest of the argument is a direct consequence of Corollaries 4.3–4.4 of [10], which completes the proof in the case when (3.4) holds.

Now, to prove (ii) we shall use the definition (2.4). Therefore we have the representation

$$(5.4)\qquad \frac{Z(t)}{e^{\alpha t}}=\frac{Z_{N(t)+1}(\beta(t))}{e^{\alpha\beta(t)}}e^{-\alpha(t-\beta(t))},$$

where $\beta(t)=t-S_{N(t)}-X_{N(t)+1}$.

It is well-known that $\Sigma p_j j\log j=\infty$ yields $Z_{ij}(t)/e^{\alpha t}\to 0$ in probability (see [3], p. 172) which implies that

$$(5.5)\qquad \frac{Z_i(t)}{e^{\alpha t}}=\sum_{j=1}^{Y_i}\frac{Z_{ij}(t)}{e^{\alpha t}}\to 0$$

in probability, as $t\to\infty$.

Using $EU_i=ET_i+EX_i=\infty$, $i\geqq 1$, by renewal theory (see [5], Ch. XI, & 3 and Ch. XIV, & 3) it follows that

$$(5.6)\qquad P\{\beta_t\leqq x\}\to 0$$

for all x as $t\to\infty$.

Finally, from (5.4) applying (5.5) and (5.6) we obtain that $W(t)\underset{t\to\infty}{\to}0$ in probability, when $\Sigma p_j j\log j=\infty$.

The theorem is proved.

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Received 10.04.1989