Mathematatica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 1

On Positive Strongly Continuous Cosine Functions

Carlos Lizama

Presented by M. Putinar

If A is a complex number then, (*) $\operatorname{Cosh} t\sqrt{A} \in \mathrm{R}_+$ for all $t \in \mathrm{R}_+$ if and only if $A \in \mathrm{R}_+$. In this note, we give a generalization of this property to the case, when A is the infinitesimal generator of a strongly continuous cosine function defined in a Hilbert space and in a Banach lattice.

1. Preliminaries

Let X be a Banach space. We denote $B(X)$ the algebra of bounded linear operators on X.

A function $C: \mathrm{R} \to B(X)$ is called a strongly continuous operator cosine function on X if

i) $C(t+s)+C(t-s)=2C(t)C(s) \quad s,\ t \in \mathrm{R}$

ii) $C(0)=\mathrm{I}$

iii) $t \to C(t)x$ is continuous on R for each fixed $x \in X$.

For an introduction to the subject of strongly continuous cosine functions, the reader is referred to [2] and the references listed in its bibliography. We state below some of the ideas we need from this theory.

The infinitesimal generator of a strongly continuous cosine function is the operator A from X to X with domain $D(A)$, defined by the conditions: $D(A)=\{x \in X / t \to C(t)x$ is twice continuously differentiable for all $t \in \mathrm{R}\}$

$$Ax=\lim_{t \to 0}(2/t^2)(C(t)x-x) \quad \text{for all} \quad x \in D(A).$$

The sine function $S(t)$, $t \in \mathrm{R}$ associated with $C(t)$, $t \in \mathrm{R}$ is defined by:

$$S(t)x=\int_0^t C(s)x\,ds \quad \text{for all} \quad t \in \mathrm{R},\ \ x \in X.$$

We define $E=\{x \in X / t \to C(t)x$ is once continuously differentiable for all $t \in \mathrm{R}\}$.

We will require the following result of J. Kisynsky [3].

Theorem 1.1. *Let $C(t)$, $t \in \mathrm{R}$ be a strongly continuous cosine function in the Banach space X with infinitesimal generator A and associated sine function $S(t)$, $t \in \mathrm{R}$. Then E under the norm*

$$\|x\|_E = \|x\| + \sup_{0 \leq t \leq 1} \|(d/dt)C(t)x\|$$

becomes a Banach space and $V(t)$, $t \in \mathbf{R}$ defined by:

$$V(t)[x, y] = [C(t)x + S(t)y, AS(t)x + C(t)y], \quad [x, y] \in E \times X$$

is a strongly continuous group in $E \times X$ with infinitesimal generator:

$$B[x, y] = [y, Ax]; \quad D(B) = D(A) \times E.$$

We denote $\sigma(T)$ and $\rho(T)$ the spectrum set and the resolvent set of a closed linear operator T respectively. If $\lambda \in \rho(T)$ then we denote $R(\lambda; T) = (\lambda - T)^{-1}$.

2. Case of a Hilbert space

Let $(H, <, >)$ be a Hilbert space. We recall that an operator $T: D(T) \subseteq H \to H$ with domain $D(T)$ is called positive if:

(2.1) $$\langle Tx, x \rangle \geqq 0 \quad \text{for all} \quad x \in D(T).$$

It is well-known that if T is bounded, then the property (2.1) is equivalent to

(2.2) $$T \text{ selfadjoint and } \sigma(T) \subseteq [0, \infty).$$

Theorem 2.1. *Let $C(t)$, $t \in \mathbf{R}$ be a strongly continuous cosine function with infinitesimal generator A. The following properties are equivalent:*

i) *$C(t)$ is positive for all $t \in \mathbf{R}$.*
ii) *A is bounded and positive.*

Proof.

i)→ii) According to (2.2) we have for every $t \in \mathbf{R}$:

(2.3) $$C(t) \text{ selfadjoint and } \sigma(C(t)) \subseteq [0, \infty).$$

Then, Theorem 1.1 and 1.2 in [4] imply:

(2.4) $$C(t) - I \text{ selfadjoint and } \sigma(C(t)) \subseteq [1, \infty),$$

(2.5) $$A \text{ selfadjoint and } \sigma(A) \subseteq [0, \infty).$$

Now, it is well-known that there are constants $M \geqq 1$ and $w \geqq 0$ such that

(2.6) $$\|C(t)\| \leqq M e^{wt}, \quad t \in \mathbf{R}$$

and the following formula holds

(2.7) $$\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} C(t)x \, dt, \quad \lambda > w, \ x \in H.$$

Then

(2.8) $$\lambda^2 R(\lambda^2; A)x - x = \int_0^\infty \lambda e^{-\lambda t} \{C(t)x - x\} dt, \quad \lambda > w, \ x \in H.$$

It follows from (2.4), (2.8) and the identity $AR(\lambda^2;\ A)=\lambda^2 R(\lambda^2;\ A)-I$ that:

$$(2.9)\langle AR(\lambda^2;\ A)x,\ x\rangle=\int_0^\infty \lambda e^{-\lambda t}\langle C(t)x-x,\ x\rangle\,dt\geqq 0,\quad \lambda>w,\ x\in H.$$

Now, let $x\in D(A)$ and $\lambda>w$ be fixed. Put $y=(\lambda^2-A)x$. As $C(t)$ is selfadjoint for every $t\in\mathrm{R}$, we obtain, by definition of the infinitesimal generator, that A is a symmetric operator. Then we have from (2.9)

$$\langle(\lambda^2-A)x,\ Ax\rangle=\langle Ax,\ (\lambda^2-A)x\rangle=\langle AR(\lambda^2;\ A)y,\ y\rangle\geqq 0$$

whence we obtain

$$\langle Ax,\ Ax\rangle=-\langle(\lambda^2-A)x,\ Ax\rangle+\lambda^2\langle Ax,\ x\rangle\leqq\lambda^2\langle Ax,\ x\rangle.$$

Therefore,

$$(2.10)\qquad \|Ax\|\leqq\lambda^2\|x\|\quad \text{for all}\quad x\in D(A).$$

Finally, using (2.10), the fact that A is a closed operator and $D(A)$ is dense in H, we obtain that $D(A)=H$. Therefore A is a bounded operator. Moreover, we obtain from (2.5) that A is a positive operator.

ii)→i). As A is bounded, the series

$$(2.11)\qquad \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}A^n\quad t\in\mathrm{R}$$

converges and defines a strongly continuous cosine function with infinitesimal generator A. Therefore, by uniqueness, we have

$$C(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}A^n\quad t\in\mathrm{R}$$

the result is now clear from the hypothesis.

Q.E.D

As an application, we have the following:

Corollary 2.2. *Let $C(t)$, $t\in\mathrm{R}$ be a strongly continuous cosine function with infinitesimal generator A. Assume that $C(t)$ is positive for all $t\in\mathrm{R}$. Then,*

i) $C(t)$ cannot be a periodic function, unless $A\equiv 0$.

ii) $-A$ is the infinitesimal generator of a periodic strongly continuous cosine function.

P r o o f.

i) Let us suppose, by absurd, that $C(t)$ is periodic with period 2π. Then, from [5], we obtain:

$$\sigma(A)\subseteq -\mathrm{N}_0^2,\quad \mathrm{N}_0:=\{0,\ 1,\ 2,\ldots\}$$

therefore, the spectral relation in (2.5) imply

$$\sigma(A)=\{0\}\quad\text{or}\quad\sigma(A)=\emptyset.$$

As A is bounded, $\sigma(A)=\{0\}$. But this implies, due to the self-adjointness of A, that A is the identically null operator.
ii) We define

$$C(t)x=(\mathrm{Cos}\, t\sqrt{A})x, \quad x\in H, \quad t\in \mathrm{R}.$$

Q.E.D.

3. Case of a Banach Lattice

Let X be an ordered Banach space with positive cone P, that is, P is a closed convex cone with $P\cap -P=0$. (This induces the order $x\leqq y$ if and only if $y-x\in P$.) An operator $T: D(T)\subseteq X\to X$ is called positive if and only if $Tx\in P$ for all $x\in D(T)_+$; where $D(T)_+=D(T)\cap P$. In such case, we denote $T\geqq 0$. For fundamental notions, we refer to [8].

Theorem 3.1. *Let $C(t)$, $t\in \mathrm{R}$ be a strongly continuous cosine function with infinitesimal generator A defined in a Banach lattice X. The following properties are equivalent:*
i) *$C(t)\geqq I$ for all $t\in \mathrm{R}$,*
ii) *A is bounded and positive.*

Proof.
As in the proof of Th. 2.1 it is clear that ii) implies i). Conversely, it is well-known that the formula

$$(3.1) \qquad T(s)x=\frac{1}{\sqrt{\pi s}}\int_0^\infty e^{-t^2/4s}\, C(t)x\, dt, \quad s>0, \quad x\in X$$

defines a C_0-semigroup with infinitesimal generator A (cf. [1], Remark 5.11, p. 92). Then, we obtain that T is a positive C_0-semigroup due to $C(t)=(C(t)-I)+I\geqq 0$ for all $t\in \mathrm{R}$ and P is closed.

On the other hand, if $x\in D(A)_+$, then it is obtained from the definition of the infinitesimal generator and the hypothesis

$$(3.2) \qquad Ax=\lim_{t\to 0} 2/t^2\ (C(t)x-x)\geqq 0.$$

Because of the relation (3.2) and the fact that $T(t)$ is a positive C_0-semigroup on a Banach lattice, we have from lemma 4.18 p. 279 in [6], that A is bounded. We give the proof for completeness.

There exists a constant $K\geqq 1$ such that $\|R(\lambda;\, A)\|\leqq K/\lambda$ for all $\lambda\geqq w_0$. Fix $\mu\geqq w_0$, then

$$AR(\mu;\, A)Ax=\mu R(\mu;\, A)Ax-Ax=\mu^2R(\mu;\, A)x-\mu x-Ax,$$

hence

$$0\leqq Ax\leqq \mu^2 R(\mu;\, A)x, \quad \text{whenever} \quad x\in D(A)_+.$$

Thus $\|Ax\|\leqq C\|x\|$ for all $x\in D(A)_+$ (C=constant). Consequently, $\|\lambda R(\lambda;\, A)x-x\|=\|AR(\lambda;\, A)x\|\leqq C\|R(\lambda;\, A)x\|\leqq KC/\lambda\|x\|$ for all $x\in P$ and all $\lambda\geqq w_0$. Hence,

$$\|\lambda R(\lambda;\ A)y-y\|\leqq KC/\lambda\ (\|y^+\|+\|y^-\|)\leqq 2KC/\lambda\|y\|$$

for all $y\in X$.
Thus $R(\lambda;\ A)$ is invertible if λ is large enough and $D(A)=\mathrm{Im}(\lambda R(\lambda;\ A))=X$.

Q.E.D.

Remark 3.2. Formula (3.1) shows that the properties of the infinitesimal generator are reduced, in general, to the case of positive C_0-semigroups. However, if the C_0-semigroup in (3.1) is positive, then, the strongly continuous cosine function in (3.1) is not necessarily positive as the example $C(t)x=(\cos bt)x$ $t\in \mathsf{R}$, $x\in \mathsf{R}$, b real and fixed, shows.

In the following result, we point out the relation between a positive C_0-semigroup and a strongly continuous positive cosine function. We consider the C_0-semigroup $V(t)$, $t\geqq 0$ defined in section 1.

Proposition 3.3. *Let $C(t)$, $t\in \mathsf{R}$ be a strongly continuous cosine function with infinitesimal generator A defined in a ordered Banach space X.*

If $C(t)\geqq I$ for all $t\in \mathsf{R}$ then $V(t)\geqq 0$ for all $t\in \mathsf{R}_+$. Conversely, if $V(t)\geqq 0$ for all $t\in \mathsf{R}_+$ then $C(t)\geqq 0$ for all $t\in \mathsf{R}$.

Proof.
i) As in section 2, from (2.8) and the identity $AR(\lambda^2;\ A)=\lambda^2 R(\lambda^2;\ A)-I$ for $\lambda>w$, we obtain, by hypothesis

$$AR(\lambda^2;\ A)\geqq 0 \quad \text{for all} \quad \lambda>w. \tag{3.3}$$

Now, we observe that if $\lambda^2\in\rho(A)$ then $\lambda\in\rho(B)$ and for each $[x,\ y]\in E\times X$ we have:

$$R(\lambda;\ B)\ [x,\ y]=[\lambda R(\lambda^2;\ A)x+R(\lambda^2;\ A)y,\quad AR(\lambda^2;\ A)x+\lambda R(\lambda^2;\ A)y].$$

Therefore, we have from (3.3) that

$$R(\lambda;\ B)\geqq 0 \quad \text{for all} \quad \lambda>w.$$

The result now follows from the well known formula for C_0-Semigroups

$$V(t)=\lim_{n\to\infty}\ (n/t)\ R(n/t;\ B)^n,\quad t>0.$$

ii) We obtain by definition of $V(t)$ that

$$V(t)\ [0,\ y]=[S(t)y,\ C(t)y] \quad \text{for all} \quad y\in X,\ t\geqq 0$$

and the result now follows from the hypothesis.

Q.E.D.

Example 3.4. There exist strongly continuous positive cosine functions with unbounded infinitesimal generator.

Let $X=C(\mathsf{R})$ be the Banach space of the bounded continuous functions on R with the sup-norm. The following set

$$P=\{f\in X/f(s)\geqq 0 \quad \text{for all} \quad s\in \mathsf{R}\}$$

defines a positive cone in X, and the formula

$$(C(t)f)x=(f(x+t)+f(x-t))/2, \quad x, \ t\in \mathrm{R}, \ f\in X$$

defines a strongly continuous positive cosine function with infinitesimal generator

$$(Af)x=f''(x), \quad x\in \mathrm{R}, \quad f\in D(A)$$
$$D(A)=\{f\in X/f''\in X\},$$

and it is clear that A is unbounded.

Let X be an ordered Banach space with positive cone P and $T(t)$, $t\in \mathrm{R}$ a strongly continuous positive group of operators defined in X with infinitesimal generator A (see [6]). Then the formula

$$C(t)x=(T(t)x+T(-t)x)/2, \quad x\in X, \quad t\in \mathrm{R}$$

defines a strongly continuous positive cosine function with infinitesimal generator A^2.

Example 3.5. Let $C(t)$, $t\in \mathrm{R}$ be a strongly continuous positive cosine function with infinitesimal generator A. Let $b\in \mathrm{R}$ be fixed. Then, the series

$$C_b(t)x=\sum_{n=0}^{\infty} b^{2n}C_n(t)x \quad x\in X, \quad t\in \mathrm{R},$$

where

$$C_0(t)=C(t), \quad C_n(t)x=\int_0^t S(t-s)C_{n-1}(s)x \, ds, \quad x\in X, \quad t\in \mathrm{R}$$

defines also a strongly continuous positive cosine function, but now with infinitesimal generator $A_b=A+b^2I$ (see [2] p. 60, lemma 4.1).

Remark 3.6. Necessary and sufficient conditions such that a strongly continuous cosine function be positive have been also considered in [7], where, moreover, it is shown that the positiveness of a strongly continuous cosine function is a sufficient condition for the positiveness of the "resolvent" of the Volterra equation

$$u(t)=f(t)+\int_0^t a(t-s)Au(s)\, ds,$$

when A is an infinitesimal generator of a strongly continuous cosine function and $a(t)$ is an appropriate kernel.

Acknowledgements. I am indebted to Professor L. Zsidó for several discussions on the subject of this paper and for their many useful suggestions, and also to DAAD for financial support.

References

1. H. O. Fattorini. Ordinary differential equations in linear topological spaces. *I. J. Differential Eq.*, 5, 1968, 72-105.
2. H. O. Fattorini. Second order linear differential equations in Banach spaces. North Holland, Mathematics studies 108, 1985.

3. J. Kisynski. On cosine operator functions and one parameter semigroups of operators. *Studia Math.*, **44**, 1972, 93-105.
4. D. Lutz. Über operatorwertige Lösungen der Funktionalgleichung des Cosinus. Math. Z., **171**, 1980, 233-245.
5. D. Lutz. Periodische operatorwertige Cosinusfunktionen. Res. Math., **4**, 1981, 75-83.
6. R. Nagel (ed.). One parameter semigroups of positive operators. Lecture Notes in Math., vol. **1184**, Springer Verlag, Berlin and New York, 1986.
7. J. Prüß. Positivity and regularity of hyperbolic Volterra equations in Banach spaces. Math. Ann., **279**, 1987, 317-344.
8. H. H. Schaefer. Banach lattices and positive operators. Springer-Verlag, New York—Heidelberg, Berlin, 1974.

Carlos Lizama
Dpto. Matématicas y C. C.
Universidad de Santiago de Chile
Casilla 5659-Correo 2
Santiago—
CHILE

Received 17.04.1989

CURRENT ADDRESS
Universität Stuttgart
Pfaffenwaldring 57
7000 Stuttgart 80
WEST GERMANY