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On the Gaps between Consecutive k-Free Numbers

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1. Introduction

This paper is concerned with the problem of determining a small value of $h=h(x)$, such that for sufficiently large x, there is a 3-free number in the interval $(x, x+h]$. This problem is a special case of the problem for the k-free numbers in short intervals. The first results are concerned with the case $k=2$ (squarefree numbers). E. Fogels [4] proved that there is a squarefree number in the interval $(x, x+x^{\theta}]$, where $\theta=2/5+\varepsilon$ (ε-fixed arbitrary positive number) and x-sufficiently large. Later K. F. Roth [10] obtained that one can take $\theta=1/4+\varepsilon$ by using only elementary methods, and $\theta=3/13+\varepsilon$ using the method of the trigonometric sums. Due to more precise estimates of exponential sums H. E. Richert [9], R. A. Rankin [8], P. G. Schmidt [11] and W. Graham and G. Kolesnik [5] obtained the improvements for h, $h=x^{\theta}$, where $\theta=2/9$, $\theta=0.221982\ldots$, $\theta=109556/494419=0.221585\ldots$ and $\theta=1057/4785=0.2208986\ldots$ respectively. In the same time results using only elementary methods were proved M. Nair [7] $\theta=1/4$, M. Filaseta [1] $\theta=3/13$. Recently, independent of each other the author in [12] and M. Filaseta [3] have given two different elementary ways for obtaining the result of H. E. Richert ($\theta=2/9$). In [12] is obtained a further improvement for $\theta-\theta=17/77=0.2207792\ldots$ using exponential sum techniques. The best result to date for $k=2$ is due to Michael Filaseta [3] $-\theta=47/217+\varepsilon=0.2165898\ldots+\varepsilon$.

Our main interests are in the more general k-free problem. First H. Halberstam and K. F. Roth [6] established that for any $\varepsilon>0$ and x sufficiently large there is a k-free number in the interval $(x, x+h]$, where $h=x^{\theta+\varepsilon}$, $\theta=1/2k$. M. Nair's approach [7] implies that $\theta=1/2k$ is permissible. This result is slightly improved in [6] $\theta=\dfrac{1}{2k+w(k)}$, where $w(k)=0(2^{-k})$ using exponential sum techniques. An important step was made by Michael Filaseta. He proved in [2] that one may take $\theta=\dfrac{1}{2k+1/3}$, when $k=3$ or 4 and $\theta=\dfrac{1}{2k+1/5}$ if $k\geqq 5$.

We prove

Theorem 1. *There is a constant c, such that for x sufficiently large, there is a 3-free number in the interval* $(x, x+c.x^{\theta}]$, *where* $\theta=7/46$.

In other words we replace the constant 1/3 with 4/7 in the case $k=3$. The author thinks that the method developed in this paper will also work for $k>3$. Section 2 includes some preliminaries. In section 3 we derive some new estimates based on polynomial identities. The final result is proved in Section 4.

Notation

$c, c_0, c_1, c_2, c_3, \ldots$ are absolute positive constants

$\theta=7/46$

$h=c.x^{\theta}$

φ is a number greater than θ, but φ may depend on x. In all cases $x^{\varphi}\geqq x^{\theta}.\sqrt{\log x}$ and $x^{\theta}=o(x^{\varphi})$.

u, u', a, b, i represent positive integers.

$\alpha, \beta, \gamma, \delta, u_1, u_2, v_1, v_2, A, B, C$ represent real numbers.

p is a prime number.

2. Preliminaries

Let S denote the number of integers in $(x, x+h]$ which are not 3-free. Then

$$S\leqq\sum_{p}\left(\left[\frac{x+h}{p^3}\right]-\left[\frac{x}{p^3}\right]\right)=S_1+S_2,$$

$$S_1=\sum_{p\leqq x^{\theta}.\sqrt{\log x}}\left(\left[\frac{x+h}{p^3}\right]-\left[\frac{x}{p^3}\right]\right),$$

$$S_2=\sum_{p>x^{\theta}.\sqrt{\log x}}\left(\left[\frac{x+h}{p^3}\right]-\left[\frac{x}{p^3}\right]\right).$$

It is evident that $S_1\leqq\sum_{p\leqq x^{\theta}.\sqrt{\log x}}\left(\frac{h}{p^3}+1\right)<\left(\sum_{n=2}^{\infty}\frac{1}{n^3}\right)+\pi(x^{\theta}.\sqrt{\log x})<\frac{1}{2}h$ for x sufficiently large.

Thus we need to prove that $S_2<c_1.h$ $(c_1<1/2)$.

Let estimate

(1) $$S_2\leqq S_2'=\sum_{x^{\theta}\sqrt{\log x}\leqq n\leqq 2.x^{1/3}}\left(\left[\frac{x+h}{n^3}\right]-\left[\frac{x}{n^3}\right]\right).$$

$n>x^{\theta}$, hence $\left[\frac{x+h}{n^3}\right]-\left[\frac{x}{n^3}\right]=0$ or 1 and $\left[\frac{x+h}{n^3}\right]-\left[\frac{x}{n^3}\right]=1$ if and only if there exists an integer k such that $k.n^3\in(x, x+h]$.

Define

(2) $$S(A, B)=\{n\in(A, B]\cap\mathbb{N}\mid\exists\, k\in\mathbb{N}: k.n^3\in(x, x+h]\}.$$

Then $S_2'=|S(x^{\theta}\sqrt{\log x}, 2x^{1/3})|$.

Now we use the following (cf. [2]).

Lemma 1. *If*

(3) $$|S(x^{\varphi},\ 2x^{\varphi})| \ll x^{\alpha-\beta\varphi} \quad for \quad u_1 \leqq \varphi \leqq v_1, \quad and \quad \beta > 0$$

then $|S(x^{u_1},\ x^{v_1})| \ll_{\beta} x^{\alpha-\beta u_1}$.

If

(4) $$|S(x^{\varphi},\ 2x^{\varphi})| \ll x^{\gamma+\delta\varphi} \quad for \quad u_2 \leqq \varphi \leqq v_2, \quad and \quad \delta > 0$$

then $|S(x^{u_2},\ x^{v_2})| \ll_{\delta} x^{\gamma+\delta v_2}$.

3. Auxiliary results

Lemma 2. *Let* $u,\ u+l_1,\ u+l_2 \in S(x^{\varphi},\ 2x^{\varphi})$, $\theta < \varphi < 1$, $l_1 < l_2$ *and* $l_2 = o(x^{\varphi})$. *Then there exists a constant* c_2, *such that* $l_1 . l_2^4 \geqq c_2 . x^{6\varphi-1}$ *for* x *sufficiently large.*

Proof. Since $u,\ u+l_1,\ u+l_2 \in S(x^{\varphi},\ 2x^{\varphi})$ there exist positive integers k_0, k_1 and k_2 such that k_0u^3, $k_1(u+l_1)^3$, $k_2(u+l_2)^3 \in (x,\ x+h]$. Then

(5) $$k_0 = \frac{x}{u^3} + 0\left(\frac{h}{u^3}\right), \quad k_1 = \frac{x}{(u+l_1)^3} + 0\left(\frac{h}{u^3}\right), \quad k_2 = \frac{x}{(u+l_2)^3} + 0\left(\frac{h}{u^3}\right).$$

Since $l_1 + l_2 = o(x^{\varphi})$ one can write the Taylor-series expansions of $\dfrac{x}{(u+l_1)^3}$ and $\dfrac{x}{(u+l_2)^3}$.

(6) $$\frac{x}{(u+l_1)^3} = \frac{x}{u^3} - \frac{3xl_1}{u^4} + \frac{6xl_1^2}{u^5} - \frac{10xl_1^3}{u^6} + \frac{15xl_1^4}{u^7} - \dots$$

$$\frac{x}{(u+l_2)^3} = \frac{x}{u^3} - \frac{3xl_2}{u^4} + \frac{6xl_2^2}{u^5} - \frac{10xl_2^3}{u^6} + \frac{15xl_2^4}{u^7} - \dots$$

Then we obtain the identities

$$k_0 - k_1 = \frac{3xl_1}{u^4} + O\left(\frac{xl_1^2}{u^5}\right) + O\left(\frac{h}{u^3}\right).$$

If $k_0 = k_1$ then $|k_0u^3 - k_0(u+l_1)^3| \geqq 3k_0u^2l_1 > u > h \quad (u \geqq x^{\varphi} \geqq x^{\theta})$ which is a contradiction with k_0u^3, $k_1(u+l_1)^3 \in (x,\ x+h]$.

Then $k_0 \neq k_1$ and $|k_0 - k_1| \geqq 1$ $(k_0,\ k_1 \in \mathbf{N})$.

But $O(h/u) = o(1)$ and $\dfrac{x . l_1^2}{u^5} = o\left(\dfrac{x . l_1}{u^4}\right)$.

Then for x sufficiently large we obtain

$$\frac{3.x.l_1}{u^4} \geqq \frac{1}{2},$$

(7) $$l_1 \geqq \frac{1}{6}.x^{4\varphi-1}.$$

Now consider the identity

(8) $$A=k_0.(-u+l_1)+k_1.(u+2l_1)=-\frac{2xl_1^3}{u^5}+\left(\frac{xl_1^4}{u^6}\right)+\left(\frac{h}{u^2}\right)$$

((8) follows from (5) and (6)).

Again $\frac{x.l_1^4}{u^6}=o\left(\frac{x.l_1^3}{u^5}\right)$ $(l_1=o(x^\varphi))$ and $\frac{h}{u^2}=o\left(\frac{l_1^3}{u^5}\right)$ (because $l_1^3>l_1 > \frac{1}{6}.x^{4\varphi-1}$ and $\varphi>\theta$).

Then the case $A=0$ is impossible for x sufficiently large. (If $A=0$ we get that $\frac{l_1^3}{u^5}=O\left(\frac{x.l_1^4}{u^6}\right)+O\left(\frac{h}{u^2}\right)$.) Then $|A|\geqq 1$ $(A\in \mathbf{Z})$ and we obtain $\frac{2.x.l_1^3}{u^5}\geqq\frac{1}{2}$,

(9) $$l_1^3 \geqq \frac{1}{4}.x^{5\varphi-1}.$$

The third identity we consider in this lemma is:

$$B=k_0.[3(l_1-l_2).u+l_2^2-l_1^2]+k_1.[3l_2.u+l_2(5l_1-l_2)]$$
$$+k_2.[-3l_1.u+l_1(l_1-5l_2)]=-\frac{5x.l_1.l_2.(l_1-l_2).(l_1^2-l_1l_2+l_2^2)}{u^6}$$
$$+O\left(\frac{x.l_1.l_2^4.(l_2-l_1)}{u^7}\right)+O\left(\frac{h.l_2^2}{u^2}\right).$$

(We can write the first remainder term $O\left(\frac{x.l_1.l_2^4.(l_2-l_1)}{u^7}\right)$ in this form because one can easily see that the coefficients of u^{-8}, u^{-9},... contain the factor $l_1.(l_1-l_2)$.)

But $\frac{x.l_1.l_2^4.(l_2-l_1)}{u^7}=o\left(\frac{x.l_1.l_2^3.(l_2-l_1)}{u^6}\right)$ and $\frac{h.l_2}{u^2}=o\left(\frac{x.l_1.l_2^3.(l_2-l_1)}{u^6}\right)$, because $\frac{x.l_1.l_2^3.(l_2-l_1)}{u^6}>\frac{l_2.l_1^3.x^{1-5\varphi}}{32.u}\geqq(l_2-l_1\geqq 1)\geqq\frac{l_2}{128.u}$ (see (9), $(l_2.h)/u^2$

$$=o(h/u)\ (\varphi>\theta) \text{ and } l_1^2-l_1l_2+l_2^2\geqq\frac{3}{4}.l_2^2).$$

Then for x sufficiently large we get $B \neq 0$,

$$\frac{5x \,.\, l_1 \,.\, l_2 \,.\, (l_2 - l_1) \,.\, (l_1^2 - l_1 l_2 + l_2^2)}{u^6} \geqq \frac{1}{2}$$

and

$$l_1 \,.\, l_2^4 \geqq \frac{1}{10} \,.\, x^{6\varphi - 1} \,. \tag{10}$$ ■

Define $t(a) = |\{u \in S(x^\varphi, 2x^\varphi)$ such that $u + a$ is the next element of $S(x^\varphi, 2x^\varphi)\}|$.

Then $S(x^\varphi, 2x^\varphi) = \sum_{a=1}^{\infty} t(a)$.

From Lemma 2 it follows that

$$\sum_{a=1}^{x_1} t(a) \leqq \sum_{a \geqq x_1}^{\infty} t(a) + 1, \quad \text{where} \quad x_1 = c_2^{1/5} \,.\, x^{\frac{6\varphi - 1}{5}}$$

(to every $a < c_2^{1/5} \,.\, x^{\frac{6\varphi - 1}{5}}$ we may relate his right "neighbour").

Then $S(x^\varphi, 2x^\varphi) \leqq 2 \,.\, \sum_{a \geqq x_1}^{\infty} t(a) + 1$.

Let u, $u+a$ and $u+a+b$, $u+2a+b$ are two "couples" of elements of the set $S(x^\varphi, 2x^\varphi)$. We want to estimate the minimal distance between them. For technical reasons we use a different notation in the following

Lemma 3. *Let $u-d$, $u-c$, $u+c$, $u+d$ $(d > c)$ are elements of $S(x^\varphi, 2x^\varphi)$ and $d = o(x^\varphi)$. Then there exists a constant c_3 such that for x sufficiently large $(d-c)^3 \,.\, d^5 \geqq c_3 \,.\, x^{8\varphi - 1}$.*

Proof. The proof is based on three polynomial identities. Since $u-d$, $u-c$, $u+c$, $u+d \in S(x^\varphi, 2x^\varphi)$ there exist integers k_0, k_1, k_2, k_3 such that $k_0(u-d)^3$, $k_1(u-c)^3$, $k_2(u+c)^3$, $k_3(u+d)^3 \in S(x^\varphi, 2x^\varphi)$. Again we can use the Taylor-series expansions of $\frac{x}{(u-d)^3}$, $\frac{x}{(u-c)^3}$, $\frac{x}{(u+c)^3}$ and $\frac{x}{(u+d)^3}$.

The first identity we consider is

$$\frac{A}{2} = k_0 \,.\, (-u + 2d - c) + k_1 \,.\, (u + d - 2c) + k_2 \,.\, (u - d + 2c) + k_3 \,.\, (-u - 2d + c)$$

$$= \frac{2x \,.\, (d-c)^3 \,.\, (d+c)}{u^6} + O\left(\frac{x \,.\, (d-c)^3 \,.\, d^3}{u^8}\right) + O\left(\frac{h}{u^2}\right).$$

(The reason to write the first error term in the form $O\left(\frac{x \,.\, (d-c)^3 \,.\, d^3}{u^8}\right)$ is that the coefficients of $u^{-(2n+1)}$ are 0, of $u^{-2n} - (4n-2)x \,.\, [(n-2)d^{2n-2} - (n-1)d^{2n-3}c + (n-1)d \,.\, c^{2n-3} - (n-2)c^{2n-2}]$ and all of them contain the factor $(d-c)^3$.)

But $\frac{(d-c)^3 \,.\, d^3}{u^8} = o\left(\frac{(d-c)^3 \,.\, d}{u^6}\right)$ $(d=o(x^\varphi))$ and $\frac{h}{u^2} = o\left(\frac{x \,.\, (d-c)^3 \,.\, d}{u^6}\right)$ (in (9) we proved that $(d-c)^3 \geqq \frac{1}{4} \,.\, x^{5\varphi-1}$ and $d \geqq 1$).

Then for x sufficiently large $A \neq 0$, $|A| \geqq 1$ ($2c$ and $2d \in \mathbb{Z}$) and $\frac{2x \,.\, (d-c)^3 \,.\, (d+c)}{u^6} \geqq \frac{1}{4}$,

$$(d-c)^3 \,.\, d \geqq \frac{1}{8} \,.\, x^{6\varphi-1}. \tag{11}$$

The second identity is

$$\begin{aligned}\frac{B}{4} &= k_0 \,.\, (-3(c+d)u + 5d^2 + 5cd - 4c^2) + k_1 \,.\, (3(c+d)u + 4d^2 - 5cd - 5c^2)\\ &\quad + k_2 \,.\, (-3(c+d)u + 4d^2 - 5cd - 5c^2) + k_3 \,.\, (3(c+d)u + 5d^2 + 5cd - 4c^2)\\ &= \frac{24x \,.\, (d-c)^3 \,.\, (d+c) \,.\, (d^2+3cd+c^2)}{u^7} + O\left(\frac{x \,.\, (d-c)^3 \,.\, d^5}{u^9}\right) + O\left(\frac{h \,.\, d}{u^2}\right).\end{aligned}$$

(The coefficients of u^{-2n} are 0, of $u^{-(2n+1)} - 8nx$. $[(n-2)d^{2n} + (n-2)d^{2n-1}c - (2n-1)d^{2n-2}c^2 + (2n-1)d^2c^{2n-2} - (n-2)d \,.\, c^{2n-1} - (n-2)c^{2n}]$ and all of them contain the factor $(d-c)^3$.)

But $\frac{(d-c)^3 \,.\, d^5}{u^9} = o\left(\frac{(d-c)^3 \,.\, d^3}{u^7}\right)$ and $\frac{h \,.\, d}{u^2} = o\left(\frac{(d-c)^3 \,.\, d^3}{u^7}\right)$ (this follows from $d \geqq 1$ and (11)).

Then for x sufficiently large $B \neq 0$ and

$$\frac{24x \,.\, (d-c)^3 \,.\, (d+c) \,.\, (d^2+3cd+c^2)}{u^7} \geqq \frac{1}{8},$$

$$(d-c)^3 \,.\, d^3 \geqq \frac{1}{1920} \,.\, x^{7\varphi-1}. \tag{12}$$

The third identity is

$$\begin{aligned}\frac{C}{8} &= (k_0[-2cd \,.\, u + c(3d^2 - c^2)] + k_1[2cd \,.\, u + d(d^2 - 3c^2)] + k_2[2cd \,.\, u - d(d^2 - 3c^2)]\\ &\quad + k_3 \,.\, [-2cd \,.\, u - c(3d^2 - c^2)] = \frac{14xc \,.\, d^4 \,.\, (d-c)^3}{u^8} + O\left(\frac{xc \,.\, d^6 \,.\, (d-c)^3}{u^{10}}\right)\\ &\quad + O\left(\frac{h \,.\, c \,.\, d}{u^2}\right) + O\left(\frac{h \,.\, d^3}{u^3}\right).\end{aligned}$$

(The coefficients of $u^{-(2n+1)}$ are 0, of $u^{-2n}-(4n-2)x.[(n-3)c.d^{2n-1}-(n-1)c^3d^{2n-3}+(n-1)c^{2n-3}d^3-(n-3)c^{2n-1}d]$ and contain the factor $c.d.(d-c)^3$.)

But
$$\frac{c.d^6.(d-c)^3}{u^{10}}=o\left(\frac{c.d^4.(d-c)^3}{u^8}\right)$$
$$\frac{h.c.d}{u^2}=o\left(\frac{c.d^4.(d-c)^3}{u^8}\right) \quad \text{(see (12))}$$
and
$$\frac{h.d^3}{u^3}=o\left(\frac{c.d^4.(d-c)^3}{u^8}\right) \quad \text{(see (11))}.$$

Then $C\neq 0$ for sufficiently large x and
$$\frac{2xc.d^4.(d-c)^3}{u^8}\geqq\frac{1}{16}, \quad (d-c)^3.d^5\geqq\frac{1}{32}.x^{8\varphi-1} \quad (d>c). \quad ■$$

In other words we have proved that, if u, $u+a$ and $u+a+b$, $u+2a+b$ are two "couples" of $S(x^\varphi, 2x^\varphi)$ then

(13) $$a^3.(a+b)^5\geqq c_3.x^{8\varphi-1}$$

for x sufficiently large. $(b=c/2,\ a=d-c)$

Corollary 1. *Let $\varphi>1/8$. Then for x sufficiently large there exists a constant c_4 such that $S(x^\varphi, 2x^\varphi)\leqq c_4.x^{\frac{1+5\varphi}{13}}$.*

Proof. We use that:

(14) $$\sum_{a=1}^{\infty} t(a)=S(x^\varphi, 2x^\varphi)$$

and

(15) $$\sum_{a=1}^{\infty} a.t(a)=x^\varphi.$$

From (15) follows
$$\sum_{a\geqq N}^{\infty} t(a)\leqq\frac{x^\varphi}{N}.$$

There are two cases to be studied.

I case: $a\geqq b$

Then
$$a^8\geqq\frac{c_3}{32}.x^{8\varphi-1}, \quad a\geqq(c_3/32)^{1/8}.x^{\varphi-1/8}=a_1$$
$$\sum_{a\geqq a_1}^{\infty} t(a)\leqq c_5.x^{1/8} \quad (c_5=(c_3/32)^{1/8}).$$

II case $a<b$.

Then $a^3 . b^5 \geqq \frac{c_3}{32} . x^{8\varphi-1}$, $b \geqq b_{\min} = (c_3/32)^{1/5} . x^{(8\varphi-1)/5} . a^{-3/5}$ and the number $t(a)$ of the "couples" u, $u+a$ is not exceeding $\frac{x^{\varphi}}{b_{\min}}+1$, $t(a) \leqq c_6 . x^{(1-3\varphi)/5} . a^{8/5} (c_6 = (32/c_3)^{1/5})$.

Let $a_2 = x^{(8\varphi-1)/13}$.

Then

$$\sum_{a=1}^{a_2} t(a) \leqq c_6 . a_2^{3/5} . x^{(1-3\varphi)/5} = c_6 . x^{(1+5\varphi)/13}$$

and

$$\sum_{a \geqq a_2} t(a) \leqq \frac{x^{\varphi}}{a_2} = x^{(1+5\varphi)/13}.$$

Finally we obtain $S(x^{\varphi}, 2x^{\varphi}) \leqq (c_6+1) . x^{(1+5\varphi)/13} + c_5 . x^{1/8}$. But $\frac{1+5\varphi}{13} > \frac{1}{8}$ $(\varphi > 1/8)$. ■

4. Main results

First we will prove (following an idea of M. Filaseta)

Lemma 4. *Let u, $u+l_1$ and $u+u_1$, $u+u_1+l_2 \in S(x^{\varphi}, 2x^{\varphi})$, $l_1 \neq l_2$. Let $l_1^2+l_2^2 = o(u)$ and $l_2 . u_1 = o(u)$. Then one of the following statements is true*

(i) $|l_1^5 - l_2^5| \geqq c_7 . x^{6\varphi-1}$,

(ii) $|l_1^5 - l_2^5| \leqq c_8 . x^{5\varphi+\theta-1}$.

Proof. There exist integers k_0, k_1 such that $k_0 u^2$, $k_1(u+l_1)^2 \in (x, x+h]$. ($u$ and $u+l_1 \in S(x^{\varphi}, 2x^{\varphi})$).

One can write the identity

$$A_1 = 6u^2(k_0-k_1) - 3ul_1(k_0+5k_1) + l_1^2(k_0-10k_1) = \frac{51x . l_1^5}{u^6} + O\left(\frac{x . l_1^6}{u^7}\right) + O(h/u), \quad (l_1 = o(u)).$$

In the same way one can write

$$A_2 = \frac{51x . l_2^5}{(u+u_1)^6} + O\left(\frac{x . l_2^6}{u^7}\right) + O(h/u)$$
$$= \frac{51x . l_2^5}{u^6} + O\left(\frac{x . l_2^5 . u_1}{u^7}\right) + O\left(\frac{x . l_2^6}{u^7}\right) + O(h/u) \quad (A_2 \in \mathbf{Z}).$$

Then

$$A_1-A_2=\frac{51x(l_1^5-l_2^5)}{u^6}+O\left(\frac{x.(l_1^6+l_2^6)}{u^7}\right)+O\left(\frac{x.l_2^5.u_1}{u^7}\right)+O(h/u).$$

There are two cases to be considered

I case. $A_1-A_2=0$

Then $$\frac{51x(l_1^5-l_2^5)}{u^6}+O\left(\frac{x.(l_1^6+l_2^6)}{u^7}\right)+O\left(\frac{x.l_2^5.u_1}{u^7}\right)=O(h/u).$$

But $$|l_1^5-l_2^5|>l_1^4+l_2^4 \quad (l_1-l_2\neq 0)$$

and

(16) $$\frac{x.(l_1^6+l_2^6)}{u^7}=o\left(\frac{x.(l_1^4+l_2^4)}{u^6}\right) \quad (l_1^2+l_2^2=o(u))$$

(17) $$\frac{x.l_2^5.u_1}{u^7}=o\left(\frac{x.l_2^4}{u^6}\right) \quad (l_2.u_1=o(u)).$$

In this case we obtain that there exists a c_8, such that

$$\frac{x.|l_1^5-l_2^5|}{u^6}\leqq c_8.\frac{h}{u}, \quad |l_1^5-l_2^5|\leqq c_8.x^{\theta+5\varphi-1}.$$

II case. $A_1-A_2\neq 0$

Then $|A_1-A_2|\neq 0$. Since $O(h/u)=o(1)$ then for sufficiently large x we get

$$\frac{51x|l_1^5-l_2^5|}{u^6}\geqq\frac{1}{2}, \quad |l_1^5-l_2^5|\geqq\frac{1}{102}.x^{6\varphi-1}$$

(see (16) and (17)). ■

Proof of the Theorem. Let $1/8<\varphi<9/35$. Let us divide the interval $[x^\varphi, 2x^\varphi]$ to equal subintervals with length $|I|=c_9.x^{(10\varphi-1)/11}$, where c_9 is a constant, such that $2.c_9^5<c_3$.

Let I is one of these subintervals.

Define t_i — the set of all intervals $[u, u_1]$ which have the properties:

1) u and u_1 are consecutive elements of $S(x^\varphi, 2x^\varphi)$,

2) u and $u_1\in I$,

3) $|u-u_1|\in[2^i.x^{\frac{6\varphi-1}{5}}, 2^{i+1}.x^{\frac{6\varphi-1}{5}})$.

Let $S_I=|I\cap S(x^\varphi, 2x^\varphi)|$.

We have proved that $|S(x^\varphi, 2x^\varphi)|\leqq 2\,\Sigma_{a\geqq x_1}t(a)+1$, where $x_1=c_2.x^{\frac{6\varphi-1}{5}}$.

In the same way we prove that

$$S_I\leqq 2.\sum_{\substack{a\geqq x_1\\ u,u+a\in I}} t(a)+1, \quad S_I\leqq 2.\sum_{i=c_{10}}^{\infty}|t_i|+1 \quad \text{where } 2^{c_{10}}<c_2.$$

Let J is the integer number, such that

$$2^J\leqq x^{\frac{3-8\varphi}{165}}<2^{J+1}.$$

Then

$$\sum_{i=c_{10}}^{\infty} |t_i| = \sum_{i=c_{10}}^{J} |t_i| + \sum_{i=J+1}^{\infty} |t_i|.$$

But

$$\sum_{i=c_{10}}^{\infty} 2^i . x^{\frac{6\varphi-1}{5}} . |t_i| \leqq |I|, \quad \text{thus} \quad \sum_{i=J+1}^{\infty} |t_i| \leqq \frac{|I|}{x^{\frac{(6\varphi-1)}{5}} . 2^J} \leqq 2c_9 . x^{\frac{3-8\varphi}{33}}.$$

Now we want to estimate $\sum_{i=c_{10}}^{\infty} |t_i|$ and for this purpose we derive an estimate for $|t_i|$.

Lemma 5. *Let $a_1, a_2, \ldots, a_m$ are different integer numbers in the finite interval K with the property: for every $1 \leqq i < j \leqq m$ $|a_i - a_j| \leqq P$ or $|a_i - a_j| \geqq Q$. Then $m \leqq (|K|/Q+1).(2P+1)$.*

Proof. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_s}$ is the maximal set such that $|a_{i_j} - a_{i_k}| \geqq Q$ for every $1 \leqq j \neq k \leqq s$. It is evident that $s \leqq |K|/Q+1$. For every a_{i_l} there is no more than $2P$ $a_i's$ such that $|a_{i_l} - a_i| \leqq P$. ■

Let $[u, u+l_1]$ and $[u+u_1, u+u_1+l_2]$ are two elements of t_i. ($u, u+l_1, u+u_1, u+u_1+l_2 \in S(x^\varphi, 2x^\varphi) \cap I$ and $l_1, l_2 \in [2^i x^{\frac{6\varphi-1}{5}}, 2^{i+1} x^{\frac{6\varphi-1}{5}})$.)

Let $i<J$. Then the conditions of Lemma 4 are satisfied.

$$l_1^2 \leqq 2^{2(J+1)} . x^{\frac{2(6\varphi-1)}{5}} \leqq 4 . x^{\frac{76\varphi-12}{33}} = o(x^\varphi) \ (\varphi < 9/35).$$

$$l_1 . u_1 \leqq 2^{J+1} . x^{\frac{6\varphi-1}{5}} . |I| \leqq 2c_9 . x^{\frac{68\varphi-9}{33}} = o(x^\varphi).$$

Then $\qquad |l_1 - l_2| \leqq \frac{c_7}{5} . 2^{-4i} x^{\frac{6\varphi-1}{5}} \quad (l_1, l_2 \in t_i)$

or $\qquad |l_1 - l_2| \geqq c_8 . 2^{-4i} . x^{\theta+(\varphi-1)/5}.$

Now we will prove that there are not elements of t_i with equal length.

Let assume that $[u, u+a]$ and $[u+b+a, u+b+2a]$ are elements of $S(x^\varphi, 2x^\varphi)$. In (13) we proved that $a^3(a+b)^5 \geqq c_3 . x^{8\varphi-1}$ for x is sufficiently large.

But $a \leqq 2^{J+1} . x^{\frac{6\varphi-1}{5}}$ $(i<J)$ and $a+b \leqq |I| = c_9 . x^{(10\varphi-1)/11}$. Then $a^3(a+b)^5 \leqq 2c_9^5 . x^{\frac{3-8\varphi}{55} + \frac{3(6\varphi-1)}{5} + \frac{5(10\varphi-1)}{11}} = 2c_9^5 . x^{8\varphi-1}$. But $2c_9^5 < c_3$ and the assumption lead us to contradiction.

From Lemma 4 and Lemma 5 there follows:

$$|t_i| \leqq \left(|I| / \left(\frac{c_7}{5} . 2^{-4i} . x^{(6\varphi-1)/5} \right) + 1 \right) . (2 . c_8 . 2^{-4i} . x^{\theta+(\varphi-1)/5} + 1)$$

$$\leqq c_{11} . (2^{5i} + 2^i . x^{\theta+(\varphi-1)/5})$$

$$((6\varphi-1)/5 < (10\varphi-1)/11, \ c_{11} = \max(20c_8/c_7, \ 10/c_7)).$$

$$\sum_{i=c_{10}}^{J} |t_i| \leqq 2c_{11} . (2^{5J} + 2^J . x^{\theta+(\varphi-1)/5}) = 2c_{11}(x^{(8\varphi-1)/33} + x^{\theta+(5\varphi-6)/33})$$

and we get that

$$S_I=|S(x^\varphi,\ 2x^\varphi)\cap I|\leqq 2(c_{11}+c_9).(x^{(8\varphi-1)/33}+x^{\theta+(5\varphi-6)/33})$$

$$c_{12}=2(c_{11}+c_9).$$

$$|S(x^\varphi,\ 2x^\varphi)|\leqq\left(\frac{x^\varphi}{|I|}+1\right).S_I\leqq\frac{2x^\varphi}{|I|}.S_I\leqq 2c_{12}(x^{(6-5\varphi)/33}+x^{\theta+(8\varphi-3)/33}).$$

But $(8\varphi-3)/33<-1/35$, $(\varphi<9/35)$.
Then for $1/6<\varphi<9/35$ we have proved

$$|S(x^\varphi,2x^\varphi)|\leqq 2c_{12}(x^{(6-5\varphi)/33}+x^{\theta-1/35}).$$

Let $\theta=7/46$.
Then

$$|S(x^{9/46},\ x^{11/46})|\leqq c_{13}.x^{(6-5.9/46)/33}+o(x^\theta)=(11/46<9/35)=c_{13}.x^{7/46}+o(x^\theta).$$

From Corollary 1, there follows

$$|S(x^\theta\sqrt{\log x},\ x^{9/46})|\leqq c_{14}.x^{(1+5.9/46)/13}=c_{14}.x^{7/46}.$$

Finally we use the estimate from [6]

$$|S(x^\varphi,\ 2x^\varphi)|\leqq c_{15}.x^{(1-\varphi)/5}.$$

$$|S(x^{11/46},\ 2x^{1/3})|\leqq c_{16}.x^{(1-11/46)/5}=c_{16}.x^{7/46}.\quad\blacksquare$$

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