Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 1

Hausdorff Distance and the Structure of Certain Function Spaces

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Hausdorff distance was successfully applied to the study of the space of all continuous functions defined on a metric space X with values in a metric space Y. In some of the considerations of this kind the compactness of X is essential. However various results which seem to be of interest may be obtained without assumption of compactness on X if we restrict to the functions "vanishing at infinity". Such results are presented in this paper. Most of them are motivated by the research of Gerald Beer ([1-3]).

1. Introduction

Let X, Y be metric spaces. $C(X, Y)$ stands for the set of all bounded continuous functions from X to Y while $C_0(X, R)=C_0(X)$ denotes the set of all continuous functions vanishing at infinity. Instead of saying that f vanishes at infinity we simply say that f vanishes, or f is vanishing. A continuous function $f: X \to R$ is said to be vanishing if for any $\varepsilon>0$ there exists a compact set $K \subset X$ such that $|f(x)|<\varepsilon$ for each $x \notin K$. A collection $\Omega \subset C_0(X)$ is said to be uniformly vanishing if to any $\varepsilon>0$ there exists a compact set $K \subset X$ such that $|f(x)|<\varepsilon$ for each $f \in \Omega$ and each $x \notin K$.

By d_1 we denote the supremum metric on $C(X, Y)$ or $C_0(X)$. If (X, d_x), (Y, d_y) are metric spaces we denote d the metric in $X \times Y$ defined by

$$d[(x_1, y_1), (x_2, y_2)]=\max\{d_x(x_1, x_2), d_y(y_1, y_2)\}.$$

The Hausdorff distance h_d for two nonempty sets A, B of a metric space (Z, d) will be defined as

$$h_d(A, B)=\inf\{\varepsilon: S_\varepsilon[A]\supset B \text{ and } S_\varepsilon[B]\supset A\},$$

where $S_\varepsilon[E]$ denotes the union of all open ε-balls whose centers run over E.

It is well-known that when restricted to the nonempty closed subsets of Z the Hausdorff distance defines (in general infinite valued) metric. Throughout the paper h_d namely will be used where d is defined as above and $Y=R$ with the usual metric. It will be usually applied to $C_0(X)$, where the function f is identified with its graph, i.e. with a subset of $X \times R$. For the sake of simplicity we will write d_2

instead of h_d. The symbols $S_r[x]$, $B_r[x]$, where $r>0$ denote open and closed balls respectively with the center x and radius r. For a subset A of a metric space the symbol $B_r[A]$ denotes the union of all closed balls with the radius r and the centers running over the set A.

The following results will be frequently used:

A. *For any metric space X the d_1-convergence in $C(X, Y)$ implies d_2-convergence.*

B. ([4] *Theorem* 1) *For any metric space X the d_1-convergence and d_2-convergence in $C_0(X)$ are equivalent.*

C. ([2] *Theorem* 2) *Let (X, d_x) be a compact metric space and let (Y, d_y) be an arbitrary metric space. Then $\Omega\subset C(X, Y)$ is d_2-totally bounded if and only if $\{(x, f(x)): x\in X, f\in\Omega\}$ is a totally bounded subset of $X\times Y$.*

D. ([3] *Lemma*) *Let (X, d_x) and (Y, d_y) be metric spaces. Let x be a limit point of X and let $\Phi: [0, 1]\to Y$ be a path such that $\Phi(0)\neq\Phi(1)$. Then for each pair of positive numbers ε and δ there exist $\{f, g\}\subset C(X, Y)$ such that*

(*i*) *both $g(S_\varepsilon[x])$ and $f(S_\varepsilon[x])$ are δ-dense in Φ ([0, 1]),*

(*ii*) *$g(z)=f(z)=\Phi(0)$ whenever $d_x(z, x)\geqq\varepsilon$,*

(*iii*) *for every z, either $f(z)=\Phi(0)$ or $g(z)=\Phi(0)$.*

2. Results

The following result is a variant of the classical Arzela-Ascoli theorem for the space $C_0(X)$.

Theorem 1. *Let X be a metric space. A set $\Omega\subset C_0(X)$ is d_1-totally bounded if and only if*

(*i*) *$\{f(x): f\in\Omega\}$ is bounded for each $x\in X$,*

(*ii*) *Ω is equicontinuous,*

(*iii*) *Ω vanishes uniformly.*

Proof. Let (i)—(iii) be satisfied. It is sufficient to prove that for any $\varepsilon>0$ there is a finite ε-d_1-dense subset of Ω. Choose a compact set $K\subset X$ such that for each $f\in\Omega$ and each $x\notin K$ we have $|f(x)|<\varepsilon/2$. The set $\{f/K: f\in\Omega\}$ is obviously equicontinuous and the set $\{(f/K)(x): f\in\Omega\}$ is bounded for every $x\in K$. Hence $\{f/K: f\in\Omega\}$ is a d_1-totally bounded set by the usual compactness criterion for $C(K, R)$. Hence ε-d_1-dense subset $\{f_1/K, \dots, f_n/K\}$ of $\{f/K: f\in\Omega\}$ exists. Taking $\{f_1, \dots, f_n\}$ we have an ε-d_1-dense subset of $\{f: f\in\Omega\}$. Thus Ω is d_1-totally bounded.

Conversely, let Ω be a d_1-totally bounded subset of $C_0(X)$. We show first that (iii) is valid. If not, then $\varepsilon_0>0$ exists such that for every compact set K there is $f\in\Omega$ and $x\in K$ such that $|f(x)|>\varepsilon_0$. Choose an arbitrary $f_1\in\Omega$. Let K_1 be such a compact set that $|f_1(x)|\leqq\varepsilon_0/2$ for every $x\notin K_1$. Then there exists $f_2\in\Omega$ and a point $x_2\notin K_1$ such that $|f_2(x_2)|>\varepsilon_0$. So

$$d_1(f_1, f_2)\geqq|f_1(x_2)-f_2(x_2)|>\varepsilon_0/2.$$

In this way it is possible to construct a sequence $\{f_n\}$ of distinct functions belonging to Ω such that $d_1(f_i, f_j) > \varepsilon_0/2$ for $i, j = 1, 2, \ldots, i \neq j$. Thus the subset $\{f_n : n = 1, 2, \ldots\}$ of the totally bounded set Ω is not totally bounded. It is a contradiction so (iii) is proved.

The validity of (i) is obvious. Now suppose that Ω is not equicontinuous at some $x \in X$. Then $\varepsilon > 0$ and sequences $\{f_n\} \subset \Omega$, $\{x_n\} \subset X$ exist such that $d_x(x_n, x) \to 0$ and $|f_n(x_n) - f_n(x)| > \varepsilon$ for $n = 1, 2, \ldots$. Since $\{f_n : n = 1, 2, \ldots\}$ is d_1-totally bounded, a d_1-Cauchy subsequence $\{g_n\}$ of the sequence $\{f_n\}$ exists. So n_0 exists such that for each $n \geqq n_0$ and each $x \in X$ we have $|g_n(x) - g_{n_0}(x)| < \varepsilon/3$. Using this, the inequality

$$|g_n(x_n) - g_n(x)| \leqq |g_n(x_n) - g_{n_0}(x_n)| + |g_{n_0}(x_n) - g_{n_0}(x)| + |g_{n_0}(x) - g_n(x)|$$

and the continuity of g_{n_0} at x, we obtain $|g_n(x_n) - g_n(x)| < \varepsilon$ for sufficiently large n. Since $\{g_n\}$ is a subsequence of $\{f_n\}$ it is a contradiction.

Another criterion of d_1-total boundedness in $C_0(X)$ involving the d_2-total boundedness is the following

Theorem 2. *A set $\Omega \subset C_0(X)$ is d_1-totally bounded if and only if*
(i*) Ω *is d_2-totally bounded,*
(ii) Ω *is equicontinuous,*
(iii) Ω *is uniformly vanishing.*

Proof. Let Ω be d_1-totally bounded. Then (i*) follows from the inequality $d_2(f, g) \leqq d_1(f, g)$ which is true for each $f, g \in C_0(X)$. The validity of (ii) and (iii) follows from Theorem 1.

Suppose now that (i*), (ii), (iii) are satisfied. We prove the d_1-total boundedness of Ω showing that to any $\varepsilon > 0$ a finite $\varepsilon - d_1$-dense subset exists. Since Ω is uniformly vanishing there exists a compact set $K \subset X$ such that for each $f \in \Omega$ and each $x \notin K$ we have $|f(x)| < \varepsilon/2$. Now for any $x \in K$ the equicontinuity of Ω at x implies the existence of $r_x > 0$ such that

(1) $\quad |f(x) - f(z)| < \varepsilon/4$ for each $z \in S_{r_x}[x]$ and each $f \in \Omega$.

There exists $\{x_1, x_2, \ldots, x_n\} \subset K$ such that $K \subset \bigcup_{i=1}^{n} S_{r_i}[x_i]$, where $r_i = r_{x_i}/2$. Put $\delta = \min\{\varepsilon/2, r_i : i = 1, 2, \ldots, n\}$.

Let $\{f_1, \ldots, f_m\}$ be a δ-d_2-dense subset of Ω. We prove that $\{f_1, \ldots, f_m\}$ is also ε-d_1-dense subset of Ω. To show this it is sufficient to prove that for each $f, g \in \Omega$ the inequality $d_2(f, g) < \delta$ implies $d_1(f, g) < \varepsilon$. So suppose $d_2(f, g) < \delta$ is true. If $x \notin K$ then $|f(x) - g(x)| < \varepsilon$. If $x \in K$, then there exists $z \in X$ such that $d[(x, g(x)), (z, f(z))] < \delta$, hence $d_x(x, z) < \delta$ and $|f(z) - g(x)| < \delta$. There exists $x_i \in K$ such that $x \in S_{r_i}[x_i]$. Evidently $z \in S_{2r_i}[x_i]$, so (1) implies $|f(x) - g(x)| \leqq |f(z) - g(x)| + |f(x) - f(z)| < \varepsilon$. So $d_1(f, g) < \varepsilon$.

Since the space $C_0(X)$ is d_1-complete we obtain from the preceding theorem and from the known criterion of compactness in metric spaces, the following compactness criterion in $C_0(X)$

Theorem 3. *A set $\Omega \subset C_0(X)$ is d_1-compact if and only if it is d_1-closed and satisfies (i*), (ii), (iii).*

Now a condition for d_2-total boundedness of $\Omega \subset C_0(x)$ seems to be of interest. Note firstly that (iii) is not necessary for d_2-total boundedness of Ω.

Example 1. Let $X=(0, \infty)$ with the usual metric. For $n=1, 2, \dots$, let $f_n: X \to R$ be such that $f_n(x)=0$ if $x \leqq 1/n$ or $x \geqq 3/n$. On the segments $[1/n, 2/n]$, $[2/n, 3/n]$ let f_n be linear and such that $f_n(2/n)=1$. The set $\Omega=\{f_n : n=1, 2, \dots\}$ is d_2-totally bounded because $\{f_n\}$ is a d_2-cauchy sequence. But Ω does not vanish uniformly.

The condition (iii) is necessary for d_2-total boundedness in certain type of metric spaces.

A metric space is said to be uniformly locally compact, if there exists $\delta>0$ such that for any $x \in X$ the set $\mathrm{cl}\, S_\delta[x]$ is compact.

Theorem 4. *Let X be uniformly locally compact metric space. Let $\Omega \subset C_0(X)$ be d_2-totally bounded. Then Ω vanishes uniformly.*

To prove Theorem 4 we first prove the following

Lemma 1. *Let X be uniformly locally compact metric space with δ such that $\mathrm{cl}\, S_\delta[x]$ is compact for each $x \in X$. Let $K \subset X$ be a compact set and $\eta<\delta$. Then $\{z \in X : d_x(z, K) \leqq \eta\}$ is compact.*

Proof. Denote $L=\{z \in X : d_x(z, K) \leqq \eta\}$. Let $\{x_n\}$ be a sequence of points belonging to L. Let $\{y_n\}$ be sequence of elements from K such that $d_x(x_i, y_i) \leqq \eta$ for $i=1, 2, \dots$. Since K is compact there exists a limit point of $\{y_n\}$, $y_0 \in K$. Without loss of generality we can suppose that $\{y_n\}$ converges to $y_0 \in K$. (In this case of course $\{x_n\}$ will be a subsequence of the original sequence which was chosen.) If n is sufficiently large then $x_n \in \mathrm{cl}\, S_\delta[y_0]$. Hence the compactness of $\mathrm{cl}\, S_\delta[y_0]$ implies that there exists a limit point x_0 of $\{x_n\}$ belonging to $\mathrm{cl}\, S_\delta[y_0]$. What is more we prove that $x_0 \in \mathrm{cl}\, S_\eta[y_0]$. If $\varepsilon<\delta-\eta$ then there exists k such that $d_x(y_0, y_k)<\varepsilon/2$ and also $d_x(x_0, x_k)<\varepsilon/2$. Then

$$d_x(x_0, y_0) \leqq d_x(x_0, x_k)+d_x(x_k, y_k)+d_x(y_k, y_0)<\eta+\varepsilon.$$

Since ε is arbitrary we have $d_x(y_0, x_0) \leqq \eta$, hence $x_0 \in L$. The compactness of L is proved.

Proof of Theorem 4. Suppose that Ω does not vanish uniformly. Let $\varepsilon>0$ be such that for each compact set $K \subset X$ there exists $f \in \Omega$ and $x \notin K$ such that $|f(x)| \geqq \varepsilon$. Choose $f_1 \in \Omega$ arbitrary and a compact set $K_1 \subset X$ such that $|f_1(x)| \leqq \varepsilon/2$ for each $x \notin K_1$. Now let $L_1=\{z \in X : d_x(z, K_1) \leqq \delta/2\}$, where δ is the positive integer from the uniform compactness of X. Since L_1 is compact, by Lemma 1, there exists $f_2 \in \Omega$ and $x_2 \notin L_1$ such that $|f_2(x_2)|>\varepsilon$. Then for every point $x \in X$

$$(2) \quad d[(x_2, f_2(x_2)), (x, f_1(x))]=\max\{d_x(x_2, x), |f_2(x_2)-f_1(x)|\}>\max\{\delta/2, \varepsilon/2\}.$$

In fact, if $x \in K_1$, then $d[(x_2, f(x_2)), (x, f_1(x))] \geqq d_x(x_2, x)>\delta/2$, if $x \notin K_1$, then $d[(x_2, f_2(x_2)), (x, f_1,(x))] \geqq |f_2(x_2)-f_1(x)|>\varepsilon/2$. Denoting $c=\max\{\delta/2, \varepsilon/2\}$ the inequality (2) implies $d_2(f_2, f_1) \geqq c>0$. Now the construction by induction in a natural way may be used to obtain a sequence $\{f_n\}$ of elements of Ω with the property $d_2(f_i, f_j) \geqq c$ for $i \neq j$, $i, j = 1, 2, \ldots$. Obviously there is no Cauchy subsequence of $\{f_n\}$. Thus Ω is not totally bounded.

The Beer's result **C.** on total boundedness may not be without a change transferred to $C_0(X)$ for arbitrary metric space X.

Example 2. Let $X = R$ with the usual metric and $f \in C_0(R)$ be an arbitrary fixed element. Then $\Omega = \{f\}$ is d_2-totally bounded but $\{(x, f(x)) : x \in R\}$ is not totally bounded subset of $R \times R$.

But we have the following result

Theorem 5. *Let X be uniformly locally compact metric space. Then the following are necessary and sufficient for $\Omega \subset C_0(X)$ to be d_2-totally bounded:*

(a) *Ω vanishes uniformly;*

(b) *for each compact set $K \subset X$ the set $\{(x, f(x)) : x \in K, f \in \Omega\}$ is totally bounded in $X \times R$.*

Proof. Let Ω be d_2-totally bounded. Then (a) is satisfied according to Theorem 4. From the d_2-total boundedness of Ω the existence of a finite 1-d_2-dense set $\{f_1, \ldots, f_n\} \subset \Omega$ follows. Because the set is obviously uniformly bounded on X we obtain uniform boundedness of Ω on X, hence for any compact set $K \subset X$ the set $\{f/K : f \in \Omega\}$ is uniformly bounded on K. So a number c exists such that $\{(x, f(x)) : x \in K, f \in \Omega\} \subset K \times [-c, c]$. Thus the set $\{(x, f(x)) : x \in K, f \in \Omega\}$ is a subset of a compact subset of $X \times R$, so it is totally bounded in $X \times R$. Thus (b) is satisfied.

Conversely let (a), (b) be satisfied. Let $\varepsilon > 0$. Since Ω vanishes uniformly, there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon/2$ for each $f \in \Omega$ and each $x \notin K$. According to the assumption $\{(x, f(x)) : x \in K, f \in \Omega\}$ is totally bounded in $K \times R$. So by the result **C.** the set $\{f/K : f \in \Omega\}$ is d_2-totally bounded in $C(K, R)$. So there are $f_1, \ldots, f_n \in \Omega$ such that $f_1/K, \ldots, f_n/K$ is ε-d_2-dense subset of $\{f/K : f \in \Omega\}$. Now it is easy to verify that $\{f_1, \ldots, f_n\}$ is ε-d_2-dense subset of Ω.

Remark 1. A careful examination of the second half of the proof of Theorem 5 shows that in this part no assumption on the metric space X was necessary.

Remark 2. Moreover in the first part of the proof of Theorem 5 the condition of the uniform compactness was used only for proving that Ω vanishes uniformly.

Taking in account Remark 1 we obtain from Theorem 5 the following

Theorem 6. *Let X be arbitrary metric space. If the conditions (a), (b) for $\Omega \subset C_0(X)$ are satisfied, then Ω is d_2-totally bounded.*

The procedure used in the proof of Theorem 5 combined with result **C.** gives also the following useful result

Theorem 7. *Let X be arbitrary metric space. Let $\Omega \subset C_0(X)$ be d_2-totally bounded set. Then for any compact set $K \subset X$ the set $\{f/K : f \in \Omega\}$ is d_2-totally bounded.*

Proof. It follows from the proof of Theorem 5 (see also Remark 2) that $\{(x, f(x)) : x \in K, f \in \Omega\}$ is totally bounded subset of $K \times R$. So by the result **C.** the set $\{f/K : f \in \Omega\}$ is d_2-totally bounded.

Note that the total boundedness of the restrictions $\{f/M : f \in \Omega\}$, where Ω is totally bounded is not always true if M is not compact. In this direction we refer the reader to Example 1 of [4] which may serve to illustrate such situation.

Using the criteria for d_2-total boundedness, the criteria for d_2-compactness of $\Omega \subset C_0(X)$, where Ω is d_2-closed may be obtained. The only thing which is necessary to guarantee is the d_2-completeness.

The conditions for d_2-completeness of $\Omega \subset C(X, Y)$ were studied in [1] and also in [2] under certain conditions on X and Y. Applying one of them we obtain a result which is an analogy to such a criterion for $C(X, Y)$ when X is compact (See [1] Theorem 1).

Theorem 8. *Let X be arbitrary metric space. Let $\Omega \subset C_0(X)$ be d_2-closed. Then Ω is d_2-compact if and only if*

(1) *Each d_2-Cauchy sequence is d_1-Cauchy;*

(2) *for any compact set $K \subset X$ the set $\{(x, f(x)) : x \in K, f \in \Omega\}$ is totally bounded in $X \times R$;*

(3) *Ω vanishes uniformly.*

We use the following Lemma in the proof of Theorem 8.

Lemma 2. *Let X be arbitrary metric space. Let $\Omega \subset C_0(X)$ be d_2-compact. Then Ω vanishes uniformly.*

Proof. The d_2-compactness in $C_0(X)$ is equivalent with d_1-compactness ([4] Theorem 1). So by Theorem 1Ω vanishes uniformly.

Proof of Theorem 8. Let Ω be d_2-compact. If $\{f_n\} \subset \Omega$ is a d_2-Cauchy sequence then d_2-converges to a function $f \in \Omega$. But then $\{f_n\}$ is d_1-convergent according to the result **B.** and so it is d_1-Cauchy. Thus (1) is true. To prove (2) observe that Ω is d_2-totally bounded so (2) follows from Theorem 5 (see also Remark 2). The condition (3) follows from Lemma 2.

Now let (1), (2), (3) be satisfied. Then by Theorem 6 Ω is d_2-totally bounded. The only thing which remains to be proved is the d_2-completeness of Ω. So let $\{f_n\} \subset \Omega$ be d_2-Cauchy sequence. Then by (2) it is d_1-Cauchy sequence and hence d_1-convergent in $C_0(X)$ because $C_0(X)$ is complete. Since by the result **A.** d_1-convergence implies d_2-convergence and Ω is d_2-closed the d_2-limit of $\{f_n\}$ exists and belongs to Ω.

Remark 3. The criterion given in Theorem 8 is obviously also a criterion for d_1-compactness (see Lemma 2). So Theorem 8 is a variant of Arzela-Ascoli Theorem for the space $C_0(X)$.

In the rest of the paper we will discuss a bit deeper the connection between uniform vanishing of $\Omega \subset C_0(X)$ and d_2-total boundedness. We present a class of metric spaces X in which the condition that $\Omega \subset C_0(X)$ is uniformly vanishing for each d_2-totally bounded set Ω is equivalent to the condition that the space is uniformly locally compact.

Lemma 3. *Let X be a locally compact metric space. Let each d_2-totally bounded subset $\Omega \subset C_0(X)$ be uniformly vanishing. Then X is complete.*

Proof. Let X be not complete. Then there exists a Cauchy sequence $\{x_n\}$ in X without a limit point in X. Consider two cases. First suppose that there exists a sequence $\{y_n\} \subset \{x_n\}$ of distinct isolated points. Define for $n=1, 2, \ldots$, $f_n : X \to R$ as

$$f_n(x) = \begin{cases} 1 & \text{if } x = y_n \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \in C_0(X)$ for $n=1, 2, \ldots$ and the set $\Omega = \{f_n : n \in \mathrm{N}\}$ is totally bounded. The set Ω does not vanish uniformly because if it is uniformly vanishing then a compact set $K \subset X$ exists such that $\{x_n\} \subset K$. But this is impossible since $\{x_n\}$ has not a limit point.

If a subsequence of distinct isolated points does not exist, then let $\{y_n\} \subset \{x_n\}$ be a subsequence of distinct points of $\{x_n\}$ each of which is an accumulation point in X. For each n let $0 < \varepsilon_n < 1/n$ be such that $\{\mathrm{cl}\, S_{\varepsilon_n}[y_n] : n \in \mathrm{N}\}$ is a collection of mutually disjoint compact sets. For each $n \in \mathrm{N}$ let $f_n : X \to R$ be a continuous function such that $f_n(X) \subset [0, 1]$, $S_{\varepsilon_n}[f_n(S_{\varepsilon_n}[y_n])] \supset [0, 1]$ and $f_n(z) = 0$ for each z for which $d_x(z, y_n) \geqq \varepsilon_n$. Such a sequence exists according to D. Then $f_n \in C_0(X)$ for $n=1, 2, \ldots$. The set $\{f_n : n \in \mathrm{N}\}$ is d_2-totally bounded but it does not vanish uniformly for the same reason as in the first case.

A metric d on a set X is said to be convex if there exists for each $x, y \in X$ such an element $z \in X$ that $d(x, z) = d(z, y) = d(x, y)/2$.

Lemma 4. *Let X be a locally compact complete metric space with a convex metric d. Then X is uniformly locally compact.*

Proof. Suppose the locally compact complete space X not to be uniformly locally compact. Then to number 1 there exists $x_1 \in X$ such that $B_1[x_1]$ is not a compact set. Let $0 \leqq \delta \leqq 1$ be the greatest lower bound of the set of all those η for which $B_\eta[x_1]$ is not a compact set. Then $B_{3\delta/4}[x_1]$ is a compact set. Now we prove that there exists $y \in B_{3\delta/4}[x_1]$ such that $B_{\delta/2}[y]$ is not compact. Suppose that for each $y \in B_{3\delta/4}[x_1]$ the set $B_{\delta/2}[y]$ is compact. Then, using the same method as in Lemma 1, the set $B_{3\delta/8}[B_{3\delta/4}[x_1]]$ may be proved to be compact. Since the metric is convex, it can be easily seen that $B_{9\delta/8}[x_1] \subset B_{3\delta/8}[B_{3\delta/4}[x_1]]$. This is a contradiction because $B_{9\delta/8}[x_1]$ is not a compact set. So there exists $x_2 \in X$ such that $d(x_1, x_2) < \delta$ and $B_{\delta/2}[x_2]$ is not a compact set. Continuing this way we get through the construction by induction a sequence $\{x_n\}$ of elements of X and a sequence $\{\delta_n\}$ of positive numbers such that $d(x_n, x_m) \to 0$ if $m, n \to \infty$ and $\delta_n \to 0$ if $n \to \infty$. So $\{x_n\}$ is a Cauchy sequence. We prove that it has not a limit point. Thus we get a contradiction which finishes the proof. So suppose x to be a limit point of $\{x_n\}$. There exists $\delta > 0$ such that $B_\delta[x]$ is a compact set. Choose $n_0 \in \mathrm{N}$ such that for $n \geqq n_0$ we have $x_n \in B_{\delta/2}[x]$. Let $n_1 \geqq n_0$ be such that $\delta_{n_1} < \delta/2$. Then $B_{\delta_{n1}}[x_{n_1}] \subset B_\delta[x]$. The set $B_{\delta_{n1}}[x_{n_1}]$ as a closed subset of a compact set is compact and it is a contradiction.

Theorem 9. *Let X be a locally compact metric space with a convex metric d. Then X is uniformly locally compact if and only if each d_2-totally bounded set $\Omega \subset C_0(X)$ uniformly vanishes.*

Proof. If X is uniformly locally compact then the assertion follows from Theorem 4. Now let X be locally compact with a convex metric and let each d_2-totally bounded set $\Omega \subset C_0(X)$ be uniformly vanishing. By Lemma 3 X is a complete metric space. According to Lemma 4 X is uniformly locally compact space.

3. Concluding remarks

Some of our considerations for the space $C_0(X)$ with X in general not compact enable to prove among others some results of G. Beer contained in [1]. However in the last mentioned paper $C(X, Y)$ with X compact and Y usually arbitrary complete metric space was considered. Putting X compact and $Y = R$ we have $C(X, R) = C_0(X)$. But in general if $Y = R$ the mentioned results do not include those of G. Beer. There is no difficulty to avoid this unpleasant situation. The only thing which is necessary is to substitute the space $C_0(X) = (C_0(X, R))$ by a space $C_0(X, Y)$ where X, Y are metric spaces. To define such a space $C_0(X, Y)$ we exhibit fixed element $y_0 \in Y$ and define the vanishing function $f: X \to Y$ as a continuous $f: X \to Y$ with the property that to any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\rho(f(x), y_0) < \varepsilon$ for each $x \notin K$, where ρ is a metric in Y. Then $C_0(X, Y)$ is defined as the set of all vanishing functions $f: X \to Y$.

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Received 15.05.1989