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Basis for Finite Element Schemes for an Inhomogeneous Fluid

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To apply FEM for an inhomogeneous liquid we need to know the explicit form and the basis of the subspace, generated by the continuity equation. In the present work the basis is obtained in term of fluxes and mesh stream function. Two and three dimensional cases are considered. The numerical implementation of the basis is discussed as well as the choice of finite elements.

1. Introduction

The inhomogeneous liquid model is useful in the study of many real objects, e. g. the atmosphere, the sea and technological processes. It is applicable, if the density is a known function of one quantity (e. g. the temperature T or the concentration C)

(1.1) $$\rho=\rho(T) \text{ or } \rho=\rho(C).$$

For the sake of concreteness we shall consider only the case (1.1.a) when the density is a function of the temperature. The continuity equation takes the form

(1.2) $$\operatorname{div}(\rho \mathrm{V})=-\partial_t(\rho)=-\partial_T \rho \partial_t(T).$$

If we consider T a known function of time t, then (1.2) is a particular case of the general inhomogeneity condition

(1.3) $$\operatorname{div} \rho \mathrm{U}=\operatorname{div} \mathrm{V}=H(\mathrm{X},\ t),\ \mathrm{X}\in \mathrm{R}^2 \text{ or } \mathrm{R}^3,$$

where $H(\mathrm{X},\ t)$ is an inhomogeneity function, $\mathrm{V}=\rho \mathrm{U}$ is a new variable for the velocity multiplied by the density – mass velocity.

A relation of the type (1.1.a) follows, e. g. from the equation of the state

(1.4) $$p=\rho R\, T;\ R \text{ is the gas constant}$$

if the changes of the pressure p in the fluid are negligible or known in advance. Such relation results from the Boussinesq approximation too [1].

In particular we need the present investigations for numerical modelling of atmospheric motions. The general meteorology models are given and discussed in [14]. They are of the type

(1.5) $$\mathrm{DV}/\mathrm{D}t=\mathrm{L}\,\mathrm{V}-\operatorname{grad} p+\varphi(\mathrm{V}),\ \partial_t \rho+\operatorname{div} \mathrm{V}=0,$$

where D/Dt is the operator of the full derivative, LV is a general energy dissipation operator (often $L \equiv 0$) and φ is a linear operator of the body forces.

The mathematical model (1.5) is excessively general. It describes not only large scale motions of meteorological significance but also the high-frequency acoustic and gravity waves. A way to filter them is to use the Boussinesq type approximation for the continuity equation. For micro and meso-scale motions this approach is developed in [15], [5] and for large scale one in [1]. The equation (1.5. b) is substituted by an equation of the type (1.3). The inhomogeneity function H is known if the temperature is considered as a known function of t. This point of view is natural for numerical methods as splitting by physical processes.

The approach considered could be used for environment and ecological problems too.

To implement the finite element method for problems (1.5. a), (1.2) we need to study the following questions: *i.* choice of finite elements, *ii.* differences and resemblance with homogeneous liquid case, *iii.* the possibility to use ideas and codes of hydrodynamical numerical methods.

To answer these questions and to develop effective numerical methods for the considered class of problems, we need to know the structure and the explicit form of the basis of the space, generated by (1.2) and FEM for any triangulation.

As far as we know, the question for the explicit form of the basis for FEM in the inhomogeneous liquid case has not been considered in the literature. Similar questions have been extensively studied in incompressible case. A construction of a basis for nonconforming tetrahedral FE in $3D$ is given in [2]. It is based on the idea for mesh stream function. Another basis is constructed in [8] which used fluxes through FE sides or faces. A numerical method based on it is proposed in [9] and the connections between different approaches and methods are proved. A review of the literature for FEM in hydrodynamics can be found in [13].

In the next section we formulate the problem. In the third and forth one a basis in terms of fluxes and mesh stream functions is obtained correspondingly. In the penultimate section comparisons with incompressible case are carried out. The last section contains conclusions from the study.

2. Formulation of the problem

In order to study FEM ($2D$ and $3D$ cases) we need to know the structure of the space

$$V(H,\ t)=\mathscr{V}_H=\{\mathrm{V}_h;\ \int_\Omega \operatorname{div}\mathrm{V}_h\,\delta p_h\,\mathrm{d}\Omega=\int_\Omega H\delta p_h\,\mathrm{d}\Omega,\ \mathrm{V}_h\in V_h,\ \delta p_h\in P_h\},$$

where X_h, P_h are FEM spaces for the mass velocity and for the pressure. In particular, we have to be sure that it is not an empty set for any triangulation $\mathscr{T}_h$. The structure of these finite dimensional spaces have to correspond to the physical structure of the space of functions, which satisfies the restriction (1.3).

This question has been extensively studied for incompressible liquid ($H \equiv 0$) [2], [13], [12]. It has been proved that the strong mass conservation is important for the accuracy of FEM and its equivalent to piecewise continuous approximation for the pressure. FE with continuous approximation for the pressure has worse conservative properties [4], [11]. This causes difficulties with the accuracy for the pressure in computations [6]. FE with discontinuous approximation for the pressure seems to be the best for incompressible liquid from theoretical and practical point of view [3], [4], [13], [11]. The problems with continuous pressure FE concern also inhomogeneous liquid because incompressible liquid is a particular case of it. So we shall study only FE with discontinuous approximation of pressure as the most promising for practical usage. This class contains many $2D$ and $3D$ conforming and nonconforming FE with different degree of accuracy [13], [10].

So the important case is

$$(2.1) \qquad \mathscr{V}_H=\{\mathrm{V}_h,\ \int_e \operatorname{div}\mathrm{V}_h q\,\mathrm{d}\Omega=\int_e Hq\,\mathrm{d}\Omega,\ e\in\mathscr{T}_h,\ q\in\mathscr{P}_n\},$$

where e is any FE from the triangulation $\mathscr{T}_h$, q is a polynomial in X, $\mathscr{P}_n$ is the set of polynomials of the degree n.

If we know the structure of the space for $n=1$

$$(2.2) \qquad \mathscr{V}_H=\{\mathrm{V}_h,\ \int_e \operatorname{div}\mathrm{V}_h\,\mathrm{d}\Omega=H_e,\ e\in\mathscr{T}_h\},\ H_e=\int_e H\,\mathrm{d}\Omega$$

the generalization for $n>1$ is standard [7].

The cases $n=1$ and 2 are the most interesting for the practical implementation. We shall consider only the case of a simply connected domain Ω as being the most interesting for meteorology, although most of the results are also true for multiply connected one.

3. Basis in term of fluxes

Consider finite element solutions from $(\mathrm{V}_h,\ p_h)$ with piecewise continuous pressure [13], [10]. The discrete continuity equations are

$$(3.1) \qquad \int_e \operatorname{div}\mathrm{V}_h\,\mathrm{d}\Omega=\Sigma Q_i=H_e,\ Q_i=\int_{F_i}\mathrm{V}_{h,n}\,\mathrm{d}s,$$

where F_i are faces (sides) of e, n is a normal to F_i (arbitrary one from the two possible but fixed for F_i), Q_i is mass flux through F_i.

Change the dependant variables in FE, introducing $(Q_i,\ C_i)$ in $2D$ case, (C_i is the circulation of V over F_i) and $(Q_i,\ C_{1,i},\ C_{2,i})$ in $3D$ case, $C_{1,i}$ are $C_{2,i}$ two generalized circulations

$$C_{s,i}=\int_{F_i}(\mathrm{V}_h,\ \tau_s)\,ds,\ s=1,\ 2;$$

where τ_1 and τ_2 are two linearly independent tangents to F_i.

This can be done for all important for the practice FE [9].

Definition 3.1. A set $\mathscr{B}$ of faces of FE from $\mathscr{T}_h$ is called Q-basis if

a) it does not contain a closed surface from $\mathscr{T}_h$.

b) it contains all faces $F\subset\partial\Omega$ but exactly one, i.e. if $a\in\partial\Omega$ and $a\notin\mathscr{B}$ then $a=a_0$ – a fixed one.

c) it is a maximal set with properties a) and b) i.e. if $\mathscr{B}\subset\mathscr{B}'$ and a) and b) hold for $\mathscr{B}'$ then $\mathscr{B}'=\mathscr{B}$.

We shall call the fluxes through sides $F\in\mathscr{B}$ basis fluxes.

The choice of $\mathscr{B}$ is equivalent to the choice of the way to solve (3.1) with respect to some Q_i. So the fluxes through sides $F \notin \mathscr{B}$ can be expressed in terms of the basis one under the conditions (3.1) and that is the reason to use the word "basis".

Remark 3.1. It is easy to see, that the set $\mathscr{B}$ is not unique and the band width of the FE system depends on its choice.

We shall often use the notion simple closed surface (SCS) for a closed surface without lines of self-crossing or bifurcation.

Lemma 3.1. *If $F \notin \mathscr{B}$ then there exists a unique SCS in $\mathscr{T}_h$, denoted by Γ_F, such that $F \in \Gamma_F$ and $\Gamma_F \setminus F \subset \mathscr{B}$. There exists a unique Ω_F – simply connected subdomain in $\mathscr{T}_h$, such that $\partial\Omega_F = \Gamma_F$.*

The proof is a direct consequence of the definition.

Let Γ_1 and Γ_2 be two closed surfaces in $\mathscr{T}_h$ with one common face at least. Define the addition for them as follows $\Gamma_1 + \Gamma_2 = \Gamma_1 \cup \Gamma_2 \setminus (\Gamma_1 \cap \Gamma_2)$. This is also a closed simple surface in $\mathscr{T}_h$.

We shall call $\Gamma_e = \partial e$ an elementary closed simple surface. For a simply connected domain each closed simple surface in $\mathscr{T}_h$ is a sum of elementary closed surfaces.

Lemma 3.2. *For every elementary simple surface $\Gamma_e = \partial e$ $\Gamma_e = \sum_{F \in C} \Gamma_F$, $C = \{F;\ F \in \partial e,\ F \notin \mathscr{B}\}$ holds.*

Proof. If $\bigcap_{F \in C} \Gamma_F = \emptyset$ then $\bigcup_{F \in C} (\Gamma_F \setminus F) \cup \{F;\ F \in \partial e,\ F \in \mathscr{B}\}$ will be a closed surface in $\mathscr{B}$, which is in contradiction with its definition. Therefore the sum $\Gamma = \sum_{F \in C} \Gamma_F$ is defined. Since $F \in \Gamma_F$ then $\bigcup_{F \in C} \Gamma_F \subset \Gamma \cap \Gamma_e$. Therefore Γ and Γ_e has a common side and $\Gamma + \Gamma_e$ is defined. Consider $F \in \Gamma + \Gamma_e$ and $F \notin \mathscr{B}$. If $F \in \Gamma_a$ and $F \notin \mathscr{B}$, then $F \in C \subset \Gamma_a$, because $\Gamma_F \setminus F \in \mathscr{B}$ by the construction of Γ_F. Hence $\Gamma + \Gamma_a$ lies in $\mathscr{B}$ and is a simple closed surface. But according to the definition of $\mathscr{B}$ this is impossible, therefore $\Gamma + \Gamma_e = \emptyset$. Hence $\Gamma = \Gamma_e$. □

Lemma 3.3. *The set $\mathscr{J} = \{\Gamma_F;\ F \notin \mathscr{B},\ \Gamma_F$ is defined in* **Lemma** *3.1$\}$ is a basis in sense of addition for the set of all closed simple surfaces in $\mathscr{T}_h$.*

Proof. Suppose that the set $\mathscr{J}$ is not linearly independent, i.e. there exists $A \subset \mathscr{J}$ such that $\sum_{F \in A} \Gamma_F = \emptyset$. Hence there exists $\Gamma_{F^1} = \sum_{F \in B} \Gamma_F$, $B = A \setminus F^1$ and there exists $F^1 \neq F^2$, $F^2 \in B$.

From Lemma 3.1 follows, that $\Gamma_{F^1} \setminus F^1$ lies in $\mathscr{B}$, hence $\sum_{F \in B} \Gamma_F \setminus F^1$ lies in $\mathscr{B}$. But $F^2 \in \sum_{F \in B} \Gamma_F \setminus F^1$ therefore $F^2 \in \mathscr{B}$, and $F^2 \in A$ hence $F^2 \notin \mathscr{B}$ – which is a contradiction. Hence $\mathscr{J}$ is a linearly independent set in the sense of addition.

Let Γ be a simple closed surface in $\mathscr{J}_h$. There exists a set A of finite elements from the triangulation, $\Gamma = \sum_{e \in A} \partial e$, ∂e – elementary simple closed surface. Therefore the desired statement follows from Lemma 3.2. □

Corollary 3.1. *The number of nonbasis fluxes are equal to the number of finite elements in the triangulation.*

Proof. The set $\{\Gamma_e; \Gamma_e = \partial e, e \in \mathcal{T}_h\}$ is a basis in sense of addition for the set of all closed simple surfaces in $\mathcal{T}_h$ and the number of its members is equal to the number of FE in the triangulation. According to Lemma 3.2, the set $\mathcal{J}$ is also a basis of the same set, hence they possess equal number of members. The number of members of $\mathcal{J}$ is equal to the number of nonbasis fluxes. □

Remark 3.2. Lemma 3.2, Lemma 3.3 and Corollary 3.1 are true in particular for $\mathcal{T}'_h = \mathcal{T}_h \cap \partial\Omega$, considered as $2D$ triangulation.

The nonbasis fluxes can be expressed in terms of basic ones.

$$Q_F = - \sum_{G \in \Gamma_F \setminus F} \varepsilon_G Q_G + \sum_{e \in \Omega_F} H_e, \quad \varepsilon_G = \pm 1, \tag{3.2}$$

where $\varepsilon_G = 1$ if the orientation of n_F and outward normal to Γ_F are the same and $\varepsilon_G = -1$ if not.

To find a basis of $\mathcal{V}_H$ we have to substitute (2.2) in FEM approximation for V_h.

Theorem 3.1. *Let V_h be a function from the space of FE approximations X_h.*
i. there exists unique representation

$$\mathrm{V}_h = \mathrm{V}_{h,1} + \mathrm{V}_{h,2} \text{ such that } \int_F (\mathrm{V}_{h,1}, n)\,\mathrm{d}F = 0 \text{ for every } F \in \mathcal{T}_h; \tag{3.3}$$

ii. Let $\mathcal{E}$ be the set of all faces of FE in $\mathcal{T}_h$. For $F \in \mathcal{E}$ defines H_F as $H_F = \int_{\Omega_F} H d\Omega$ for $F \in \mathcal{E} \setminus \mathcal{B}$ and $H_F = 0$ for $F \in \mathcal{B}$.

The discretisation of the continuity equation (2.1) is satisfied iff

$$\mathrm{V}_{h,2} = \sum_{F \in B} Q(F, \mathrm{V}) \bar{\Phi}_F + \mathrm{V}_{h,H} \quad \mathrm{V}_{h,H} = \sum_{F \in E} H_F \Phi_F, \tag{3.4}$$

where $Q(F, \mathrm{V})$ is the flux through the face F, Φ_F are basis functions of FE, which corresponds to Q_F, $\bar{\Phi}_F$ are basis function of the space $\mathcal{V}_0$ (i.e. basis functions in a homogeneous fluid case) in terms of fluxes [8] [11], Ω_F is the set mentioned in Lemma 3.1 such that $\partial Q_F = \Gamma_F$.

Proof. The uniqueness of the representation (2.3) is proved as in the homogeneous liquid case [11].

Necessity. Let V_h satisfy (2.1) for every $e \in \mathcal{T}_h$. Let E be the set of all faces of FE in $\mathcal{T}_h$. We change the variables in $\mathcal{V}_h$ as in the homogeneous case [8], [9] and use (3.2)

$$\mathrm{V}_{h,2} = \sum_{F \in E} Q(F, \mathrm{V}) \Phi_F = \sum_{F \in B} Q_F \Phi_F + \sum_{F \in E \setminus B} Q_F \Phi_F$$

$$\mathrm{V}_{h,2} = \sum_{F \in B} Q_F \Phi_F + \sum_{F \in E \setminus B} \Big(- \sum_{\in \Gamma_F \setminus F} \varepsilon_G Q_G + H_F\Big) \Phi_F. \tag{3.5}$$

The set $\Gamma_F \backslash F$ lies in $\mathscr{B}$, hence only basis fluxes take part in the representation. After grouping basis fluxes, we see that new basis functions are the same as in the homogeneous case. Hence (3.4) holds.

Sufficiency. Suppose V_h satisfies (2.4). Hence it satisfies its equivalent form (3.5). Since Φ_F is the basis function for Q_F then $\int\limits_G \Phi_{F,n} ds = \varepsilon_G \delta_{F,G}$, $\delta_{F,G} = 0$ if $F \neq G$ and $\delta_{F,F} = 1$. Consider $F \in E \backslash B$ and uniquely defined Ω_F in Lemma 3.1.

$$\int\limits_{\Omega_F} \operatorname{div} V_{h,2} d\Omega = \int\limits_{\Gamma_F} V_{h,2,n} ds = \sum_{G \in \Gamma_F} \int\limits_G V_{h,2,n} ds = \Big(\sum_{G \in \Gamma_F \backslash F \subset B} + \sum_{F=G} \Big) \int\limits_G V_{h,2,n} ds$$

$$= \sum_{G \in \Gamma_F \backslash F} \varepsilon_G Q_G - \sum_{G \in \Gamma_F \backslash F} \varepsilon_G Q_G + H_F = \int\limits_{\Omega_F} H d\Omega.$$

Hence (2.1) is satisfied for every Γ_F, $F \notin \mathscr{B}$. Using Lemma 3.3. we obtain that for every $e \in \mathscr{T}_h$ its surface ∂e is a sum of several Γ_F. Hence $\int\limits_{\partial e} V_{h,2,n} d\Omega = \int\limits_e H d\Omega$ and (2.1) holds for every $e \in \mathscr{T}_h$. $\square$

Remark 3.3. $\{\Phi_F\}$ are linearly independent and the number of nonbasis faces in $\mathscr{T}_h$ is equal to the number of FE in $\mathscr{T}_h$. In $2D$ and $3D$ cases the bases in term of fluxes are similar. They are with a nonlocal support.

4. Basis in terms of mesh stream function

The basis functions of the space V_H, corresponding to mass fluxes Q_i in (2.4), are with nonlocal carrier. So their direct use does not produce classical FEM. To obtain basis of V_H, which generates FEM, we have to change mass fluxes variables with mesh stream function ones. Roughly speaking, we shall do a change of variables for $\{Q_i\}_{i \in \mathscr{B}}$, to which new basis functions correspond (they are linear combinations of the old basis functions Φ_F) with local supports. The principal difference with a homogeneous case is that the mesh stream function depends on the choice of the basis $\mathscr{B}$. Some analogues with the theory of homogeneous case (see e. g. [13]) will be used. We have to define paths in $\mathscr{T}_h$, which has to possess an orientation to introduce the mesh stream function. The definition of the orientation is the same as in the homogeneous case and will be only sketched.

4.1. Mesh stream function for a closed path in the triangulation

Consider the set of all edges a of FE from $\mathscr{T}_h$. In $3D$ we shall specify a fixed orientation of the tangent to every edge a. We will need it to define the mesh stream function in $3D$. In $2D$ a are the vertexes A (i. e. only the ends of the sides of FE, not the mesh points on the middle of the sides) in the triangulation and they need no orientation. This is an important difference between the $2D$ and $3D$ cases.

Consider an arbitrary triangulation $\mathscr{T}_h$ of the domain Ω, which consists of triangular and rectangular finite elements (including probably both types mixed and isoparametric). We shall call a curve without self-crossing and bifurcations a simple curve — SC.

Definition 4.1. Let γ be a SC in $\mathcal{T}_h$, which is composed by different edges of a FE from $\mathcal{T}_h$, taken by sign $+1$ if its local orientation coincides with the positive orientation of γ and -1 if not. For the definition of positive orientation of paths in $\mathcal{T}_h$ see Remark 4.3. We shall call it a path in $\mathcal{T}_h$. If it is closed we shall call it a closed path (CP) in $\mathcal{T}_h$.

Definition 4.2. An elementary path (EP) in $\mathcal{T}_h$ is CP such that $\gamma=\partial F$ and F is a face of a FE from $\mathcal{T}_h$.

The positive orientation of elementary paths in $\mathcal{T}_h$ is defined in the following way. Take any fixed face F_0, $\partial F_0=\gamma_0$ — EP and specify an orientation to it, which will be considered positive. Consider an EP γ_1, which possesses common edge a with γ_0 and chose its positive orientation so that the orientation of a in γ_1 will be the opposite to that in γ_0. Using this procedure recursively, we define the positive orientation for every elementary path. It is easy to see that this definition is correct.

Remark 4.1. We need the orientation of a path γ to define uniquely the mesh stream function for it, see **Definition 4.3.**

Remark 4.2. We consider $2D$ plane and axially symmetric case as a particular case of $3D$ one. Hence consider $2D$ plane or axially symmetric triangulation as $3D$ one. The sides of FE will be stripes — see Fig. 1 and Fig. 2.

In $2D$ edges of the faces of FE are the vertexes $\{A\}$ from the triangulation. The elementary paths are point sets of the type $\{A_1, A_2\}$, where $A_1 A_2$ is a side of a FE. They are closed at the infinity. All curves are sets of form $\{A_1\}$ or $\{A_1, A_2\}$, A_1, A_2 are vertexes of $\mathcal{T}_h$ and all are closed. The curves of the form $\{A_1\}$ have always mesh stream function zero (see Definition 4.3). They are trivial and they are not considered as closed ones.

Assume γ_1 and γ_2 two CP in $\mathcal{T}_h$ with at least one common side. We define their sum as $\gamma_1+\gamma_2=\gamma_1\cup\gamma_2\backslash(\gamma_1\cap\gamma_2)$. It is easy to see that $\gamma_1+\gamma_2$ is a CP too. The positive orientation of $\gamma_1+\gamma_2$ is correctly defined by the positive orientation of γ_1 and γ_2 (see the definition of the orientation for elementary paths).

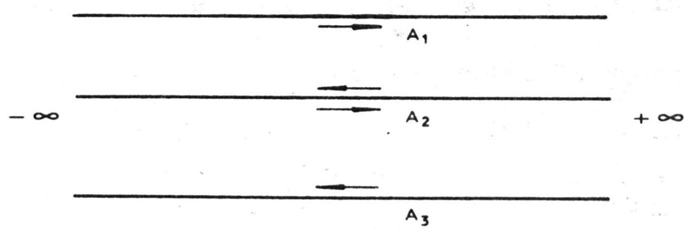

Fig. 1

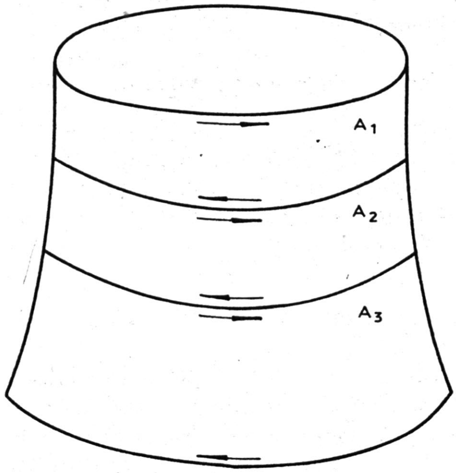

Fig. 2

In $2D$ the sum of two paths $\{A_1, A_2\}$ and $\{B_1, B_2\}$ is defined, iff $A_2 = B_1 - \{A_1, B_2\}$ or $A_1 = B_2 - \{B_1, A_2\}$.

Lemma 4.1. *For every CP γ in $\mathcal{T}_h$ there exists a set $\mathcal{A}_\gamma$ of elementary paths γ_i, such that $\gamma = \Sigma\gamma_i$, $i \in \mathcal{A}_\gamma$, the equality is understood without orientation.*

The statement of Lemma 4.1 is verified directly. Usually the set $\mathcal{A}_\gamma$ is not unique.

Remark 4.3. The positive orientation of any path can be defined from Lemma 4.1. It is easy to see, that the definition does not depend on the choice of the set $\mathcal{A}_\gamma$. Hence the statement of Lemma 4.1 holds including the orientation also.

From Lemma 4.1 it follows that there exists a set $\mathcal{G}$ of sides F such that $\gamma = \partial\Gamma$, $\Gamma = \bigcup\limits_{G} F$. This set $\mathcal{G}$ is not unique. The next theorem states that $\mathcal{G}$ is defined uniquely, if we want $\mathcal{G}$ to be a subset of $\mathcal{B}$.

Theorem 4.1. *Let γ be a CP in $\mathcal{T}_h$. There exists a unique simple (i.e. without self-crossing and bifurcations) surface (SS) Γ in $\mathcal{T}_h$, such that $\gamma = \partial\Gamma$ and $\Gamma = \cup F$, $F \in \mathcal{B}$.*

Proof. **Uniqueness.** Assume Γ_1 and Γ_2 two simple surfaces in $\mathcal{T}_h$, Γ_1, $\Gamma_2 \subset \mathcal{B}$ and $\gamma = \partial\Gamma_1 = \partial\Gamma_2$. Consider $\Gamma = \Gamma_1 \cup \Gamma_2 \setminus (\Gamma_1 \cap \Gamma_2)$. Evidently it is a closed surface in $\mathcal{B}$, which is a contradiction with the definition of $\mathcal{B}$.

Existence. First assume that γ is an EP, i.e. $\gamma = \partial F$. If $F \in \mathcal{B}$ then put $\Gamma = F$. If $F \notin \mathcal{B}$ then according to Lemma 3.1 there exists Γ_F such that $\Gamma_F \setminus F \subset \mathcal{B}$. Put $\Gamma = \Gamma_F \setminus F$, $\gamma = \partial\Gamma$, $\Gamma \subset \mathcal{B}$ and Γ is a simple surface.

In the general case according to Lemma 4.1 $\gamma = \Sigma\gamma_i$, γ_i are EP. Hence, there exist simple surfaces $\Gamma_i \subset \mathcal{B}$ such that $\gamma_i = \partial\Gamma_i$. Put $\Gamma = (\cup\Gamma_i) \setminus [\cup(\Gamma_i \cap \Gamma_j)]$.

Evidently $\Gamma \subset \mathscr{B}$ and $\partial\Gamma=\gamma$. From the definition of $\mathscr{B}$ follows, that there are no self-crossing surfaces in it. If there are bifurcation lines in it, then $\partial\Gamma$ will be not a path. Hence Γ is a simple surface. □

Remark 4.4. If Ω is a multiply connected domain then the surface Γ in $\mathscr{B}$ is not unique.

We call a set $\mathscr{A}$ of paths linearly dependent, if one of them is a sum of others or of a part of them.

Theorem 4.2. *Let $\mathscr{A}$ be a maximal set of linearly independent paths in $\mathscr{T}_h$. Then the set $\mathscr{A}$ is isomorphic to the set of basis faces $\mathscr{B}$, i.e. there exists a one-to-one correspondence between $\mathscr{A}$ and $\mathscr{B}$.*

The proof follows directly from the investigations in the homogeneous case [2] and Theorem 3.1.

Now we shall define a mesh stream function for closed paths in $\mathscr{T}_h$.

Definition 4.3. Let γ be a closed path in $\mathscr{T}_h$. According to Theorem 4.1 there exists a unique simple surface $\Gamma\in\mathscr{B}$ such that $\gamma=\partial\Gamma$. Fix one of the possible unit normals n_Γ to Γ using the orientation of γ in the standard way [16]. We defined a mesh stream function for closed paths γ as

$$\Psi(\gamma, \mathsf{V})=\sum_{F\in\Gamma}\varepsilon_F Q_F,\quad \varepsilon_F=\pm 1, \tag{4.1}$$

where $\varepsilon_F=+1$ if the prescribed unit normal n_F is equal to n_Γ and $\varepsilon_F=-1$ if not.

Remark 4.5. We need the orientation of γ only to define uniquely the unit normal n in $3D$.

Remark 4.6. In the homogeneous liquid case this definition coincides with the usual definition of the flux through the closed curve

$\Psi(\gamma, \mathsf{V})=\int_\Gamma \mathsf{V}_n d\gamma$, which is independent of the choice of the surface Γ, $\partial\Gamma=\gamma$.

In the incompressible case there exists a simple connection between vector potential and stream mesh function for closed paths.

Remark 3.7. If A is a vector potential in $3D$ (it is not unique), such that $\mathsf{V}=\operatorname{rot}\mathsf{A}$, then

$$\Psi(\gamma, \mathsf{V})=\int_\Gamma(\operatorname{rot}\mathsf{A}, n)\,d\sigma=\int_\gamma(\mathsf{A}, \tau)\,d\gamma,\quad \partial\Gamma=\gamma, \tag{4.2}$$

where τ is the positive tangent to γ with respect to n.

Hence, the mesh stream function for closed paths in the incompressible case can be defined only on the basis of any stream vector field A and is independent of its choice. If $\mathsf{V}=\operatorname{rot}\mathsf{A}_1=\operatorname{rot}\mathsf{A}_2$ then for each closed curve γ holds

$$\int_\gamma(\mathsf{A}_1, \tau)\,d\gamma=\int_\gamma(\mathsf{A}_2, \tau)\,d\gamma. \tag{4.3}$$

Theorem 4.3. *The mesh stream function for closed paths in $\mathscr{T}_h$ does not depend on the choice of the set $\mathscr{B}$ iff $H_e=0$ for every $e\in\mathscr{T}_h$.*

Proof. **Necessity.** Consider $e \in \mathcal{T}_h$. Let γ be a closed path on ∂e in $\mathcal{T}_h$ and $\partial e = \bigcup_{i=1}^{m} F_i$, $m \geqq 3$, F_i are sides of FE e. The path γ divides ∂e into two parts $-$ $\partial e_1 = \{F_i\}_{i \in A_1}$ and $\partial e_2 = \{F_i\}_{i \in A_2}$ and $meas\ (\partial e_1 \cap \partial e_2) = 0$. Using Definition 3.1 we get that there exist two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\{F_i\}_{i \in A_k} \subset \mathcal{B}_k$, $k = 1, 2$. So $\gamma = \partial\Gamma_1 = \partial\Gamma_2$, $\Gamma_k = \cup F_i$, $i \in A_k$, $k = 1, 2$ and Γ_1 and Γ_2 are the unique surfaces mentioned in Theorem 4.1 with respect to the two bases $\mathcal{B}_1$ and $\mathcal{B}_2$. The two mesh stream functions for γ are

$$\Psi_k(\gamma, V) = \sum_{F \in \Gamma_k} \varepsilon_F Q_F, \quad k = 1, 2.$$

As $\Psi_1(\gamma, V) = \Psi_2(\gamma, V)$ then $0 = \sum_{F \in \Gamma_1} \varepsilon_F Q_F - \sum_{F \in \Gamma_2} \varepsilon_F Q_F = \pm \sum_{F \in \partial e} Q_F = \pm H_e$.

Sufficiency. From (3.1) there follows that $H_e = \sum_{F \in \partial e} Q_F$ for every $e \in \mathcal{T}_h$. Using assumptions we get $\sum_{F \in \Gamma} Q_F = 0$ for every Γ – CSS in $\mathcal{T}_h$. Therefore

$\Psi(\gamma, V) = \sum_{F \in \Gamma_1} \varepsilon_F Q_F = \sum_{F \in \Gamma_2} \varepsilon_F Q_F$, where Γ_1 and Γ_2 are arbitrary CSS in $\mathcal{T}_h$ such that $\partial\Gamma_1 = \partial\Gamma_2 = \gamma$. Hence the definition of $\Psi(\gamma, V)$ is independent of the choice of surface Γ and hence from the choice of $\mathcal{B}$. □

Corollary 4.1. If the fluid is inhomogeneous, i.e. H is not identically zero, then mesh stream function depends on the choice of $\mathcal{B}$.

4.2. Mesh stream function for edges of FE

Now we want to establish one to one correspondence between basis fluxes and a set of edges in $\mathcal{T}_h$. After that we shall define mesh stream function for this set of edges. It can not be defined for all edges of FE in $\mathcal{T}_h$.

Definition 4.4. If $\mathcal{K}$ is a maximal set of edges in $\mathcal{T}_h$, such that it does not contain a closed path and possesses the maximal possible number of elements on $\partial\Omega$, then it is called a set of non-basis edges.

Denote by E the set of all edges in $\mathcal{T}_h$. We shall call the elements of $E \setminus \mathcal{K}$ basis edges.

Using Remark 4.2, we obtain that in $2D$ $E \setminus \mathcal{K}$ is the set of all vertexes of $\mathcal{T}_h$ except one, which has to be on $\partial\Omega$.

Lemma 4.2. *For every $a \in E \setminus \mathcal{K}$ (basis edges) there exists a unique closed path γ_a in $\mathcal{T}_h$ such that $\gamma_a \setminus a \subset \mathcal{K}$.*

The proof is direct consequence of Definition 4.4.

Lemma 3.3. *The set $\{\gamma_a\}_{a \in E \setminus \mathcal{K}}$ is a basis in the sense of addition in the set of paths in $\mathcal{T}_h$.*

The proof is a direct consequence of the investigations in the homogeneous case [2].

Lemma 4.4. *The set $\{\Gamma_a\}_{a\in E\setminus\mathscr{K}}$ is a basis in the sense of addition for $\{\Gamma_F\}_{F\in\mathscr{E}}$.*

Proof. According to Lemma 4.3 $\{\gamma_a\}_{a\in E\setminus\mathscr{K}}$ is a basis in the sense of addition for the set of elementary paths $\{\gamma_F\}_{F\in\mathscr{E}}$. Hence Γ_F can be represented as a sum of elements of $\{\Gamma_a\}_{a\in E\setminus\mathscr{K}}$. This surface is simple because of the construction of $\mathscr{B}$. □

4.3. Application of boundary conditions

The statement of Lemma 4.2 is true for every maximal set of edges in $\mathscr{T}_h$, which does not contain a closed path. We want $\mathscr{K}$ to contain the maximal possible number of edges on $\partial\Omega$ to ensure the possibility of computing all mesh stream functions unknown for edges on $\partial\Omega$ knowing only mass fluxes Q_i for faces on $\partial\Omega$. To prove that and to give algorithm for practical computations we need some examinations.

Lemma 4.5. *Let $E'=\{a, a\in E, a\in\partial\Omega\}$. Then the set $\mathscr{K}'=\mathscr{K}\cap E'$ is maximal set in E', which does not contain a closed path in E'.*

The proof is direct consequence of the definition of $\mathscr{K}$.

Definition 4.5. We shall call sets $\mathscr{K}'$ with the property, mentioned in Lemma 4.4, set of nonbasis edges on the boundary.

In $2D$ the set of non-basis edges in the whole domain and non-basis edges on the boundary coincides, i.e. $\{\mathscr{K}\}=\{\mathscr{K}'\}$. Every $\mathscr{K}'$ set is $\mathscr{K}$ set and conversely.

Corollary 3.2. *For every basis edge $a\in E'\setminus\mathscr{K}'$ on $\partial\Omega$ the unique closed path γ_a, introduced in Lemma 4.2, lies on $\partial\Omega$ and $\gamma_a\setminus a\subset\mathscr{K}'$.*

Theorem 4.4. *Let a be a basis edge on the boundary $\partial\Omega$. Consider closed path γ_a mentioned in Lemma 4.2 and a simple surface Γ_a, mentioned in Theorem 4.1. Then*

i) If $F\in\Gamma_a$ then $F\in\Gamma$, i.e. Γ_a lies on the boundary.

ii) The set of basis edges on the boundary is isomorphic to the set of FE's faces on $\partial\Omega$, except one. There exists one to one function f which maps $\{a;\ a\in E'\setminus\mathscr{K}'\}$ into $\{F;\ F\in\mathscr{B},\ F\subset\partial\Omega,\ F\neq F_0\}$ for every fixed $F_0\in\partial\Omega$.

iii) If we know $V_n|_{\partial\Omega}$ then $\Psi(\gamma_a, V)$ is defined.

Proof. *i)* Let $\mathscr{B}'=\{F;\ F\subset\mathscr{B},\ F\subset\partial\Omega\}$. It's easy to see that for γ_a – CC on $\partial\Omega$ there exist unique simple surface $\Gamma\subset\mathscr{B}'$ such that $\gamma_a=\partial\Gamma$. But $\mathscr{B}'\subset\mathscr{B}$ and from Theorem 4.1 follows that $\Gamma=\Gamma_a$.

ii) Consider $\mathscr{T}'_h=\mathscr{T}_h\cap\partial\Omega$ as $2D$ triangulation. $3D$ edges in it will play now the role of $2D$ sides. Consider the set of nonbasis $3D$ edges on the boundary. According to Definition 4.5, Definition 3.1 and Lemma 4.5, it is a set of basis $2D$ faces in $\mathscr{T}'_h$. For if we apply Corollary 3.1 with Remark 3.2 and obtain that the number of $2D$ FE in $\mathscr{T}'_h$ is equal to the number of nonbasis $2D$ sides in $\mathscr{T}'_h$, which coincides with the set of $3D$ basis edges on $\partial\Omega$. Going back to $3D$ notations this means that the number of FE's faces on the boundary is equal to the number of basis edges on the boundary.

The statement of *iii)* is a consequence of *i)* and Definition 4.4 □

Definition 4.6. A stream function for basic edges $a \in E \setminus \mathscr{K}$ *is* defined using Lemma 4.2 and Definition 4.3 for mesh stream function for closed paths as follows

(4.4) $$\Psi_a = \Psi(a, \mathsf{V}) = \Psi(\gamma_a, \mathsf{V}), \quad a \in E \setminus \mathscr{K}.$$

Remark 3.8. If $H \neq 0$ then Ψ_a depends on the choice of $\mathscr{B}$.

Next, we have to be sure, that $\{\Psi_a, a \subset \partial\Omega\}$ can be computed only on the base of $\mathsf{V}_n | \partial\Omega$ to apply the boundary conditions.

Theorem 4.5. *The set* $\mathscr{A} = \{\Psi_a : a \in E' \setminus \mathscr{K}'\}$ *is a change of variables for the set* $\mathscr{D} = \{Q_F : F \neq F_0, F \in \mathscr{B} \cap \partial\Omega\}$ *and can be evaluated only on the base of* $\mathsf{V}_n | \partial\Omega$.

Proof. In $2D$ the statement of the theorem is evident.

Using Definition 4.1, Definition 4.4 and Theorem 4.4 *i*), we obtain

$$\Psi_a = \Psi(\gamma_a, \mathsf{V}) = \Psi(\Gamma_a, \mathsf{V}) = \Sigma \varepsilon_F Q_F, \quad F \in \Gamma, \ F \subset \partial\Omega.$$

Hence, the mesh stream function can be computed on the basis of the fluxes on Γ.

Now we have to prove that basis fluxes on Γ can be computed on the basis of $\{\Psi_a\}_{a \in \mathscr{A}}$. Fix any $F \in \mathscr{D}$. Consider the elementary path γ_F. Using Lemma 4.3 for $\mathscr{T}_h \cap \Gamma$ we obtain that $\{\gamma_a\}_{a \in E' \setminus \mathscr{K}'}$ is a basis in sense of addition for the set of all paths in $\mathscr{T}_h \cap \Gamma$. Hence there exist $\gamma_F = \sum_{\mathscr{A}'} \gamma_a$, $\mathscr{A}' \subset E' \setminus \mathscr{K}'$ and $\Psi(\gamma_F) = \Psi(F, \mathsf{V}) = \sum_{\mathscr{A}'} \Psi(\gamma_a)$. □

4.4. Basis in terms of mesh stream functions for FE faces

We shall use the notations of Theorem 3.1. The use of mesh stream function changes only the second statement.

Theorem 3.6. *Let* Ψ_a *are the basic functions of FEM for mesh stream function variables in incompressible case* [2] *and* $\mathsf{V}_{h,H}$ *is the inhomogeneous "tail" defined in Theorem* 3.1.

A function V_h *belongs to* $\mathscr{V}_H$ *iff*

(4.5) $$\mathsf{V}_{h,2} = \sum_{a \in E \setminus \mathscr{K}} \varepsilon_a \Psi(a, \mathsf{V}) \Phi_a + \mathsf{V}_{h,H},$$

where $\Psi(a, \mathsf{V})$ *is the mesh stream function in inhomogeneous case (see Definition* 4.3), ε_a *is* $+1$ *if the orientation of the side* a *is the same as in* Γ_a *and* -1 *if not.*

Proof. **Necessity.** Let V_h satisfies (2.1) for every $e \in \mathscr{T}_h$, i.e. $\int_e \operatorname{div} \mathsf{V}_h d\Omega = \int_e H d\Omega$. We want to prove that the representation of the type (4.5) exists and is unique.

(4.6) $$\mathsf{V}_{h,2} = \sum_{a \in E \setminus \mathscr{K}} U_a \Phi_a + \sum_{F \in \mathscr{E} \setminus \mathscr{B}} H_F \Phi_F.$$

Fix $\ell \in E \setminus \mathscr{K}$ and consider γ_ℓ.

$$\Psi(\gamma_\ell, V_{h,2}) = \Psi(\gamma_\ell, V_h) = \Psi(\gamma_\ell, V) = \sum_{a \in E \setminus \mathscr{K}} U_a \Psi(\gamma_\ell, \Phi_a) + \sum_{F \in \mathscr{E} \setminus \mathscr{B}} H_F \Psi(\gamma_\ell, \Phi_F)$$

$\Psi(\gamma_\ell, \Phi_F) = \int_{\Gamma_\ell} \Phi_{F,n} ds = 0$ since $\Gamma_\ell \subset \mathscr{B}$ hence $F \notin \Gamma_\ell$ for each $F \in \mathscr{E} \setminus \mathscr{B}$. $\Psi(\gamma_\ell, \Phi_a) = \delta_{a,\ell} \varepsilon_a$ which is a consequence of the definition of Φ_a and is obtained as in the homogeneous case. Hence $\Psi(\gamma_\ell, V_{h,2}) = \sum_{a \in E \setminus \mathscr{K}} U_a \delta_{a,\ell} \varepsilon_a = U_\ell \varepsilon_\ell$. Therefore $\varepsilon_\ell \Psi(\gamma_\ell, V_{h,2}) = U_\ell$ for each $\ell \in E \setminus \mathscr{K}$.

From Theorem 3.1 there follows that to prove the existence of the representation (4.6) it is equivalent to prove this for each $F \in \mathscr{B}$

$$\Phi_F = \sum_{a \in E \setminus \mathscr{K}} \varepsilon_a \Psi(\gamma_a, \Phi_F) \Phi_a, \tag{4.7}$$

which is a consequence of the definitions of Φ_a and Ψ and is verified as in the homogeneous case [13].

Sufficiency. Let (4.5) holds.

$$\int_e \operatorname{div} V_h ds = \sum_{a \in E \setminus \mathscr{K}} \varepsilon_a \Psi(a, V) \int_e \operatorname{div} \Phi_a ds + \int_e \operatorname{div} V_{h,H} ds = \int_{\partial e} V_{h,H,n} ds$$

because of the construction of Φ_a. Hence it remains to prove that $\int_{\partial e} V_{h,H,n} ds = H_e$.

If we prove that $\int_{\Gamma_G} V_{h,H,n} ds = H$ for any $G \in \mathscr{E} \setminus \mathscr{B}$ than it follows from Lemma 3.3.

$$\int_{\Gamma_G} \Phi_{F,n} ds = \int_{\Gamma_G \setminus G} \Phi_{F,n} ds + \int_G \Phi_{F,n} ds = \delta_{F,G} \text{ since } F \in \mathscr{E} \setminus \mathscr{B},\ \Gamma_F \setminus G \subset \mathscr{B}$$

and the first integral is zero. Therefore

$$\int_{\Gamma_G} V_{h,H,n} ds = \sum_{F \in \mathscr{E} \setminus \mathscr{B}} H_F \delta_{F,G} = H_G. \square$$

The representation (4.5) consists only of functions with local support and hence it generates classical FEM. Its singular feature is that it contains known parameters H_F weighted by mass flux basis functions.

5. Discussions and comments

The results of §3 and §4 show, that FE with discontinuous pressure [3], [10], [13] constructed and practically tested for an incompressible liquid, are good for the inhomogeneous case too.

The basis in the inhomogeneous case differs from the incompressible one by an additional inhomogeneity term. The physical meaning of some notions and variables changes in the inhomogeneous case.

The direct use of the flux form (3.4) of the basis of $\mathscr{V}_h$ is not more complicated than in the homogeneous liquid case. It differs from it only by the term $V_{h,H}$, which acts only for nonbasis flux variables. The inhomogeneous corrections $H_F \Phi_F$ contain only classical local shape functions of the initial FE. The integrals H_F over Ω_F can be computed by addition of H_e, which are evaluated in a standard way.

For a general triangulation the basis set $\mathscr{B}$ and the subdomains $\{\Omega_F\}_{\mathscr{E}\setminus\mathscr{B}}$ have to be automatically computed previously as in the homogeneous case. For triangulations, topologically isomorphic to regular grid, the explicit form of $\mathscr{B}$ and Ω_F are known [8], [11]. So the condensation method technique [9] is directly applicable for the solution of inhomogeneous liquid problems.

So the idea for constructing numerical methods in velocity – pressure approach of "divergence-free" type is applicable in the inhomogeneous case too. The solution is obtained in terms of basis velocity variables.

The computation of the nonbasis flux variables can be performed on the base of the knowledge of Ω_F and H_F. This is practically the same as in the homogeneous case. The computation of V variables will be discussed later.

The computation of the pressure after the use of basis (3.5) can be done only using the pressure restoration path [9], [11]. Note that now other path does not yield the same result as in the homogeneous case. This remark concerns also the basis in therms of mesh stream function.

The basis in term of mesh stream function (4.6) can be very useful if we possess a mesh stream function package for solving $3D$ incompressible liquid problems and want to generalize it for the inhomogeneous case. Although mesh stream function's variables are with different meanings, only the "tail" $V_{h,H}$ has to be added to the existing code. This operation is easy and standard in FEM, if the coefficients H_F are computed ones. For this purpose FE's sets Ω_F have to be previously computed and stored in an appropriate way as in the methods based on Theorem 3.1.

If the inhomogeneity H depends on the time (for example as in the meteorology), then H_F has to be computed on each time step.

Suppose we have computed the solution in terms of Ψ_a variables. The next important task is to find it in primitive velocity variables. It is the form, in which we can interprete the results physically. This is also needed to solve coupled heat transport equations as in meteorology or to apply Eulerian-Lagrangian type methods. For this purpose we have to compute Q_F variables using Ψ_a ones. If it is done on the base of the (Q_F, G_F) variables on one face, we can easily compute V variables on this face.

The second principal difference with the incompressible liquid is in the computation of mass flux variables Q_F.

To solve this task in the incompressible case we need to know only the set $\{\gamma_a\}_{a\in E\setminus\mathscr{K}}$. The fluxes through every simple surface Γ, $\partial\Gamma=\gamma_a$ equal to Ψ_a. Next we have to find the representation of Lemma 4.3 for elementary paths γ_F and using it to compute fluxes Q_F. Hence for unsteady problems we have to compute and store the set $\{\gamma_a\}_{E\setminus\mathscr{K}}$ and the representations of γ_F by it. The Ψ_a variables have the physical sense of fluxes through the closed curve γ_a.

In the inhomogeneous case the flux through γ_a can not be defined correctly.

The variables Ψ_a are fluxes through the surface $\Gamma_a \subset \mathscr{B}$ which corresponds to γ_a. So to compute Q_F fluxes we have to use not Lemma 4.3 but Lemma 4.4. Therefore we have to evaluate in the set $\{\Gamma_a\}_{E\setminus\mathscr{X}}$ and the representations of the type $\Gamma_F = \Sigma\Gamma_a$ in explicit form and to store them.

6. Conclusions

The basis for FEM for an inhomogeneous liquid is constructed in the form of fluxes. It differs from the incompressible case by an additional term, corresponding to the inhomogeneity.

The mesh stream function form of the basis is also constructed. The mesh stream function possesses different physical meanings than in the homogeneous case, but the form of its part in the basis remain the same. The inhomogeneous "tail" is presented as in the flux form of the basis.

These two forms of the basis can be used directly to produce numerical methods. But they are more complicated than in the homogeneous case. A price for treating the inhomogeneity has to be paid.

The investigations show that (V, p) FE good for the incompressible liquids, are also good for the inhomogeneous ones.

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