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On the C*-Algebras of Multivariable Wiener-Hopf Operators

Nikolay P. Buyukliev

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In this paper we consider the C^*-algebra $\mathscr{B}$, generated by the Wiener-Hopf operators in subsemigroup P of a locally compact group G. Here G is a subgroup of R^n endowed with discrete or other locally compact topology, and P is a polyhedral cone. We use the groupoid approach of P. Muhly and J. Renault.

We construct a composition sequence for $\mathscr{B}$ and investigate the type of the ideals of $\mathscr{B}$.

Introduction

Let G be a locally compact group and P be a subsemigroup of G. The Wiener-Hopf operator $W(f)$ with symbol $f \in C_c(G)$ is the operator in $L^2(P)$, defined by the formula

$$(W(f)\xi)(t) = \int_P f(t-s)\xi(s)\,\mathrm{d}\lambda(s).$$

The C^*-algebra, generated by $W(f)$ when f runs through $C_c(G)$ is denoted by $\mathscr{B}$.

The algebra $\mathscr{B}$ and the similar C^*-algebra of the Toeplitz operators on bounded symmetric domains are studied by A. Dynin in [1], [2] and by H. Upmeier in [3], [4]. Their methods are very different and it seems impossible to investigate $\mathscr{B}$, if it is discrete. Special cases are considered in [5] and [6].

We follow the approach, suggested by P. Muhly and J. Renault. They prove in [7], that $\mathscr{B}$ is isomorphic to a groupoid C^*-algebra $C^*(\theta)$, where θ is an explicitly constructed groupoid. They applicate this result to study the structure of $\mathscr{B}$, when $G = R^n$, endowed with the usual topology and P is polyhedral or homogeneous cone. Here we use this isomorphism. We obtain similar results when G is a subgroup of R^n endowed with a locally compact topology and P is an intersection of G and a polyhedral cone in R^n. The most important examples occured, when $G = Z^n$ or $G = R^n$ and the topology is discrete, or $G = R^n$ with the usual topology and a direct product of the above groups.

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We recall some facts from [7] and [8] about groupoid C^*-algebras. A groupoid θ is a set endowed with a product map $\theta^{(2)} \to \theta$, where $\theta^{(2)}$ is a subset of $\theta \times \theta$ called the set of composable pairs and an inverse map $x \to x^{-1}$ with some relations. If $x \in \theta$, then we denote: $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$. $\theta^0 = r(\theta) = d(\theta)$ is the unit space of θ. For $u, v \in \theta^0$ the relation $u \sim v$ iff $r^{-1}(u) \cap d^{-1}(v) \neq \emptyset$ is an equivalence relation on the unit space θ^0.

An important example for a groupoid is the transformation group $Y \times G$, where G is a locally compact group and G acts continuously on a locally compact space Y. The image of the point $y \in Y$ by the transformation $t \in G$ is denoted by $y+t$. We may define the following groupoid structure: $(y, t)^{-1} = (y+t, -t)$; (y, t) and (z, s) are composable iff $y+t=z$ and in this case $(y, t)(y+t, s) = (y, t+s)$. The unit space of $Y \times G$ is the set $\{(y, 0) : y \in Y\}$ and we identify this set with Y. If θ is a groupoid and E is a subset of θ^0, then $\theta | E = \{x \in \theta : r(x) \in E;\ d(x) \in E\}$ is a subgroupoid of θ with unit space E. $\theta | E$ is called the reduction of θ by E.

In such groupoids one may define a family of measures $\{\lambda^u : u \in \theta^0\}$ satisfying the left Haar system axioms. In [9] A. Nica investigates whether those axioms are valid after a reduction of a groupoid. Using the Haar system one may construct the groupoid C^*-algebras $C^*(\theta)$ and $C^*_{red}(\theta)$. In this case, which we consider here, $C^*(\theta) = C^*_{red}(\theta)$ (see [7], Proposition 2.15). The following proposition, concerning the structure of $C^*_{red}(\theta)$ is proved in [8], p. 101:

Proposition I. *Let θ be a locally compact groupoid with Haar system.*

(i) If E is an invariant open subset of θ^0, then there exists an ideal $I(E)$ of $C^*_{red}(\theta)$, which is isomorphic to $C^*_{red}(\theta | E)$ and such that the quotient is isomorphic to $C^*_{red}(\theta | (\theta^0 \setminus E))$.

(ii) The correspondence $I \to I(E)$ is a one to one, order preserving map from the lattice of invariant open subsets of θ^0 into the lattice of ideals of the $C^*_{red}(\theta)$.

Let G be a locally compact group and P is a normal subsemigroup of G, such that P is a closure of its interior, $O \in P$, $P \cap (-P) = \{0\}$ and P spans G. P. Muhly and J. Renault construct in [7], §3 a locally compact space Y and its subspace X (details can be found in §2). Let $\theta = (Y \times G) | X$.

Proposition 2. (*i*) (Theorem 3.7 of [7]) *There exists an isomorphism W between $C^*_{red}(\theta)$ and $\mathscr{B}$.*

(*ii*) (Lemma 3.3 of [7]) *Each orbit in Y meet X.*

We will use (i) and we will investigate $C^*_{red}(\theta) = C^*(\theta)$. If we have a good model of Y, then we can determine the invariant open subsets of Y and X. If E is an open invariant subset of X, then there exists a corresponding ideal in $C^*(\theta)$ and in $\mathscr{B}$ and we may apply the groupoid technique in finding the type of the ideals and the quotients.

In §I we study the space U of pointwise limits of the translations of the cone P in the terms of the facial structure of P and its properties are given in Theorem I. In §2 we make a model for Y, i.e. we construct a space, which is homeomorphic to Y. If G is a subgroup of R^n and the topology of G is discrete, then a model of Y is a subspace U_G of U. If G is a subgroup of R^n and G is endowed with a locally compact topology, then the model of Y consists of classes of equivalent a.e. elements of U_G. In §3 we construct a composition sequence for $\mathscr{B}$. We investigate whether $\mathscr{K}$-the ideal of the compact operators is contained in $\mathscr{B}$ and we study the type of the ideals and the quotients in the above composition sequence. We discuss some interesting examples.

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I. Let P be a polyhedral cone in R^n. Let S be the space of all characteristic functions $\chi(M)$ of nonvoid subsets M of R^n, endowed with the topology w of the pointwise convergence. R^n acts on the space S by translations. In this part we will study U-the minimal invariant under action of R^n, w-closed subset of S, such that $\chi(-P)\in U$. We will define U explicitly using the lattice of faces of P and we will prove in Theorem I that U has the above properties. We will use U in §2 and §3 to study the ideal structure of $\mathscr{B}$.

By definition, a polyhedral cone P in R^n is a closed solid cone, generated by finite number of points of R^n. We will assume that P spans R^n. In our analysis we will fix a minimal set $\tilde{P}=\{l_1,\ldots, l_N\}$, of linear functionals, such that $x\in P$ iff $l(x)\geqq 0$ for all $l\in\tilde{P}$.

We need some notations and definitions. Let $\tilde{F}\subset\tilde{P}$. F and $\langle F\rangle$ are subsets of R^n, such that $x\in F$ iff $l(x)=0$ for $l\in\tilde{F}$ and $l(x)\geqq 0$ for $l\in\tilde{P}\backslash\tilde{F}$ and $x\in\langle F\rangle$, iff $l(x)=0$ for $l\in\tilde{F}$. We define $F^{\perp}$ as $R^n\ominus\langle F\rangle$. We note that if $\tilde{F}=\emptyset$, then $F=P$, $\langle F\rangle=R^n$ and if $\tilde{F}=\tilde{P}$, then $F=\{0\}$ and $\langle F\rangle=\{0\}$.

If $\tilde{F}=\{l_1,\ldots, l_k\}\subset\tilde{P}$ and μ_i is -1, 0 or 1 for $i=1,\ldots,k$, we define a subset $D(\tilde{F}, \mu_1,\ldots, \mu_k)$ of $F^{\perp}$, called a determining set; $x\in D(\tilde{F}, \mu_1,\ldots, \mu_k)$ iff $l_i(x)$ is $\leqq$, $=$ or >0 when μ_i is equal to -1, 0 or 1 respectively. If $\tilde{F}=\{l_1,\ldots,l_k\}\subset\tilde{P}$, σ_i is 0 or 1 for $i=1, 2,\ldots, k$ and $x\in F^{\perp}$, then $\chi(\tilde{F}, \sigma_1,\ldots, \sigma_k, x)$ is the characteristic function of the set of all $y\in R^n$, such that $l_i(y)\leqq l_i(x)$ when $\sigma_i=1$ and $l_i(y)<l_i(x)$ when $\sigma_i=0$, where $i=1,\ldots, k$.

Definition. Let $\tilde{F}\subset\tilde{P}$. We will say that $\tilde{F}$ determines a face F of P if:

(1) there exists $y\in R^n$, such that $l(y)=0$ for $l\in\tilde{P}$ and $l(y)>0$ for $l\in\tilde{P}\setminus\tilde{F}$.

(2) if $l\in\tilde{P}$ and l is in the linear span of $\tilde{F}$, then $l\in\tilde{F}$.

We will assume, that $\tilde{P}$ determines the face $\{0\}$ of P; $\emptyset\subset\tilde{P}$ determines the face P of P.

Definition. U is the set, whose elements are the characteristic function $\chi(\tilde{F}, \sigma_1,\ldots, \sigma_k, x)$, such that:

(3) $\tilde{F}$ determines a face of P.

(4) $UD(\tilde{F}, \mu_1,\ldots,\mu_k)\neq\emptyset$, where μ_i is 0 or 1, when $\sigma_i=1$ and μ_i is -1, when $\sigma_i=0$, for $i=1,\ldots, k$.

If L is a linear subspace of R^n, then $\Pr(L, x)$ will denote the orthogonal projection of $x\in R^n$ onto L. The convergence of a net of points of R^n is always the convergence in the Euclidean space R^n. We note, that if $\tilde{F} \subset \tilde{P}$ satisfies (2), then the convergence in $F^{\perp}$ is equal to the $\tilde{F}$-weak convergence (i.e. $x_\alpha \to x_0$ for $x_\alpha \in F^{\perp}$ iff $l(x_\alpha) \to l(x_0)$ for each $l \in \tilde{F}$).

Lemma I. Let $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x_\alpha)_{\alpha\in A}$ *be a net in* U. *We assume, that* $\tilde{H} = \{l_1, \ldots, l_k\} \subset \tilde{F} = \{l_1, \ldots, l_m\}$ *and there exists* $x_0 \in H^{\perp}$ *and* $D(\tilde{H}^{\perp}, \mu_1, \ldots, \mu_k)$, *such that*:
(i) $l(x_\alpha) \to l(x_0)$ *when* $l \in \tilde{F}$.
(ii) $l(x_\alpha) \to \infty$ *when* $l \in \tilde{P} \setminus \tilde{F}$.
(iii) $\Pr(H^{\perp}, x_\alpha) \in x_0 + D(\tilde{H}, \mu_1, \ldots, \mu_k)$ *for each* $\alpha \in A$.

Then $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x_\alpha) \to \chi(\tilde{H}, \delta_1, \ldots, \delta_k, x_0) \in U$, *where* δ_i, $i = 1, 2, \ldots, k$ *are given in the following table*:

$$\begin{aligned} &\delta_i = 1 \quad \text{when} \quad \sigma_i = 1 \quad \text{and} \quad \mu_i \in \{0, 1\} \\ &\delta_i = 1 \quad \text{when} \quad \sigma_i = 0 \quad \text{and} \quad \mu_i = 1 \\ &\delta_i = 0 \quad \text{when} \quad \sigma_i = 1 \quad \text{and} \quad \mu_i = -1 \\ &\delta_i = 0 \quad \text{when} \quad \sigma_i = 0 \quad \text{and} \quad \mu_i \in \{-1, 0\}. \end{aligned} \tag{5}$$

Proof: We will verify, that H satisfies (2). Let us assume, that $l \in \tilde{P} \setminus \tilde{H}$ and $l = \sum_{i=1}^{k} \lambda_i l_i$. Then $l(x_\alpha) = \sum_{i=1}^{k} \lambda_i l_i(x_\alpha)$. But by (i) $l(x_\alpha) \to \infty$ and by (ii) $\sum_{i=1}^{k} \lambda_i l_i(x_\alpha) \to \sum_{i=1}^{k} \chi_i l_i(x_0)$, this is a contradiction and (2) is fulfilled. The note before this lemma and (i) imply that $\Pr(H^{\perp}, x_\alpha) \to x_0$.

We need $y \in R^n$, satisfying (I). Let us put $y_\alpha = x_\alpha - x_0$.

$$y_\alpha = \Pr(\langle H \rangle, y_\alpha), + \Pr(H^{\perp}, y_\alpha).$$

We have $l(\Pr(\langle H \rangle, y_\alpha)) = 0$ by the definition of $\langle H \rangle$, where $l \in \tilde{H}$. Thus if $l \in \tilde{H}$, then $l(\Pr(H^{\perp}, y_\alpha)) = l(y_\alpha) = l(x_\alpha - x_0) \to 0$, hence $\Pr(H^{\perp}, y_\alpha) \to 0$.

Let $l \in \tilde{P} \setminus \tilde{H}$. Since $\Pr(H^{\perp}, y_\alpha) \to 0$, then $l(\Pr(H^{\perp}, y_\alpha)) \to 0$ and $l(\Pr(\langle H \rangle, y_\alpha)) = l(y_\alpha) - l(\Pr(H^{\perp}, y_\alpha)) \to \infty$.

We choose $y = \Pr(\langle H \rangle, y_\alpha)$, where $l(\Pr(\langle H \rangle, y_\alpha)) > 0$ for all $l \in \tilde{P} \setminus \tilde{H}$. Thus $\tilde{H}$ determines a face H of P.

Now we will verify (4). Let $y \in D(H^{\perp}, \mu_1, \ldots, \mu_k)$. We may assume, that $l_i(y) > 1$, when $\mu_i = 1$; $l_i(y) = 0$, when $\mu_i = 0$ and $l_i(y) < -1$, when $\mu_i = -1$. $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x_\alpha) \in U$ and by (4) there exists $z \in F^{\perp}$ such that $l(z) \geqq 0$ when $\sigma_i = 1$ and $l_i(z) < 0$ when $\sigma_i = 0$. We may assume, that $|l_i(z)| < 1$. We put $t = y + \Pr(H^{\perp}, z)$. It is easy to verify that y belongs to the union of sets, described in (4); thus (4) is satisfied and $\chi(\tilde{H}, \delta_1, \ldots, \delta_k, x_0) \in U$.

The proof of the assertion, concerning the convergence is very long, but elementary and we only sketch it. We choose $y \in R^n$, such that $\chi(\tilde{H}, \delta_1, \ldots, \delta_k, x_0)(y) = 1$. We will prove that for sufficiently large $\alpha \in H$ we have $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x_\alpha)(y) = 1$, i.e. we have to verify some inequalities. There are some cases: when $l_i \in \tilde{H}$(i.e. $i=1,\dots,k$) and σ_i is 1 or 0 and $\mu_i=1$, 0 or -1 and when $l_i \in \tilde{F} \backslash \tilde{H}$ (i.e. $i=k+1,\dots,m$). In all cases we observe, that $l(y-x_\alpha) \leqq 1$ when $\sigma_i=1$ and that $l_i(y-x_\alpha)<0$ when $\sigma_i=0$ and α is sufficiently large, thus $\chi(\tilde{F}, \sigma_1,\dots,\sigma_n, x_\alpha)(y)=1$.

By the same way one may prove that if $\chi(\tilde{H}, \delta_1,\dots,\delta_k, x_0)(y)=0$ then $\chi(\tilde{F}, \sigma_1,\dots,\sigma_n, x_\alpha)(y)=0$ for sufficiently large α.

Thus $\chi(\tilde{F}, \sigma_1,\dots,\sigma_m, x_\alpha) \underset{w}{\rightarrow} \chi(\tilde{H}, \delta_1,\dots,\delta_k, x_0)$.

Theorem I. *(i) U is the minimal w-closed, invariant under action of R^n subset of S, such that $\chi(-P) \in U$.*
(ii) U is Hausdorff, first countable, loccally compact space.
(iii) $\{\chi(\tilde{F}, \sigma_1,\dots,\sigma_m, x_n)\}_{n=1}^{\infty}$ is w-convergent in S iff there exists $\chi(\tilde{H}, \delta_1,\dots\delta_k, x_0) \in U$ such that:

(a) $\tilde{H} \subset \tilde{F}$,

(b) $l(x_n) \rightarrow l(x_0)$ *if* $l \in \tilde{H}$,

(c) $l(x_n) \rightarrow \infty$ *if* $l \in \tilde{F} \backslash \tilde{H}$,

(d) for any sufficiently large n, $\Pr(H^{\perp}, x_n) \in x_0 + \cup D(\tilde{H}, \mu_1,\dots,\mu_k)$, *where μ_i, $i=1,2,\dots,k$ in the above union satisfy the following conditions:*

$$\text{(6)} \qquad \begin{array}{llll} \mu_i=1 & \text{if } \delta_i=1 & \text{and} & \sigma_i=0 \\ \mu_i \in \{0,1) & \text{if } \delta_i=1 & \text{and} & \sigma_i=1 \\ \mu_i=-1 & \text{if } \delta_i=0 & \text{and} & \sigma_i=1 \\ \mu_i \in \{-1,0\} & \text{if } \delta_i=0 & \text{and} & \sigma_i=0. \end{array}$$

(iv) The family $\{\chi(x-P): x \in R^n\}$ is dense in U.
(v) The closure in U of $\{\chi(x-P): x \in x_0+P\}$ is compact for any $x_0 \in R^n$.

Proof: We begin with (iii). In Lemma I we proved the sufficient condition.

Let $\chi(\tilde{F}, \sigma_1,\dots,\sigma_m, x_\alpha)$ be w-convergent in S. We may choose a subnet x_β, such that $l(x_\beta)$ is convergent or $l(x_\beta) \rightarrow \infty$ for $l \in \tilde{F}$. (If we assume that $l(x_\beta) \rightarrow -\infty$, then the limit is $\chi(\emptyset)$ — a contradiction.) Let $l \in \tilde{H}$ iff $l(x_\beta)$ is convergent. Obviously, $\tilde{H}$ satisfies (2) and by the note before Lemma I there exists $x_0 \in H^{\perp}$, such that $l(x_\beta) \rightarrow l(x_0)$ for $l \in \tilde{H}$.

The union of all characteristic sets $D(\tilde{H}, \mu_1,\dots,\mu_k)$, corresponding to $\tilde{H}$ is $H^{\perp}$. Thus we may choose a characteristic set $D(\tilde{H}, \mu_1,\dots,\mu_k)$ and a subnet of x_β, denoting by x_γ, such that $\Pr(H^{\perp}, x_\gamma) \in x_0 + D(\tilde{H}, \mu_1,\dots,\mu_k)$.

The assumptions of Lemma I are valid, hence $\chi(\tilde{F}, \sigma_1,\dots,\sigma_m, x_\gamma) \underset{w}{\rightarrow}$ $\chi(\tilde{H}, \delta_1,\dots,\delta_k, x_0)$, where δ are as in (5). A different choice of $\tilde{H}$ and x_0 leads us to a contradiction with the w-convergence of the net; a different choice of the determining set $P(\tilde{H}, \mu_1,\dots,\mu_k)$ is possible and the comparison with (5) gives us (6).

(iv) Let $(\tilde{F}, \sigma_1,\dots,\sigma_m, x_0) \in U$. Let $z_n \rightarrow 0$ and z_n is in the union of sets, described in (4). Let $y_n = ny$, where y is the point described in (I) and let us put $x_n = z_n + y_n + x_0$. By Lemma I $\chi(\tilde{P}, 1, \dots, 1, x_n) \underset{W}{\rightarrow} \chi(F, \sigma_1, \dots, \sigma_m, x_0)$.

(v) and the locally compactness of U is easy to prove using subsequences, the proof of the left over statements is trivial and we omit it.

2. In [7] P. Muhly and J. Renault define a topological space Y and its subspace X. Using these spaces, they construct a groupoid θ. It was indicated in the introduction, that these spaces describe the ideal structure of $C^*(\theta)$ and $\mathscr{B}$ (see Proposition I). In this part we will elusidate the connections between Y and U — the space considered in §I and we will construct a model of Y.

Up to the end we will assume that G is a subgroup of R^n endowed with a locally compact topology and G span R^n. Let P be a polyhedral cone in R^n, $P \cap (-P) = \{0\}$. Let the subsemigroup $P_G = P \cap G$ of G satisfies the following conditions:

(7) $$P_G - P_G = G$$
$$P_G = \overline{\mathrm{Int}(P_G)}.$$

We extend the Haar measure λ of G up to a measure λ on R^n, such that $\lambda(P^n \backslash G) = 0$. P_G is closed and hence P_G and P are λ-measurable. By Theorem I the elements of U are λ-measurable.

We denote (following [7]) by A the C^*-algebra of bounded functions $\varphi : G \rightarrow C$ generated by $\chi(-P) * f$, when f runs trough $L^1(G)$ and

$$\chi(-P) * f(t) = \int_{t-P} f(s) \, d\lambda(s).$$

Definition. Y is the spectrum of the C^*-algebra A.

There exists a continuous imbedding $\tau : G \rightarrow Y$, such that $\tau(t)\varphi = \varphi(t)$, where $t \in G$ and $\varphi \in A$. The formula $j(t) = \chi(t - P)$ defines a map $j : G \rightarrow U$ (this map is not always continuous). Let U_G be the closure of $j(G)$. Let $f \in L^1(G)$. Ψ_f is a function, defined on U_G by the formula:

(8) $$\Psi_f(\chi(M)) = \int_M f(s) \, d\lambda(s).$$

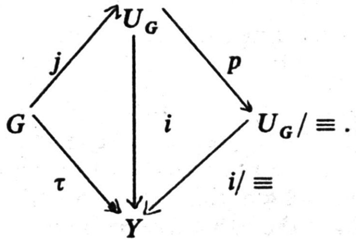
We note, that

$$\Psi_f(\chi(t-P)) = \int_{t-P} f(s) \, d\lambda(s) = \chi(-P) * f(t) = \tau(t)(\chi(-P) * f).$$

Let us define a map $i : \chi(t-P) \rightarrow \tau(t) \in Y$. Since the domain of this map is dence in U_G, then by Lebesgue theorem we may extend this map (and we save the same name) to a map $i : U_G \rightarrow Y$ and i is continuous.

If $\chi(M_1)$, $\chi(M_2) \in U_G$ and $\chi(M_1)$ is λ-a.e. equal to $\chi(M_2)$, then $i(\chi(M_1)) = i(\chi(M_2))$. Let $U_G/\equiv$ be the set of the equivalence classes of a.e. coinciding elements of U_G and $i/\equiv$ is the corresponding factormap. By Lebesgue theorem $i/\equiv$ is continuous.

Thus, we obtain the following commutative diagram:

(9)


We will show, that $i/\equiv$ is a homeomorphism, i.e. $U_G/\equiv$ is a model of Y. We note, that if $G=R^n$ with the discrete topology, then $U_G=U$.

The following lemma concerns the functions Ψ_f (see (8)).

Lemma 2. *Let* $f\in L^1(G)$. *Then* $\Psi_f\in C_0(U_G)$.

Proof: The Lebesgue theorem implies continuity of Ψ_f. Let $\varepsilon>0$. We will prove, that there exists a compact subset K of U_G, such that $|\Psi_f(\chi(M))|<\varepsilon$ if $\chi(M)\in U_G\setminus K$. We may choose a compact set $L\subset G$, such that $\int_{G\setminus L}|f|d\lambda<\varepsilon$.

Since P_G has a non-void interior, then $\{t+\mathrm{Int}(P_G):t\in G\}$ is an open cover of L. Thus there exists a finite cover of L:

$$L\subset\bigcup_{i=1}^{n}(g_i+\mathrm{Int}(P_G))\subset\bigcup_{i=1}^{n}(g_i+P_G)\subset g_0+P,\ g_0\in G$$

and therefore:

$$j(L)\subset j(t_0+P_G)\subset j(t_0+P)\doteq K.$$

If we choose $\chi(M)\in U_G\setminus K$, then $\chi(M)=w-\lim\chi(s_n-P)$ where $s_n\in G\setminus(t_0+P)$. If $s_n\in G\setminus(t_0+P)$, then $(s_n-P)\cap(t_0+P)=\emptyset$, hence $(s_n-P)\cap L=\emptyset$ and therefore $|\Psi_f(s_n-P)|<\varepsilon$. By Lebesgue theorem $|\Psi_f(\chi(M))|\leqq\varepsilon$.

Theorem 2. $i/\equiv:U_G/\equiv\ \to Y$ *is a homeomorphism.*

Proof: We observe that there exists an isometric isomorphism between C^*-subalgebra of $C_0(U_G)$, generated by the family $\{\Psi_f:f\in L^1(G)\}$ and the C^*-algebra A. The correspondence is $\Psi_f\to\varphi_f=\chi(P)*f\in A$ (the simple proof of this fact is omitted). We note that if $g\in G$, then

(10) $$\Psi(j(g))=\Psi(\chi(g-P))=\varphi(\tau(g)).$$

Let us choose $y\in Y$ and $V_1\subset\subset V_2$ be compact neighbourhoods of y. By Urison theorem ([9], part 4) there exists a continuous function φ, such that $\varphi_{|V_1}=1$ and $\varphi_{|Y\setminus V_2}=0$. Let $\Psi\in C_0(U_G)$ is the function, which is corresponding to φ. We denote $K=\Psi^{-1}([\frac{1}{2},1])$. Since $\Psi\in C_0(U_G)$, then K is a compact subset of U_G.

If $y^1=\tau(g)\in V_1$, then $y^1=i(j(g))$. Since $\varphi(y^1)=1$ and using (10) we obtain $\Psi(j(g))=1$, i.e. $j(g)\in K$. Thus $i(K)$ contains $\tau(G)\cap V_1$, which is dense in V_1. $i(K)$ is a continuous image of a compact set and hence $i(K)$ is closed. Thus $i(K)\supset V_1$.

We verify, that i is onto and hence $i/\equiv$ is onto. If $i(\chi(M_1))=i(\chi(M_2))$, where $\chi(M_1)$, $\chi(M_2)\in U_G$, then $\int_{M_1} f\,d\lambda=\int_{M_2} f\,d\lambda$ for any $f\in L^1(R^n, \lambda)$, and M_1 is a.e.-equal to M_2. Thus $i/\equiv$ is a bijection.

By the definition of p (see (9)), p is a factor map (i.e. if $f: U_G\to Z$, $g: U_G/\equiv\,\to Z$, $f=gp$ and f is a continuous map in the topological space Z, then g is a continuous map); we conclude, that $i/\equiv$ is a continuous map. Therefore the topology in $U_G/\equiv$ is stronger then the inverse image of the topology in Y. Thus $U_G/\equiv$ is Hausdorff topological space.

Let $K'=p(K)$. K' is a compact in $U_G/\equiv$. The restriction of $i/\equiv$ on K' is a continuous bijection between K' and $i(K)$ and hence is homeomorphism. Thus $i/\equiv\,: U_G/\equiv\,\to Y$ is a homeomorphism.

3. In this part we obtain some applications, concerning the Wiener-Hopf algebra $\mathscr{B}$. Let $X=\tau(P_G)$, where $\tau: G\to Y$ (see (9)). By the Theorem 2 we may identify X with the closure in $U_G/\equiv$ of $p(j(P_G))$. The Proposition I, (i) explains the importance of X: if we construct an increasing sequence of an open invariant subsets of X, then we obtain an increasing sequence of ideals of $\mathscr{B}$ and Proposition I, (ii) gives us "a presentation" of the ideals and the quotients as grupoid C^*-algebras.

An orbit in X is an intersection of X with an orbit in Y. It is not difficult to show, that if $\tilde{F}=\tilde{H}$, then $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x)$ is not a.e.-equal to $\chi(\tilde{H}, \delta_1, \ldots, \delta_s, x_0)$. Hence it is correct to define subsets X_k of X as follows: the equivalence class of $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x)\in U_G$ belongs to X_k iff $\dim\langle F\rangle=k$. Here we will use for the equivalence class of $\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x_0)$ the same sign.

Let $\{\chi(\tilde{F}, \sigma_1, \ldots, \sigma_m, x_n)\}_{n=1}^{\infty}$ be a convergent sequence in $\bigcup_{k\geq l} X^k$, where $0\leq l\leq n$ and its limes be $\chi(\tilde{H}, \delta_1, \ldots, \delta_s, x_0)$. Then $\tilde{H}\subset\tilde{F}$ by the Theorem I, hence $\dim\langle H\rangle\geq\dim\langle F\rangle$ and $\chi(\tilde{H}, \delta_1, \ldots, \delta_s, x_0)\in\bigcup_{k\geq l} X_k$. Therefore $\bigcup_{k\geq l} X_k$ is closed, for any $l=0, 1, \ldots, n$. Thus $X_l'=\bigcup_{k\leq l} X_k$ is open in X, for any $l=0, 1, \ldots, n$. It is obvious, that X_l' is invariant. Let I_l is the corresponding ideal in $\mathscr{B}$. That is the proof of the following theorem.

Theorem 3. $I_0\subset I_1\subset\ldots\subset I_n=\mathscr{B}$ *is an increasing sequence of ideals of* $\mathscr{B}$, *such that* I_{k+1}/I_k *is isomorphic to* $C^*(Y\times G\,|\,X_k)$.

Before specifying this result, we need a result of Muhly and Renault.

Proposition 3. (*i*) (3.7.2 of [7]). *If* $\theta=\theta_1\times K$, *where* θ_1 *is a reduction of a transformation group and* K *is a locally compact group such that the Haar measures in* θ *are the product of the Haar measures in* θ_1 *and the Haar measure of* K, *then* $C^*_{red}(\theta)=C^*_{red}(\theta_1)\hat{\otimes} C^*_{red}(K)$.

(*ii*) (3.7.3 of [7]). *If* X *is a regular compactification of* P, *then* $\mathscr{K}$ — *the ideal of the compact operators is contained in* $\mathscr{B}$.

Corollary 1. I_{n-1} *is the commutator ideal of* $\mathscr{B}$; I_n/I_{n-1} *is isomorphic to* $C_0(\hat{G})$, *where* $\hat{G}$ *is the dual group of* G.

Proof: The set $X'_n \setminus X'_{n-1}$ has only one element $-x_\infty = \chi(R^n)$ and the elements of I_{n-1} vanish in x_∞. As in [7], 3.5 one may prove, that the commutators generate I_{n-1}. The ideal I_n/I_{n-1} is isomorphic to $C^*_{\mathrm{red}}(\theta \mid \{x_\infty\})$. But G acts trivially on x_∞ and by Proposition 3 I_n/I_{n-1} is isomorphic to $C^*(G)$, which is $C_0(\hat{G})$.

Corollary 2. *Let G be a discrete group. X is a regular compactification of* P_G *iff there exists* ε*-neighbourhood* $\mathcal{O}(0, \varepsilon)$ *of* $0 \in R^n$, *such that* $P \cap \mathcal{O}(0, \varepsilon) \cap G = \{0\}$. *In this case* $\mathcal{K} \subset \mathcal{B}$.

We omit the easy proof. If G is not discrete, the problems when X is a regular compactification of P_G and whether $\mathcal{K} \subset \mathcal{B}$ are still open.

Example I. Let $G = Z^n$ and P be a polyhedral cone in G, then X is a regular compactification of P and thus $\mathcal{K} = I_0$. We note, that if the cone is not rational, then $\mathcal{B}$ is not of the type I.

Example 2. Let $G = R^2$, endowed with a discrete topology. Let l_1 and l_2 be linear functionals, which determine P. The elements of X_0 are the characteristic functions of the sets of the following type:

$$\{x : l_1(x - x_0) < 0;\ l_2(x - x_0) < 0\}, \quad \text{where} \quad x_0 \in \mathrm{Int}(P)$$
$$\{x : l_1(x - x_0) \leqq 0;\ l_2(x - x_0) < 0\}, \quad \text{where} \quad l_1(x_0) \geqq 0;\ l_2(x_0) > 0$$
$$\{x : l_1(x - x_0) < 0;\ l_2(x - x_0) \leqq 0\}, \quad \text{where} \quad l_1(x_0) > 0;\ l_2(x_0) \geqq 0$$
$$\{x : l_1(x - x_0) \leqq 0;\ l_2(x - x_0) \leqq 0\}, \quad \text{where} \quad x_0 \in P.$$

Here we have 4 orbits, and each orbit is dense in X_0. By [8], 4.6 is a simple C^*-algebra.

Let the elements of Z_1 be the half-spaces of the type:

$$\{x \in R^n : l_1(x - x_0) < 0\}, \quad \text{if} \quad l_1(x_0) > 0$$
$$\{x \in R^n : l_1(x - x_0) \leqq 0\}, \quad \text{if} \quad l_1(x_0) \geqq 0.$$

The functional l_2 determines a similar set Z_2. $X'_0 \cup Z_1$ and $X'_0 \cup Z_2$ are open invariant subsets of X. $X'_0 \cup Z_1 \cup Z_2$ and the isotropy group of any element of $X'_1 \setminus X'_0$ is R. Thus there are the following open invariant subsets of X: X'_0, $X'_0 \cup Z_1$, $X'_0 \cup Z_2$, $X'_1 = X'_0 \cup Z_1 \cup Z_2$ and $X_0 \cup Z_1 \cup Z_2 \cup \{\infty\}$ and the corresponding ideals are I_0, I'_1, I''_1, I_1, $I_2 = \mathcal{B}$. We have $I'_1 . I''_1 = I_0$, $I'_1 + I''_1 = I_1$. I_0 is a simple C^*-algebra, which does not contain $\mathcal{K}$. The quotients are:

$$I_1/I_0 \cong C^*(\theta \mid Z_1) \oplus C^*(\theta \mid Z_2) \cong (C(\hat{R}) \hat{\otimes} C_1) \oplus (C(\hat{R}) \otimes C_2), \quad I_2/I_1 \cong C(\hat{G}),$$

where C_1 and C_2 are simple C^*-algebras, which do not contain $\mathcal{K}$, and $\hat{G}$ is the dual group of G.

We are not able to answer whether Proposition I describes all ideals of $\mathcal{B}$. By Proposition 4.6 of [7], if G is discrete and acts transitively on X, then the above statement is valid, but not always I_0 is a simple C^*-algebra.

E x a m p l e 3. Let $G=Z\times R$ and P be generated by the points $(1, -1)$ and $(1, 1)$. There exists an open invariant proper subset of X'_0; by Corollary 3 $\mathscr{K}\subset\mathscr{B}$; the quotient $I'_0/\mathscr{K}$ is a simple C^*-algebra, which does not contain $\mathscr{K}$.

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Faculty of Mathematics
and Informatics
Sofia University
A. Ivanov Boul. 5
Sofia
BULGARIA

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