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Scattering for Nonlinear Wave Equation with Small Initial Data

Krasimir G. Ianakiev

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We construct a global solution to the nonlinear wave equation $u_{tt}-\Delta_x u=F(u, \nabla u)$ with small initial data and we prove the existence of the scattering operator.

1. Introduction

In this work we study the existence of the scattering operator for the nonlinear wave equation

(1.1) $$\Box u(x, t) \equiv u_{tt}-\Delta_x u=F(u, \nabla u) \text{ in } \mathsf{R}^n_x \times \mathsf{R}_t.$$

Here u is a scalar function, $n \geqq 3$ is odd and $\nabla u=(\partial_t u, \nabla_x u)$. The function $F(y)$ with $y=(y_0, y_1, \dots, y_{n+1})$ satisfies in a neighborhood of $y=0$ one of the following three assumptions:

(H1) $\begin{cases}\text{For } n>3,\ F(y) \text{ is a polinomial with respect} \\ \text{to } y=(y_0, y_1, \dots, y_{n+1}) \text{ such that } \partial^\alpha_y F(0,\dots,0)=0 \text{ for} \\ |\alpha| \leqq p \text{ and } p \geqq n.\end{cases}$

(H2) For $n>3$, $F=F(y)$ depends only on one variable and the estimates

$$\left|\frac{\partial^s F}{\partial y^s}(y)\right|=0(|y|^{p-s}),$$

$$\left|\frac{\partial^s F}{\partial y^s}(y_1)-\frac{\partial^s F}{\partial y^s}(y_2)\right| \leqq C|y_1-y_2|(|y_1|^{p-s+1}+|y_2|^{p-s+1})$$

hold for $s=0, 1, \dots, n-1$ and $p>\backslash(n-1+\sqrt{n^2-2n+5})/2$.

In the case $p<n$, if $s=n-1$, then we have the inequality

$$\left|\frac{\partial^{n-1} F}{\partial y^{n-1}}(y_1)-\frac{\partial^{n-1} F}{\partial y^{n-1}}(y_2)\right| \leqq C|y_1-y_2|^{p-n+1}.$$

$$\text{(H3)} \quad \begin{cases} \text{For } n=3, \ F \text{ is of the type} \\ F(y_0, y_1, \dots, y_4) = \sum_{i,j=0}^{4} c_{i,j} y_i^{p-1} y_j, \\ \text{where } p>4+\sqrt{2} \text{ and } \{c_{i,j}\} \text{ are constants.} \end{cases}$$

The problem (1.1) with nonlinear term $F(u, \nabla u)$ including the derivatives of the solution is closely related to the interactions in quantum field theory. Also, this is a typical equation in the nonlinear theory of elasticity.

To our knowledge the analysis concerning the existence of the scattering operator for the nonlinear wave equation was mainly connected with nonlinear terms involving $F(u)$. For $n=3$, W. Strauss [25] obtained the existence of the scattering operator under the assumption $F(u)=\lambda|u|^p$ with $p>2{,}686$. In [19] this result has been improved for $p>2{,}535$. H. Pecher [20] got suitable $L^\infty - L^\infty$ "weighted" estimates, suitably adapted to the different behaviour of the solution in a neighborhood of the characteristic cone and away from it.

Another approach, based on conformal conservation law for wave equation, was proposed by J. Ginibre and G. Velo (see [10], [11]). They investigated the problem for existence of the scattering operator in arbitrary space dimension. The main result in [10], [11] is the energy decay of the solution in the form

$$||\varphi(t)||_e \leqq Ct^{-(n-1)(1/2-1/l)}$$

for $2 \leqq l \leqq 2n/(n-2)$. Here φ is the solution to the nonlinear wave equation $\square\varphi=\lambda|\varphi|^p$ with initial data in the space $X_e=H^1 \oplus L^2$.

In contrast to the results cited above, in this work we study nonlinear terms $F(u, \nabla u)$ including first derivatives $u_t, u_{x_1}, \dots, u_{x_n}$ of the solution u and we cover the case of space dimensions $n \geqq 3$.

The novelty in our paper are some suitable $L^\infty - L^2$ and $L^2 - L^2$ "weighted" estimates. Combining these estimates with $L^\infty - L^\infty$ "weighted" estimates, we obtain a possibility for new proof of the existence of a global solution to the nonlinear wave equation. For some other results concerning global existence of solution to (1.1) see [3], [8], [17]. In these works various approaches are presented such as the use of the generators of the conformal group in $\mathbb{R}^{n+1}$, suitable conformal transformations, etc. Moreover, we obtain the following decay estimates for the solution and its derivatives:

$$\sup_{(x,t)\in\mathbb{R}^{n+1}} (1+|x|+|t|)^a(1+||x|-|t||)^b|\partial_t^\alpha \partial_x^\gamma u(x, t)| \leqq \delta,$$

where for $n>3$, n odd, $a=1$, $b=p-n+1$ and for $n=3$, $a=1/2$, $b=(\theta+\sqrt{2}-1)/2$ with sufficiently small $0<\theta$. For the proof of $L^\infty - L^\infty$ estimates we exploit some ideas due to H. Pecher [20]. We generalize a part of their results in the case $n \geqq 3$, n odd.

One of the basic assumptions is that the initial data are small and smooth in a suitable sense and decrease rapidly at infinity. So we obtain a solution, whose behaviour in the sense of the energy norm is the same as that of $u_0^\pm$ at $t \to \pm\infty$. More precisely, we have

$$\|(u-u_0^{\pm})(t)\|_e \leqq (1+|t|)^{-B},$$

where $B=p-n+1$ for $n>3$ and $B=(p-4-\sqrt{2})/2$ for $n=3$. Here $u_0^{\pm}$ are solutions of free wave equation (see section 2 for a description of $u_0^{\pm}$). Our main result is the following.

Theorem 1.1. *There exists a sufficiently small neighborhood U of zero in $W_{p-n+2,n+2} \times W_{p-n+3,n+1}$ (respectively in the case $n=3$ in $W_{q/2,5} \times W_{q/2+1,4}$) such that the scattering operator $S:(f, g) \in U \to (u_0^+(x, 0), \partial_t u_0^+(x, 0)) \in W_{p-n+2,(n+3)/2} \times W_{p-n+3,(n+1)/2}$ (respectively $W_{q-1/2,3} \times W_{q-1/2+1,2}$) exists in U in the sense of the energy norm. Here $W_{k,N}$ denotes the "weighted" Sobolev spaces.*

Note that even in the case, when F satisfies **(H2)**, this result is new. The initial data of the Cauchy problem for nonlinear wave equation studied by J. Ginibre and G. Velo in [10], [11] belong to the energy space $H^1 \oplus L^2$. On the other, the result of Theorem 1.1 enables one to treat solutions with initial data, which are not in $H^1 \oplus L^2$. Also, our $L^\infty - L^\infty$ "weighted" estimates are different from those, obtained by H. Pecher in [12], because in our case the order of the derivatives in the left-hand side is less than the order of the derivatives in the right-hand side of the $L^\infty - L^\infty$ estimates. This fact causes the basic difficulty in the proof of the global solution to (1.1). We overcome this difficulty by using a conservation law for nonlinear wave equation, so that the lost of derivatives can be compensated.

The plan of our work is as follows. In section 2 we give some preliminary inequalities. By using them, we derive the $L^\infty - L^\infty$, $L^\infty - L^2$ and $L^2 - L^2$ estimates. There we study the behaviour of the solution and its derivatives of the wave equation with small initial data. In section 3 we construct a global classical solution of the Cauchy problem for the nonlinear wave equation with small initial data and we obtain suitable weighted decay estimates. Finally, in this section we establish the existence of the scattering operator.

A part of results of this work has been announced in [12].

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2. Some preliminary results

In this paragraph we prove some inequalities, which will be used for the construction of a global solution to the nonlinear wave equation with sufficiently small and smooth initial data. Besides, we establish estimates for the solution of the free wave equation and its derivatives with respect to t and x.

Denote by ∂^γ, $|\gamma|=\gamma_1+\gamma_2+\dots\gamma_n$ the composition of the differential operators $\partial_i^{\gamma_i}$, where $\partial_i=\partial/\partial x_i$ and $\partial_t=\partial/\partial t$. Set $\sigma=(n-1+\sqrt{n^2-2n+5})/2$.

We start with the following

Lemma 1. *Let $p>\sigma$ and let*

$$g(\lambda, s)=\lambda(\lambda-|r-t+s|)^{\frac{n-3}{2}}(\lambda+|r-t+s|)^{\frac{n-3}{2}}((r+t-s)-\lambda)^{\frac{n-3}{2}}$$
$$\times (r+t-s+\lambda)^{\frac{n-3}{2}}(1+\lambda+|s|)^{-p}(1+|\lambda-|s||)^{-p(p-n+1)}.$$

Then the inequality

$$\int_{-\infty}^{t}\left(\int_{|r-t+s|}^{r+t-s} g(\lambda, s)\,d\lambda\right)ds \leqq Cr^{n-2}(1+r+|t|)^{-1}(1+|r-|t||)^{-p+n-1}$$

holds for any $r\geqq 0$ *and any* $t\in \mathbf{R}$.

Proof. Taking into account the inequalities

$$\lambda+|r-t+s|\leqq 2\lambda \text{ and } (r+t-s)-\lambda\leqq(r+t-s)-|r-t+s|\leqq 2r,$$

we obtain the estimate

$$((r+t-s)+\lambda)^{\frac{n-3}{2}}=[(r+t-s-\lambda)+2\lambda]^{\frac{n-3}{2}}$$
$$\leqq C_1\sum_{k=0}^{\frac{n-3}{2}}((r+t-s)-\lambda)^k\lambda^{\frac{n-3}{2}-k}\leqq C_2\sum_{k=0}^{\frac{n-3}{2}} r^k\lambda^{\frac{n-3}{2}-k}.$$

Having in mind the above inequalities, we get

$$\int_{-\infty}^{t}\left(\int_{|r-t+s|}^{r+t-s} g(\lambda, s)\,d\lambda\right)ds$$
$$\leqq C_1\sum_{k=0}^{\frac{n-3}{2}}\int_{-\infty}^{t}\left(\int_{|r-t+s|}^{r+t-s}\lambda\lambda^{\frac{n-3}{2}}\lambda^k r^{\frac{n-3}{2}-k} r^{\frac{n-3}{2}} r^k\lambda^{\frac{n-3}{2}-k}\right.$$
$$\left.\times(1+\lambda+|s|)^{-p}(1+|\lambda-|s||)^{-p(p-n+1)}\,d\lambda\right)ds$$
$$\leqq C_2 r^{n-3}\int_{-\infty}^{t}\left(\int_{|r-t+s|}^{r+t-s}\lambda(1+\lambda+|s|)^{-p+n-3}(1+|\lambda-|s||)^{-p(p-n+1)}\,d\lambda\right)ds$$
$$=C_2 r^{n-2} I(r, t).$$

Using the ideas, developed by H. Pecher [20], we deduce

$$I(r, t)\leqq Cr(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)}.$$

This completes the proof.■

Another important estimate is given in the following

Lemma 2. *Let* $n>3$, $p>\sigma$ *and* $A=2(n-3-v)$, v *being a fixed integer such that* $0\leqq v\leqq(n-3)/2$. *Then for any* $r\geqq 0$ *and* $t\in\mathbf{R}$ *we have*

$$\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}\int_{|r-t+s|}^{r+t-s}\lambda(\lambda^2-|r-t+s|^2)^{\frac{n-3}{2}}((r+t-s)^2-\lambda^2)^{\frac{n-3}{2}}$$

$$\times(1+\lambda+|s|)^{-2(p-1)}(1+|\lambda-|s||)^{-2(p-1)(p-n+1)}d\lambda\Big)^{1/2}ds$$

$$\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)}.$$

Proof. Denote by $I(r, t)$ the integral term in the left-hand side of the above inequality. We consider five cases:

1) case: $|t|-1\leqq r\leqq |t|+1$.

If $r\geqq 1$, then using the inequalities

$$A\geqq(n-3)/2+1,\ n-2\geqq(n-3)/2+2 \text{ and } 2p-n>2,$$

we arrange the following estimates:

$$I(r, t)\leqq C_1\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}(t-s)^{\frac{n-3}{2}+1}r^{\frac{n-3}{2}}(1+|s|)^{-2p+n}\Big)^{1/2}ds$$

$$\leqq C_2 r^{-1}\int_{-\infty}^{+\infty}(1+|s|)^{-(2p-n)/2}ds=C_3 r^{-1}$$

$$\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-p+n-1}.$$

If $r\leqq 1$, then we proceed similarly and taking advantage of the inequality $2p-(n+3)/2>2$, we derive

$$I(r, t)\leqq C_1\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}((t-s)r)^{\frac{n-3}{2}}2r r^{\frac{n-3}{2}}(1+|s|)^{-2p+(n+3)/2}\Big)^{1/2}ds$$

$$\leqq C_2\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-p+n-1}.$$

2) If $t\geqq 0$ and $0\leqq r\leqq |t|-1$, then $t\geqq 1$. Again we consider two cases:

2a) In the case $1+t-r\geqq(1+r+t)/2$ the inequalities

$$2p-2-(n-1)/2>2(p-n+2)+1 \text{ and } A\geqq(n-3)/2+1$$

give the estimates

$$I(r, t)\leqq C_1\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}r^{\frac{n-3}{2}}(t-s)^{\frac{n-3}{2}}r^{\frac{n-3}{2}}2r$$

$$\times(1+t-r)^{-2(p-n+2)}(1+|s|)^{-1-2\alpha}\Big)^{1/2}ds$$

$$\leqq C_2(1+t-r)^{-(p-n+2)}\int_{-\infty}^{t-1}(t-s)^{-1/2}(1+|s|)^{-1/2-\alpha}ds$$

$$\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)},$$

because the integral $\int_{-\infty}^{t-1}(t-s)^{-1/2}(1+|s|)^{-1/2-\alpha}\,ds$ is bounded. Here $\alpha=(3n-13)/2>0$.

2b) In the case $1+t-r\leqq(1+r+|t|)/2$, obviously $1+t\leqq 3r$ and by similar arguments as before we obtain

$$I(r,\ t)\leqq C_1\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}(4r(t-s))^{\frac{n-3}{2}}2(t-s)$$
$$\times(1+t-r)^{-2(p-n+1)}(1+|s|)^{-n+2}\Big)^{1/2}ds$$

$$\leqq C_2 r^{-1}(1+t-r)^{-(p-n+1)}\int_{-\infty}^{+\infty}(1+|s|)^{(2-n)/2}ds$$

$$\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)}.$$

3) If $t\geqq 0$ and $r\geqq|t|+1$, then analogously to the previous cases we get

$$I(r,\ t)\leqq C_1\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}(4r(t-s))^{\frac{n-3}{2}}2(t-s)(1+t-r)^{-2(p-n+1)}$$
$$\times(1+|s|)^{-n+2}\Big)^{1/2}ds\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-p+n-1}.$$

4) If $t<0$ and $0\leqq r\leqq|t|-1$, then as in the case 2 we obtain

$$I(r,\ t)\leqq C_1\int_{-\infty}^{t-1}\Big((t-s)^{-A}r^{2-n}r^{\frac{n-3}{2}}(t-s)^{\frac{n-3}{2}}r^{\frac{n-3}{2}}2r$$
$$\times(1+|r-t|)^{-2(p-n+2)}(1+|s|)^{-1-2\alpha}\Big)^{1/2}ds$$

$$\leqq C_2(1+|r-t|)^{-(p-n+2)}\int_{-\infty}^{t-1}(t-s)^{-1/2}(1+|s|)^{-1/2-\alpha}ds$$

$$\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)}.$$

5) If $t\leqq 0$ and $r\geqq|t|+1$, then as in the case 3 we get

$$I(r,\ t)\leqq C_1 r^{-1}(1+r+|t|)^{-p+n-1}\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)}.$$

The proof is complete.■

An analogue of Lemmas 1 and 2 in the three dimensional space is the following

Lemma 3. *Let* $p>4+\sqrt{2}$ *and* $n=3$. *Denote by* $\theta_0=(p-4-\sqrt{2})/2$ *and* $q=1+\sqrt{2}+\theta$, *where* $0<\theta<\theta_0$ *is sufficiently small. Then*

$$\int_{-\infty}^{t} r^{-1}\left(\int_{|r-t+s|}^{r+t-s} \lambda(1+\lambda+|s|)^{-p/2}(1+|\lambda-|s||)^{-p(q-2)/2}\,d\lambda\right)ds$$
$$\leq C(1+r+|t|)^{-1/2}(1+|r-|t||)^{-(q-2)/2}$$

holds for any $r\geq 0$ *and all* $t\in\mathbb{R}$.

Lemma 4. *Let* $p>4+\sqrt{2}$, $n=3$ *and let* q *be defined as in Lemma* 3. *Then the inequality*

$$\int_{-\infty}^{t}\left(r^{-1}\int_{|r-t+s|}^{r+t-s} \lambda(1+\lambda+|s|)^{1-p}(1+|\lambda-|s||)^{(1-p)(q-2)}\,d\lambda\right)^{1/2}ds$$
$$\leq C(1+r+|t|)^{-1/2}(1+|r-|t||)^{-(q-2)/2}$$

holds for any $r\geq 0$ *and all* $t\in\mathbb{R}$.

The proofs of these two lemmas are similar to the proofs of Lemmas 1 and 2 respectively and we omit the details.

The following lemma gives possibility to replace the integration on a sphere in $\mathbb{R}^n$ by an integration over some interval in $\mathbb{R}$.

Lemma 5. *Let* $r=|x|$, $n=3$ *be odd and let* $C=\mu(S^{n-2})/2^{n-3}$, *where* $\mu(S^{n-2})$ *is the Euclidean measure of the unit sphere in* $\mathbb{R}^{n-1}$. *If* h *is a continuous function, then the equality*

$$\int_{|y-x|=t} h(|y|)\,dS_y=Ctr^{2-n}\int_{|r-t|}^{r+t}\lambda(\lambda-|r-t|)^{\frac{n-3}{2}}(\lambda+|r-t|)^{\frac{n-3}{2}}$$
$$\times((r+t)-\lambda)^{\frac{n-3}{2}}((r+t)+\lambda)^{\frac{n+3}{2}}h(\lambda)\,d\lambda$$

holds for any $t\geq 0$.

Now, let us consider the Cauchy problem for the linear wave equation

$$(1)\qquad \begin{cases}\Box u=u_{tt}-\Delta u=0 \text{ in } \mathbb{R}^n_x\times\mathbb{R}_t,\\ u(x,0)=f,\\ u_t(x,0)=g.\end{cases}$$

The following lemma gives a precise estimate for the behaviour of the solution of (1) with initial data, decreasing sufficiently rapidly as $|x|\to\infty$.

Lemma 6. *Let* $n\geq 3$ *be odd and let* γ *be the arbitrary multiindex. Let* $f\in C^{|\gamma|+(n+1)/2}$ *and* $g\in C^{|\gamma|+(n-1)/2}$ *satisfy the estimates*:

$$|\partial^{\alpha+\gamma}f(x)|\leq\varepsilon(1+|x|)^{-k-|\alpha|},\ |\partial^{\beta+\gamma}g(x)|\leq\varepsilon(1+|x|)^{-k-|\beta|-1}$$

for $0\leq|\alpha|\leq(n+1)/2$ *and* $0\leq|\beta|\leq(n-1)/2$. *Then the solution* $u(x,t)$ *of* (1) *with initial data* (f,g) *satisfies the inequality*

$$|\partial_t^{\alpha}\partial_x^{\gamma}u(x,t)|\leq C\varepsilon(1+|x|+|t|)^{-1}(1+||x|-|t||)^{-k+1}$$

for any $x\in\mathbb{R}^n$, $t\in\mathbb{R}$ *and* $\alpha=0$ *or* 1.

Proof. We shall prove the lemma in the case $\gamma=0$ and $t\geqq 0$. The other cases can be considered similarly.

The solution of (1) is given by the equality

$$u(x,\ t)=\sum_{\nu=0}^{(n-3)/2}(\nu+1)a_\nu t^\nu\left(\frac{\partial^\nu}{\partial t^\nu}Q_1\right)(x,\ t)$$

$$+\sum_{\nu=0}^{(n-3)/2}a_\nu t^{\nu+1}\left[\left(\frac{\partial^{\nu+1}}{\partial t^{\nu+1}}Q_1\right)(x,\ t)+\left(\frac{\partial^\nu}{\partial t^\nu}Q_2\right)(x,\ t)\right],$$

where $\{a_\nu\}_{\nu=0}^{(n-3)/2}$ are constants.

$$Q_1(x,\ t)=[\mu(S^{n-1})]^{-1}\int_{|\xi|=1}f(x+t\xi)\,dS_\xi,$$

$$Q_2(x,\ t)=[\mu(S^{n-1})]^{-1}\int_{|\xi|=1}g(x+t\xi)\,dS_\xi.$$

Consequently,

$$u(x,\ t)=\sum_{\nu=0}^{(n-1)/2}t^\nu\Bigg(\sum_{\alpha,\beta}\int_{|\xi|=1}[c_{\alpha,\nu}\xi^\alpha\partial^\alpha f(x+t\xi)$$

$$+d_{\beta,\nu}\xi^\beta\partial^\beta g(x+t\xi)]\,dS_\xi\Bigg),$$

where $|\alpha|=\nu$, $|\beta|=\nu-1$, and $c_{\alpha,\nu}$ and $d_{\beta,\nu}$ are constants. The latter equality gives:

$$u(x,\ t)\leqq|c_{0,0}|\int_{|\xi|=1}|f(x+t\xi)|\,dS_\xi+|t|\sum_{|\alpha|=1}|c_{\alpha,1}|\int_{|\xi|=1}|\partial^\alpha f(x+t\xi)|\,dS_\xi$$

$$+|d_{0,0}|\cdot|t|\int_{|\xi|=1}|g(x+t\xi)|\,dS_\xi+\sum_{\alpha,\nu}|c_{\alpha,\nu}|\cdot|t|^\nu\int_{|\xi|=1}|\partial^\alpha f(x+t\xi)|\,dS_\xi$$

$$+\sum_{\beta,\nu}|d_{\beta,\nu}|\cdot|t|^\nu\int_{|\xi|=1}|\partial^\beta g(x+t\xi)|\,dS_\xi.$$

Here ν is an integer in the interval $[2,\ (n-1)/2]$.

Note that in the case $n=3$ the first three terms take part in the right-hand side of the above inequality only.

Now we shall estimate separately the different terms.

1) first case: $\nu=0$.

$$\int_{|\xi|=1}|f(x+t\xi)|\,dS_\xi=t^{1-n}\int_{|y-x|=t}f(|y|)\,dS_y\leqq t^{1-n}\int_{|y-x|=t}\varepsilon(1+|y|)^{-k}\,dS_y$$

$$\leqq C_1\varepsilon t^{1-n}tr^{2-n}\int_{|r-t|}^{r+t}\lambda(\lambda^2-|r-t|^2)^{\frac{n-3}{2}}((r+t)^2-\lambda^2)^{\frac{n-3}{2}}(1+\lambda)^{-k}\,d\lambda$$

$$\leqq C_2\varepsilon(tr)^{2-n}(4rt)^{n-3}\int_{|r-t|}^{r+t}\lambda(1+\lambda)^{-k}\,d\lambda=C_3\varepsilon(rt)^{-1}\int_{|r-t|}^{r+t}\lambda(1+\lambda)^{-k}\,d\lambda.$$

Here we use the inequalities

$$\lambda^2-|r-t|^2\leqq 4rt \text{ and } (r+t)^2-\lambda^2\leqq 4rt.$$

The estimate

$$C_3\,\varepsilon(rt)^{-1}\int_{|r-t|}^{r+t}\lambda(1+\lambda)^{-k}\,d\lambda\leqq C(1+r+t)^{-1}(1+|r-t|)^{-k+1}$$

is due to H. Pecher [20].

2) second case: $1\leqq \nu\leqq (n-1)/2$.

Note that the terms with a representation $|t|^\nu\int_{|\xi|=1}|\partial^\alpha f(x+t\xi)|\,dS_\xi$ or $|t|^\nu\int_{|\xi|=1}|\partial^\beta g(x+t\xi)|\,dS_\xi$, where $|\alpha|=\nu$ and $|\beta|=\nu-1$, can be estimated in the same way.

$$|t|^\nu\int_{|\xi|=1}|\partial^\alpha f(x+t\xi)|\,dS_\xi=t^\nu t^{1-n}\int_{|y-x|=t}|\partial^\alpha f(y)|\,dS_\xi$$

$$\leqq \varepsilon t^\nu t^{1-n}\int_{|y-x|=t}(1+|y|)^{-k-\nu}\,dS_\xi=C\varepsilon t^\nu t^{1-n}\,t\,r^{2-n}$$

$$\times\int_{|r-t|}^{r+t}\lambda(\lambda^2-|r-t|^2)^{\frac{n-3}{2}}((r+t)^2-\lambda^2)^{\frac{n-3}{2}}(1+\lambda)^{-k-\nu}\,d\lambda=C\varepsilon t^\nu(tr)^{2-n}J(r,\ t).$$

We divide $R^n_x\times R_t$ into five domains.

a) In the case $t\leqq r$ and $1\leqq r$ we obtain

$$t^\nu(tr)^{2-n}J(r,\ t)\leqq C_1t^\nu(tr)^{2-n}t^{\frac{n-3}{2}}r^{\frac{n-1}{2}-(\nu-1)}(rt)^{\frac{n-3}{2}}$$

$$\times\int_{|r-t|}^{r+t}\lambda^\nu(1+\lambda)^{-k-\nu}\,d\lambda\leqq C_1t^{\nu-1}r^{-\nu}\int_{|r-t|}^{+\infty}(1+\lambda)^{-k}\,d\lambda$$

$$\leqq C_2r^{-1}(1+|r-t|)^{-k+1}\leqq C(1+r+t)^{-1}(1+|r-t|)^{-k+1}.$$

b) In the case $t\leqq r\leqq 1$, we derive the inequalities:

$$t^\nu(tr)^{2-n}J(r,\ t)\leqq C_1t^\nu(tr)^{2-n}(tr)^{n-3}\int_{|r-t|}^{r+t}\lambda(1+\lambda)^{-k-\nu}\,d\lambda$$

$$\leqq C_1t^\nu.(tr)^{-1}2t\,2r(1+|r-t|)^{-k-\nu}\leqq C_2(1+|r-t|)^{-k+1}$$

$$\leqq C(1+r+t)^{-1}(1+|r-t|)^{-k+1}.$$

c) In the case $r\leqq t\leqq 1$, as in the case b), we get

$$t^\nu(tr)^{2-n}J(r,\ t)\leqq C_1t^\nu.(tr)^{2-n}(tr)^{n-3}\int_{|r-t|}^{r+t}\lambda(1+\lambda)^{-k-\nu}\,d\lambda$$

$$\leqq C_2t^{\nu-1}r^{-1}2r(1+|r-t|)^{-k-\nu+1}\leqq C(1+r+t)^{-1}(1+|r-t|)^{-k+1}.$$

d) In the case $t\geqq 1$ and $r\leqq t/2$ similarly we have

$$t^\nu(tr)^{2-n}J(r,\ t)\leqq C_1t^\nu(tr)^{2-n}(tr)^{n-3}2r(1+|r-t|)^{-k+1}(1+t/2)^{-\nu-1}$$

$$\leqq C_2.(2+t).(1+|r-t|)^{-k+1}\leqq C(1+r+t)^{-1}(1+|r-t|)^{-k+1}.$$

Here we use the inequality $1+\lambda \geqq 1+|r-t| \geqq 1+t/2$.

e) In the case $t/2 \leqq r \leqq t$ and $t \geqq 1$, as in the case a), we obtain

$$t^{\nu}(tr)^{2-n}J(r,\ t) \leqq C_1 t^{\nu}(tr)^{2-n} r^{\frac{n-3}{2}} t^{\frac{n-1}{2}-(\nu-1)}(rt)^{\frac{n-3}{2}}$$

$$\times \int\limits_{|r-t|}^{r+t} \lambda^{\nu}(1+\lambda)^{-k-\nu}\,d\lambda \leqq C_2 r^{-1} \int\limits_{|r-t|}^{r+t} (1+\lambda)^{-k}\,d\lambda$$

$$\leqq C_3 r^{-1}(1+|r-t|)^{-k+1} \leqq C(1+r+t)^{-1}(1+|r-t|)^{-k+1}.$$

Thus

$$|u(x,\ t)| \leqq C\varepsilon(1+|x|+|t|)^{-1}(1+||x|-|t||)^{-k+1}.$$

For the second inequality (when $\alpha=1$), it suffices to differentiate the solution $u(x,\ t)$ and by similar arguments we can establish the desired inequality. The lemma is proved. ∎

The following proposition makes it possible to replace the integration on the unit sphere by an integration on the unit ball.

Proposition 7. *Assume, that for any* $t \in \mathbb{R}$, $h(x,\ t) \in L^2(\mathbb{R}^n_x)$ *and* $\nabla_x h(x,t) \in L^2(\mathbb{R}^n_x)$. *Then for* $\tau,\ t \in \mathbb{R}$ *we have the inequality*:

$$\int\limits_{|\xi|=1} h^2(x+(t-\tau)\xi,\ \tau)\,dS_\xi \leqq C \int\limits_{|\xi|\leqq 1} h^2(x+(t-\tau)\xi,\ \tau)\,d\xi$$

$$+C(t-\tau)\left[\int\limits_{|\xi|\leqq 1} h^2(x+(t-\tau)\xi,\ \tau)\,d\xi + \sum_{i=1}^{n} \int\limits_{|\xi|\leqq 1} (\partial_i h)^2(x+(t-\tau)\xi,\ \tau)\,d\xi\right],$$

where C *is a constant.*

Proof. Note, that the vector normal to the unit sphere in the point $\xi=(\xi_1, \xi_2, \ldots, \xi_n)$ is just equal to $\xi=(\xi_1, \xi_2, \ldots, \xi_n)$. Using Gaussian divergence theorem, we obtain

$$\int\limits_{|\xi|=1} h^2(x+(t-\tau)\xi,\ \tau)\,dS_\xi = \int\limits_{|\xi|=1} (\xi_1^2+\xi_2^2+\ldots+\xi_n^2)h^2(x+(t-\tau)\xi,\ \tau)\,dS_\xi$$

$$= \int\limits_{|\xi|=1} \sum_{i=1}^{n} \xi_i(\xi_i h^2)(x+(t-\tau)\xi,\ \tau)\,dS_\xi$$

$$= \int\limits_{|\xi|\leqq 1} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}(\xi_i h^2)(x+(t-\tau)\xi,\ \tau)\,d\xi = n \int\limits_{|\xi|\leqq 1} h^2(x+(t-\tau)\xi,\ \tau)\,d\xi$$

$$+2 \int\limits_{|\xi|\leqq 1} \sum_{i=1}^{n} \xi_i^2(t-\tau)h(x+(t-\tau)\xi,\ \tau)\,\partial_i h(x+(t-\tau)\xi,\ \tau)\,d\xi$$

$$\leqq n \int\limits_{|\xi|\leqq 1} h^2(x+(t-\tau)\xi,\ \tau)\,d\xi$$

$$+n(t-\tau) \int\limits_{|\xi|\leqq 1} h^2(x+(t-\tau)\xi,\ \tau)\,d\xi + \sum_{i=1}^{n} (t-\tau) \int\limits_{|\xi|\leqq 1} (\partial_i h)^2(x+(t-\tau)\xi,\ \tau)\,d\xi.$$

The proof is complete. ∎

3. Existence of the scattering operator

In this section we construct a global solution of the nonlinear wave equation

(2) $$\Box u = u_{tt} - \Delta_x u = F(u, \nabla u) \quad (x, t) \in R^n_x \times R_t.$$

Here u is a scalar function, $n \geqq 3$ is odd, $\nabla u = (\partial_t u, \nabla_x u)$ and F satisfies one of the hypotheses **(H1)**, **(H2)** and **(H3)**. Moreover, we obtain suitable "weighted decay estimate" for the solution of (2). Our main aim is to establish existence of the scattering operator for (2). One of the basic assumptions is that the initial data are sufficiently smooth and small.

Introduce the norm

$$\|u\|_{a,b} = \sup_{(x,t) \in R^{n+1}} [(1+|x|+|t|)^a (1+||x|-|t||)^b |u(x, t)|].$$

Denote by V the Banach space of all functions $u \in C^0(R^n_x \times R_t)$, such that

$$\|u\|_v = \|u\|_{1, p-n+1} < \infty.$$

Define by LF the operator

$$LF(u)(x, t) = \int_{-\infty}^{t} \sum_{v=0}^{(n-3)/2} \alpha_v (t-\tau)^{v+1}$$

$$\partial_t^v \left[\int_{|\xi|=1} F(u(x+(t-\tau)\xi, \tau), \nabla u(x+(t-\tau)\xi, \tau)) \, dS_\xi \right] d\tau,$$

where $\{\alpha_v\}$ are constants, such that $LF(u)(x, t)$ is the solution of linear wave equation $\Box v = F(u, \nabla u)$ with vanishing initial data at $t = -\infty$.

Let $u_0 = u$, $u_i = \dfrac{\partial u}{\partial x_i}$ for $i = 1, 2, \ldots, n$ and let $u_{n+1} = \dfrac{\partial u}{\partial t}$.

Performing the differentiation, we derive

$$LF(u)(x, t) = \sum_{v, s, \alpha, \{\beta^0_{i,j}\}} a_v \int_{-\infty}^{t} \left\{ (t-\tau)^{v+1} x \int_{|\xi|=1} \left[\frac{\partial^s F}{\partial y^\alpha} (u(x+(t-\tau)\xi, \tau), \nabla u(x+(t-\tau)\xi, \tau)) \right.\right.$$

$$\left.\left. \times \prod_{i=0}^{n+1} \prod_{j=0}^{\alpha_i} \left[\frac{\partial^{\overset{0}{\gamma}_{i,j}} u_i}{\partial x^{\overset{0}{\beta}_{i,j}}} (x+(t-\tau)\xi, \tau) \right]^{c_{i,j}} \xi^{\overset{0}{\beta}_{i,j}} \right] dS_\xi \right\} d\tau.$$

In the above equality v varies over $\{0, 1, \ldots (n-3)/2\}$ and s belongs to $\{0, 1, \ldots, v\}$, $|\alpha| = s$. The numbers $\{\gamma^0_{i,j}\}$ are such that $\sum_{i,j} \gamma^0_{i,j} = v$ and just s of them is different from 0. For the multiindexes $\{\beta^0_{i,j}\}$ we have $|\beta^0_{i,j}| = \gamma^0_{i,j}$, and for $\{c_{i,j}\}$ we have $c_{i,j} = 0$ if $\gamma^0_{i,j} = 0$ and $c_{i,j} = 1$ if $\gamma^0_{i,j} > 0$.

Introduce the Banach space X of all functions u, such that $\partial_t^\alpha \partial_x^\gamma u \in C^0(R^n_x \times R_t)$ and $\|\partial_t^\alpha \partial_x^\gamma u\|_v \leqq \infty$ for $\alpha = 0$ or 1 and $0 \leqq \alpha + |\gamma| \leqq (n+3)/2$ and $X_\delta = \{u \in X / \|\partial_t^\alpha \partial_x^\gamma u\|_v \leqq \delta$ for $\alpha = 0$ or 1 and $0 \leqq \alpha + |\gamma| \leqq (n+3)/2\}$.

Our first result concerns with the existence of a solution to the integral equation

(3) $$u(x, t)=u_0^-(x, t)+LF(u)(x, t)$$

in the space X_δ. Here u_0^- is a solution to (1).

The key result is the following

Theorem 8. *Let $n>3$ be odd and let* **(H1)** *be fulfilled. Let u_0^- be the solution to* (1) *with initial data $f\in C^{n+2}$, $g\in C^{n+1}$, satisfying the estimates*

$$|\partial^{\alpha+\gamma}f(x)|\leqq\varepsilon(1+|x|)^{-p+n-2-|\alpha|},\quad |\partial^{\beta+\gamma}g(x)|\leqq\varepsilon(1+|x|)^{-p+n-3-|\beta|}$$

for any $x\in\mathrm{R}^n$, $0\leqq|\alpha|\leqq(n-1)/2$, $0\leqq|\beta|\leqq(n-3)/2$ and $0\leqq|\gamma|\leqq(n+3)/2$. Assume that $\partial^\alpha f(x)\in L^2(\mathrm{R}^n)$ for $|\alpha|=n+2$ and $\partial^\beta g(x)\in L^2(\mathrm{R}^n)$ for $|\beta|=n+1$.
Then there exists a real number ε_0, so that for any $\varepsilon\in(0, \varepsilon_0]$ the integral equation (3) *has a unique solution in X_δ for suitable chosen $\delta>0$.*

Proof. Define the space $Y_{\delta, R}$ of all functions u, such that $\partial_t^\alpha\partial_x^\gamma u\in C^0(\mathrm{R}_x^n\times\mathrm{R}_t)$ and $\|\partial_t^\alpha\partial_x^\gamma u\|_v\leqq\delta$ for $\alpha=0$ or 1 and $0\leqq\alpha+|\gamma|\leqq(n+3)/2$ and for any $t\in\mathrm{R}$, $\partial_t^\alpha\partial_x^\gamma u(t)\in L^2(\mathrm{R}_x^n)$, $\max_{t\in\mathrm{R}}\|\partial_t^\alpha\partial_x^\gamma u(t)\|_{L^2(\mathrm{R}_x^n)}\leqq R$ for $\alpha=0,1$ and $(n+5)/2\leqq\alpha+|\gamma|\leqq n+2$.

First we prove the implication "$u\in Y_{\delta, R}\Rightarrow LF(u)\in Y_{\delta, R}$".

Taking into account the expression for LF, we derive

$$\partial_x^a LF(u)(x, t)=\int_{-\infty}^{t}\int_{|\xi|=1}\Phi(x, t, \tau, \xi)\,dS_\xi\,d\tau,$$

where

$$\Phi(x, t, \tau, \xi)=\sum_{v,s,\alpha,\beta,m,\{\beta_{i,j}^k\}}a_v(t-\tau)^{v+1}$$

$$\times\frac{\partial^{(s+m)}F}{\partial y^{\alpha+\beta}}(u, \nabla u)(x+(t-\tau)\xi, \tau))\prod_{i=0}^{n+1}\prod_{j=0}^{\beta_i}\left[\frac{\partial^{\gamma_{i,j}^1}u_i}{\partial x^{\beta_{i,j}^1}}(x+(t-\tau)\xi, \tau)\right]^{d_{i,j}}$$

$$\times\prod_{i=0}^{n+1}\prod_{j=0}^{\alpha_i}\left[\frac{\partial^{\gamma_{i,j}^0+\gamma_{i,j}^2}u_i}{\partial x^{\beta_{i,j}^0+\beta_{i,j}^2}}(x+(t-\tau)\xi, \tau)\right]^{c_{i,j}}\xi^{\beta_{i,j}^0}.$$

Here v, s, α, $\{c_{i,j}\}$, $\{\beta_{i,j}^0\}$ and $\{\gamma_{i,j}^0\}$ vary as in the formula for $LF(u)$. For another indices we have corresponding domains. Namely m is integer in $[0, |\omega|]$, $|\beta|=m$, $\sum_{i,j,k}\beta_{i,j}^k=\omega$ and $\beta_{i,j}^k\geqq0$. For $\{\gamma_{i,j}^k\}$ we have $|\beta_{i,j}^k|=\gamma_{i,j}^k$, $\{d_{i,j}\}$ are constants such that $d_{i,j}=0$ if $\gamma_{i,j}^1=0$ and $d_{i,j}=1$ if $\gamma_{i,j}^1>0$.

Similar formulae we derive for $\partial_t^\alpha\partial_x^\gamma LF(u)$. By using the form of F, we conclude, that it is enough to estimate the integrals, in the representation

$$\int_{-\infty}^{t}(t-\tau)^{v+1}\left\{\int_{|\xi|=1}\prod_{i=0}^{n+1}(u_i)^{k_i}(x+(t-\tau)\xi, \tau)\prod_{j=1}^{m_i}(\partial^{\gamma_i^j}u_i)(x+(t-\tau)\xi, \tau)\,dS_\xi\right\}d\tau,$$

where $p=\sum_{i=0}^{n+1}(k_i+m_i)$, $k_i\geqq 0$, $m_i\geqq 0$ and $\sum_{i=0}^{n+1} m_i\leqq n$. Note, that if $m_i=0$, then $\prod_{j=1}^{m_i}(\partial^{\gamma_i^j}u_i)=1$. Set $r=|x|$,

$$I_1(r,\ t)=\int_{-\infty}^{t-1}(t-\tau)^{\nu+1}\left\{\int_{|\xi|=1}\left[\prod_{i=0}^{n+1}(u_i)^{k_i}\prod_{j=1}^{m_i}(\partial^{\gamma_i^j}u_i)\right](x+(t-\tau)\xi,\ \tau)\,dS_\xi\right\}d\tau,$$

$$I_2(r,\ t)=\int_{t-1}^{t}(t-\tau)^{\nu+1}\left\{\int_{|\xi|=1}\left[\prod_{i=0}^{n+1}(u_i)^{k_i}\prod_{j=1}^{m_i}(\partial^{\gamma_i^j}u_i)\right](x+(t-\tau)\xi,\ \tau)\,dS_\xi\right\}d\tau.$$

Define the numbers $\{h_i\}_{i=0}^{n+1}$ so that $h_i=0$ if $i=0$ and $h_i=1$ if $i=1,2,\dots,n+1$. Let the pair $(k,\ m)$ be such that $|\gamma_k^m|+h_k\geqq|\gamma_i^j|+h_i$ for any $(i,\ j)$. Note, that $\sum_{i,j}|\gamma_i^j|=\nu+|\omega|$, $\nu\leqq(n-3)/2$ and $|\omega|\leqq(n+3)/2$. Hence, for any $(i,\ j)\neq(k,\ m)$ the inequality $|\gamma_i^j|\leqq(n+1)/2$ is fulfilled. We consider two cases.

I) If $|\gamma_k^m|+h_k\leqq(n+3)/2$, then

$$(1+r+|t|)(1+|r-|t||)^{p-n+1}.I_2(r,\ t)$$
$$\leqq(1+r+|t|)(1+|r-|t||)^{p-n+1}\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}.\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right)$$
$$\times\int_{t-1}^{t}\int_{|\xi|=1}[(1+|x+(t-\tau)\xi|+|\tau|)^{-p}(1+||x+(t-\tau)\xi|-|\tau||)^{-p(p-n+1)}]\,dS_\xi\,d\tau.$$

In the case $|x|\leqq 2$ and $|t|\leqq 2$, by using the inequalities $1+r+|t|\leqq 5$ and $1+|r-|t||\leqq 3$, we get

$$\|I_2(r,\ t)\|_v\leqq C\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}.\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right).\tag{4}$$

In another case ($|x|\geqq 2$ or $|t|\geqq 2$), the inequality

$$(1+|x+(t-\tau)\xi|+|\tau|)^{-p}\leqq(1+r+|t|)^{-1}(1+|r-|t||)^{-p+n-1}$$

gives (4).

Now let us estimate the term $I_1(x,\ t)$.

$$I_1(r,\ t)\leqq C_1\int_{-\infty}^{t-1}(t-\tau)^{\nu+2-n}\int_{|y-x|=t-\tau}(1+|y|+|\tau|)^{-p}$$
$$\times(1+||y|-|\tau||)^{-p(p-n+1)}\,dS_\xi\,d\tau\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}.\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right)$$
$$\leqq C_2r^{2-n}\int_{-\infty}^{t}\left(\int_{|r-t+s|}^{r+t-s}\lambda(\lambda^2-|r-t+s|^2)^{\frac{n-3}{2}}((r+t-s)^2-\lambda^2)^{\frac{n-3}{2}}(1+\lambda+|s|)^{-p}\right.$$
$$\left.(1+|\lambda-|s||)^{-p(p-n+1)}\,d\lambda\right)ds\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}.\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right)$$
$$\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-(p-n+1)}\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}.\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right).$$

Here we have used Lemma 1. Hence,

$$\|I_1\|_v \leqq C \prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\overset{j}{\gamma_i}} u_i\|_v\right).$$

II) $|\overset{m}{\gamma_k}|+h_k \geqq (n+5)/2$.

Denote by $(k, m)\prod_{i,j} a_{i,j}$ the product of $\{a_{i,j}\}_{i,j}$ without $a_{k,m}$. Then

$$(1+r+|t|)(1+|r-|t||)^{p-n+1} . I_2(r, t) \leqq C_1(1+r+|t|)(1+|r-|t||)^{p-n+1}$$

$$\times \int_{t-1}^{t} \int_{|\xi|=1} |\partial^{\overset{m}{\gamma_k}} u_k(x+(t-\tau)\xi, \tau)|(1+|x+(t-\tau)\xi|+|\tau|)^{1-p}$$

$$\times(1+||x+(t-\tau)\xi|-|\tau||)^{-p(p-n+1)} \mathrm{d}S_\xi \mathrm{d}\tau . (k, m) \prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i} \prod_{j=1}^{m_i} \|\partial^{\overset{j}{\gamma_i}} u_i\|_v\right).$$

As in the first case, we obtain

$$\|I_2(r, t)\|_v \leqq C_1(k, m) \prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\overset{j}{\gamma_i}} u_i\|_v\right)$$

$$\int_{t-1}^{t} \int_{|\xi|=1} |\partial^{\overset{m}{\gamma_k}} u_k(x+(t-\tau)\xi, \tau)| \mathrm{d}S_\xi \mathrm{d}\tau$$

$$\leqq C_2(k, m) \prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\overset{j}{\gamma_i}} u_i\|_v\right)$$

$$\int_{t-1}^{t}\left(\int_{|\xi|=1} |\partial^{\overset{m}{\gamma_k}} u_k(x+(t-\tau)\xi, \tau)|^2 \mathrm{d}S_\xi\right)^{1/2} \mathrm{d}\tau$$

$$\leqq C_3 \int_{t-1}^{t}\left[\left(\int_{|\xi|\leqq 1} |\partial^{\overset{m}{\gamma_k}} u_k(x+(t-\tau)\xi, \tau)|^2 \mathrm{d}\xi\right)^{1/2}\right.$$

$$\left.+\left(\int_{|\xi|\leqq 1} |\nabla_x \partial^{\overset{m}{\gamma_k}} u_k(x+(t-\tau)\xi, \tau)|^2 \mathrm{d}\xi\right)^{1/2}\right] \mathrm{d}\tau$$

$$(k, m) \prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i} \prod_{j=1}^{m_i} |\partial^{\overset{j}{\gamma_i}} u_i\|_v\right)$$

$$\leqq C_4\left(\max_{t\in\mathbf{R}} \|g^{\overset{m}{\gamma_k}} u_k(t)\|_{L^2(\mathbf{R}_x^n)} + \max_{t\in\mathbf{R}} \|\nabla_x \partial^{\overset{m}{\gamma_k}} u_k(t)\|_{L^2(\mathbf{R}_x^n)}\right)$$

$$(k, m) \prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\overset{j}{\gamma_i}} u_i\|_v\right).$$

It remains to estimate $I_1(r, t)$.

As in the preceding estimate, we derive

$$I_1(r,\ t)\leqq \int_{-\infty}^{t-1} (t-\tau)^{\nu+1}\left\{\int_{|\xi|=1}\left[(k,\ m)\prod_{i=0}^{n+1}(u_i)^{k_i}(x+(t-\tau)\xi,\ \tau)\right.\right.$$

$$\left.\left.\times\prod_{j=1}^{m_i}(\partial^{\gamma_i^j}u_i)(x+(t-\tau)\xi,\ \tau)\right]^2 dS_\xi\right\}^{\frac{1}{2}}\left(\int_{|\xi|=1}|\partial^{\gamma_k^m}u_k(x+(t-\tau)\xi,\ \tau)|^2 dS_\xi\right)^{\frac{1}{2}}d\tau$$

$$\leqq C_1\int_{-\infty}^{t-1}(t-\tau)^{\nu+1}\left\{\int_{|y-x|=t-\tau}(t-\tau)^{1-n}\left[(k,\ m)\prod_{i=0}^{n+1}(u_i)^{k_i}(y,\ \tau)\right.\right.$$

$$\leqq C(k,\ m)\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}\cdot\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right)$$

$$\times\left(\max_{t\in\mathbf{R}}\|\partial^{\gamma_k^m}u_k(t)\|_{L^2(\mathbf{R}_x^n)}+\max_{t\in\mathbf{R}}\|\nabla_x\partial^{\gamma_k^m}u_k(t)\|_{L^2(\mathbf{R}_x^n)}\right)$$

$$\times\int_{-\infty}^{t-1}\left\{(t-\tau)2(\nu+1)+1-2(n-1)\int_{|y-x|=t-\tau}(1+|y|+|\tau|)^{2(1-p)}\right.$$

$$\left.\times(1+||y|-|\tau||)^{2(1-p)(p-n+1)}dS_y\right\}^{1/2}d\tau\leqq C(1+r+|t|)^{-1}(1+|r-|t||)^{-p+n-1}$$

$$\times(k,\ m)\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}\cdot\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right)$$

$$\times\left(\max_{t\in\mathbf{R}}\|\partial^{\gamma_k^m}u_k(t)\|_{L^2(\mathbf{R}_x^n)}+\max_{t\in\mathbf{R}}\|\nabla_x\partial^{\gamma_k^m}u_k(t)\|_{L^2(\mathbf{R}_x^n)}\right).$$

Hence the estimate

$$\|\partial_t^\alpha\partial_x^\gamma LF(u)\|_v\leqq C\Sigma\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}\cdot\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right) \tag{5}$$

$$+C\Sigma(k,\ m)\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}\cdot\prod_{j=1}^{m_i}\|\partial^{\gamma_i^j}u_i\|_v\right)$$

$$\left(\max_{t\in\mathbf{R}}\|\partial^{\gamma_k^m}u_k(t)\|_{L^2(\mathbf{R}_x^n)}+\max_{t\in\mathbf{R}}\|\nabla_x\partial^{\gamma_k^m}u_k(t)\|_{L^2(\mathbf{R}_x^n)}\right)$$

holds.

In the first sum we have included the terms, such that $|\gamma_i^j|+h_i\leqq(n+3)/2$, for any $(i,\ j)$, while in the second sum we include all terms, such that $\max_i\max_j|\gamma_i^j|+h_i\geqq(n+5)/2$. Here $\alpha+|\gamma|\leqq(n+3)/2$ and $\alpha=0,1$ and $\alpha+|\gamma|\leqq(n+3)/2$.

Let us estimate the term $\max_{t\in\mathbf{R}}\|\partial_t^\alpha\partial_x^\gamma LF(u)(t)\|_{L^2(\mathbf{R}_x^n)}$ for $\alpha+|\gamma|\geqq(n+5)/2$ and $\alpha=0$ or 1. Without loss of generality we can suppose $\alpha=1$. The case $\alpha=0$ can be considered in a similar way.

Note, that $\partial^\gamma LF(u)$ is the solution of Cauchy problem for linear wave equation $\Box v = \partial^\gamma F(u, \nabla u)$ with vanishing initial data at $t = -\infty$.

An application of basic energy inequality gives

$$\|\partial_t^\alpha \partial_x^\gamma LF(u)(t)\|_{L^2(\mathbf{R}_x^n)} \leqq \|\nabla_{x,t} \partial_x^\gamma LF(u)(t)\|_{L^2(\mathbf{R}_x^n)}$$

$$\leqq \int_{-\infty}^{t} \|\partial_x^\gamma F(u, \nabla u)(\tau)\|_{L^2(\mathbf{R}_x^n)} d\tau$$

$$\leqq C \sum_{k_i, m_i, \gamma_i^j} \int_{-\infty}^{t} \left(\int_{x \in \mathbf{R}^n} \left[\prod_{i=0}^{n+1} (u_i)^{k_i}(x, \tau) \prod_{j=1}^{m_i} (\partial^{\gamma_i^j} u_i)(x, \tau) \right]^2 dx \right)^{1/2} d\tau.$$

Here $\{k_i, m_i, \gamma_i^j\}$ vary over a set, determined by $\sum_{i,j} \gamma_i^j \leqq \gamma$, $\sum_i m_i \leqq |\gamma|$ and $\sum_i (k_i + m_i) = \rho$. Denote by $I_{\{k_i, m_i, \gamma_i^j\}}$ the integral term

$$\int_{-\infty}^{t} \left(\int_{x \in \mathbf{R}^n} \left[\prod_{i=0}^{n+1} (u_i)^{k_i}(x, \tau) \prod_{j=1}^{m_i} (\partial^{\gamma_i^j} u_i)(x, \tau) \right]^2 dx \right)^{1/2} d\tau.$$

Again we consider two cases

1) $\max_i \max_j |\gamma_i^j| + h_i \leqq (n+3)/2$.

Then

$$I_{\{k_i, m_i, \gamma_i^j\}} \leqq C_1 \prod_{i=0} \left(\|u_i\|_v^{k_i} \cdot \prod_{j=1} \|\partial^{\gamma_i^j} u_i\|_v \right)$$

$$\times \int_{-\infty}^{t} \left(\int_{x \in \mathbf{R}^n} (1 + |x| + |\tau|)^{-2p} dx \right)^{1/2} d\tau \leqq C_2 \prod_{i=0}^{n+1} \left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v \right).$$

2) $\max_i \max_j |\gamma_i^j| + h_i \geqq (n+5)/2$.

Let (k, m) be such that $|\gamma_m^k| + h_k = \max_i \max_j |\gamma_i^j| + h_i$. In this case, as before, we have

$$I_{\{k_i, m_i, \gamma_i^j\}} \leqq C_1(k, m) \prod_{i=0}^{n+1} \left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v \right)$$

$$\times \int_{-\infty}^{t} \left(\int_{x \in \mathbf{R}^n} (1 + |x| + |\tau|)^{-2(p-1)} |\partial^{\gamma_k^m} u_k(x, \tau)|^2 dx \right)^{1/2} d\tau$$

$$\leqq C(k, m) \prod_{i=0}^{n+1} \left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v \right) \max_{t \in \mathbf{R}} \|\partial^{\gamma_k^m} u_k(t)\|_{L^2(\mathbf{R}_x^n)}.$$

Therefore,

$$\text{(6)} \qquad \max_{t \in \mathbf{R}} \|\partial_t^\alpha \partial_x^\gamma LF(u)(t)\|_{L^2(\mathbf{R}_x^n)} \leqq C \sum \prod_{i=0}^{n+1} \left(\|u_i\|_v^{k_i} \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v \right)$$

$$+ C \sum (k, m) \prod_{i=0}^{n+1} \left(\|u_i\|_v^{k_i} \cdot \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v \right) \max_{t \in \mathbf{R}} \|\partial^{\gamma_k^m} u_k(t)\|_{L^2(\mathbf{R}_x^n)}.$$

In the first sum we include all terms, for which $\max_i \max_j |\gamma_i^j| + h_i \leqq (n+3)/2$, and in the second sum we include the remaining terms. Taking into account the estimates (5) and (6), we deduce that if $u \in Y_{\delta, R}$, then $LF(u) \in Y_{\delta, R}$.

Assume now, that $u \in Y_{\delta, R}$ and $w \in Y_{\delta, R}$. Then we can estimate the difference $LF(u) - LF(w)$ and its derivatives.

Similarly to the preceding estimate (5) and (6), we obtain

$$\begin{aligned} \|\partial_t^\alpha \partial_x^\gamma (LF(u) - LF(w))\|_v \leqq C\Sigma\Bigg(\sum_l \Bigg(\prod_{i=0}^{l-1}\Bigg(\|u_i\|_v^{k_i} \\ \times \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v\Bigg) \cdot \Bigg(\|u_l - w_l\|_v (\|u_l\|_v^{k_l-1} + \|w_l\|_v^{k_l-1}) \prod_{j=1}^{m_l} \|\partial^{\gamma_l^j} w_l\|_v \\ + \sum_{s=1}^{m_l} \|u_l\|_v^{k_l} \prod_{j=1}^{s-1} \|\partial^{\gamma_l^j} u_l\|_v \|\partial^{\gamma_l^s}(u_l - w_l)\|_v \prod_{j=s+1}^{m_l} \|\partial^{\gamma_l^j} w_l\|_v\Bigg) \\ \times \prod_{i=l+1}^{n+1}\Bigg(\|w_i\|_v^{k_i} \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} w_i\|_v\Bigg)\Bigg)\Bigg). \end{aligned} \tag{7}$$

Here $\alpha = 0$ or 1, $0 \leqq \alpha + |\gamma| \leqq 2$, $\{k_i, m_i, \gamma_i^j\}$ are such that $\sum_{i,j} \gamma_i^j \leqq \gamma$, $\sum_i m_i \leqq |\gamma|$ and $\sum_i (k_i + m_i) = p$.

In the case $\alpha = 0$ or 1, and $3 \leqq \alpha + |\gamma| \leqq (n+3)/2$, as in the proof of the estimate (5), we derive

$$\begin{aligned} \|\partial_t^\alpha \partial_x^\gamma (LF(u) - LF(w))\|_v \leqq C\Sigma\Bigg(\sum_l \Bigg(\prod_{i=0}^{l-1}\Bigg(\|u_i\|_v^{k_i} \\ \times \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v\Bigg) \cdot \Bigg(\|u_l - w_l\|_v (\|u_l\|_v^{k_l-1} + \|w_l\|_v^{k_l-1}) \prod_{j=1}^{m_l} \|\partial^{\gamma_l^j} w_l\|_v \\ + \sum_{s=1}^{m_l} \|u_l\|_v^{k_l} \prod_{j=1}^{s-1} \|\partial^{\gamma_l^j} u_l\|_v \|\partial^{\gamma_l^s}(u_l - w_l)\|_v \prod_{j=s+1}^{m_l} \|\partial^{\gamma_l^j} w_l\|_v\Bigg) \\ \times \prod_{i=l+1}^{n+1}\Bigg(\|w_i\|_v^{k_i} \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} w_i\|_v\Bigg)\Bigg)\Bigg) + C\Sigma\Bigg((k, m)\sum_l \Bigg(\prod_{i=0}^{l-1}\Bigg(\|u_i\|_v^{k_i} \\ \times \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} u_i\|_v\Bigg) \cdot \Bigg(\|u_l - w_l\|_v (\|u_l\|_v^{k_l-1} + \|w_l\|_v^{k_l-1}) \prod_{j=1}^{m_l} \|\partial^{\gamma_l^j} w_l\|_v \\ + \sum_{s=1}^{m_l} \|u_l\|_v^{k_l} \prod_{j=1}^{s} \|\partial^{\gamma_l^j} u_l\|_v \|\partial^{\gamma_l^s}(u_l - w_l)\|_v \prod_{j=s}^{m_l} \|\partial^{\gamma_l^j} w_l\|_v\Bigg) \\ \times \prod_{i=l+1}^{n+1}\Bigg(\|w_i\|_v^{k_i} \prod_{j=1}^{m_i} \|\partial^{\gamma_i^j} w_i\|_v\Bigg)\Bigg)\Bigg) \\ \times \Bigg(\max_{t \in \mathbf{R}} \|\partial^{\gamma_k^m} w_k(t)\|_{L^2(\mathbf{R}_x^n)} + \max_{t \in \mathbf{R}} \|\nabla_x \partial^{\gamma_k^m} w_k(t)\|_{L^2(\mathbf{R}_x^n)}\Bigg) \end{aligned} \tag{8}$$

$$+C\Sigma(k,\ m)\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}\cdot\prod_{j=1}^{m_i}\|\partial^{\overset{j}{\gamma_i}}u_i\|_v\right)$$

$$\times\left(\max_{t\in\mathsf{R}}\|\partial^{\overset{m}{\gamma_k}}(u_k-w_k)(t)\|_{L^2(\mathsf{R}_x^n)}+\max_{t\in\mathsf{R}}\|\nabla_x\partial^{\overset{m}{\gamma_k}}(u_k-w_k)(t)\|_{L^2(\mathsf{R}_x^n)}\right).$$

If $n+2\geqq\alpha+|\gamma|\geqq(n+5)/2$ and $\alpha=0$ or 1, then

$$\max_{t\in\mathsf{R}}\|\partial_t^\alpha\partial_x^\gamma(LF(u)-LF(w))(t)\|_{L^2(\mathsf{R}_x^n)}\leqq C\Sigma\left(\sum_l\left(\prod_{i=0}^{l-1}\left(\|u_i\|_v^{k_i}\right.\right.\right.\tag{9}$$

$$\left.\times\prod_{j=1}^{m_i}\|\partial^{\overset{j}{\gamma_i}}u_i\|_v\right)\cdot\left(\|u_l-w_l\|_v(\|u_l\|_v^{k_l-1}+\|w_l\|_v^{k_l-1})\prod_{j=1}^{m_l}\|\partial^{\overset{j}{\gamma_l}}w_l\|_v\right.$$

$$\left.+\sum_{s=1}^{m_l}\|u_l\|_v^{k_l}\prod_{j=1}^{s}\|\partial^{\overset{j}{\gamma_l}}u_l\|_v\|\partial^{\overset{s}{\gamma_l}}(u_l-w_l)\|_v\prod_{j=s}^{m_l}\|\partial^{\overset{j}{\gamma_l}}w_l\|_v\right)$$

$$\left.\left.\left.\times\prod_{i=l+1}^{n+1}\left(\|w_i\|_v^{k_i}\prod_{j=1}^{m_i}\|\partial^{\overset{j}{\gamma_i}}w_i\|_v\right)\right)\right)+C\Sigma\left((k,\ m)\sum_l\left(\prod_{i=0}^{l-1}\left(\|u_i\|_v^{k_i}\right.\right.\right.$$

$$\left.\times\prod_{j=1}^{m_i}\|\partial^{\overset{j}{\gamma_i}}u_i\|_v\right)\cdot\left(\|u_l-w_l\|_v(\|u_l\|_v^{k_l-1}+\|w_l\|_v^{k_l-1})\prod_{j=1}^{m_l}\|\partial^{\overset{j}{\gamma_l}}w_l\|_v\right.$$

$$\left.+\sum_{s=1}^{m_l}\|u_l\|_v^{k_l}\prod_{j=1}^{s}\|\partial^{\overset{j}{\gamma_l}}u_l\|_v\|\partial^{\overset{s}{\gamma_l}}(u_l-w_l)\|_v\prod_{j=s}^{m_l}\|\partial^{\overset{j}{\gamma_l}}w_l\|_v\right)$$

$$\left.\left.\left.\times\prod_{i=l+1}^{n+1}\left(\|w_i\|_v^{k_i}\prod_{j=1}^{m_i}\|\partial^{\overset{j}{\gamma_i}}w_i\|_v\right)\right)\right)\cdot\max_{t\in\mathsf{R}}\|\partial^{\overset{m}{\gamma_k}}w_k(t)\|_{L^2(\mathsf{R}_x^n)}$$

$$+C\Sigma(k,\ m)\prod_{i=0}^{n+1}\left(\|u_i\|_v^{k_i}\cdot\prod_{j=1}^{m_i}\|\partial^{\overset{j}{\gamma_i}}u_i\|_v\right)\max_{t\in\mathsf{R}}\|\partial^{\overset{m}{\gamma_k}}(u_k-w_k)(t)\|_{L^2(\mathsf{R}_x^n)}.$$

Now, we consider the sequence $\{u_m\}$ defined by the equalities $u_0=u_0^-$ and $u_{m+1}=u_0^-+LF(u_m)$. We shall use the well-known energy equality:

$$\|\nabla_{x,t}v(t_1)\|_{L^2(\mathsf{R}_x^n)}=\|\nabla_{x,t}v(t_2)\|_{L^2(\mathsf{R}_x^n)},$$

where v is a solution of the free wave equation $\square u=0$.

Taking in mind the assumption of f and g and above conservation law, we conclude, that $\partial_t^\alpha\partial_x^\gamma u_0^-(t)\in L^2(\mathsf{R}_x^n)$ for any $t\in\mathsf{R}$, $\alpha=0,1$ and $(n+5)/2\leqq\alpha+|\gamma|\leqq n+2$. Let $M\in\mathsf{R}$ be such that $\|\partial_t^\alpha\partial_x^\gamma u_0^-(t)\|_{L^2(\mathsf{R}_x^n)}\leqq M$. Denote by Y_δ the space $Y_{\delta,M}$. Then exploiting Lemma 6, we deduce $u_0^-\in Y_\delta$ for sufficiently small $\varepsilon>0$. Hence $u_m\in Y_\delta$. Denote by N the maximal number of terms in the right-hand side of the inequalities (7), (8) and (9). Let δ be chosen so that $C\delta^{p-1}M\leqq\rho N^{-1}$ and $C\delta^p\leqq\rho N^{-1}$ hold for some $\rho<1$. Applying the inequalities $\|\partial_t^\alpha\partial_x^\gamma u_m(t)\|_v\leqq\delta$ if $0\leqq\alpha+|\gamma|\leqq(n+3)/2$ and $\alpha=0,1$ and $\max_{t\in\mathsf{R}}\|\partial_t^\alpha\partial_x^\gamma u_m(t)\|_{L^2(\mathsf{R}_x^n)}\leqq M$ for $n+2\geqq\alpha+|\gamma|\geqq(n+3)/2$ and $\alpha=0,1$, we obtain

$$\max\left(\max_{\alpha,\beta} \|\partial_t^\alpha \partial_x^\beta (u_{m+1} - u_m)\|_v, \max_{\alpha,\gamma,t} \|\partial_t^\alpha \partial_x^\gamma (u_{m+1} - u_m)(t)\|_{L^2(\mathbb{R}_x^n)}\right)$$

$$\leqq \rho \max\left(\max_{\alpha,\beta} \|\partial_t^\alpha \partial_x^\beta (u_m - u_{m-1})\|_v, \max_{\alpha,\gamma,t} \|\partial_t^\alpha \partial_x^\gamma (u_m - u_{m-1})(t)\|_{L^2(\mathbb{R}_x^n)}\right),$$

where $\alpha=0,1$, $0 \leqq \alpha+|\beta| \leqq (n+3)/2$, $n+2 \geqq \alpha+|\gamma| \geqq (n+3)/2$. By induction with respect to m we derive $\|\partial_t^\alpha \partial_x^\beta (u_m - u_{m-1})\|_v \leqq C\rho^m$. Hence $\{u_m\}$ tend to u in X_δ and u is a solution of the integral equation (3). The proof of the Theorem is complete.∎

Remark that the solution, which we have constructed in Theorem 8, is obviously twice continuously differentiable to t. Indeed, the fact $u \in X$ shows that u is continuously differentiable. We differentiate formally twice and use the facts $\partial_t^\alpha \partial_x^\beta u \in V$ for $\alpha=0,1$ and $0 \leqq \alpha+|\beta| \leqq (n+3)/2$. So we obtain that the function in the right-hand side in differentiated equality is continuous. Then u is twice continuously differentiable. This implies the following

Theorem 9. *Let f and g satisfy the assumptions of Theorem 8 and let u be a solution to the integral equation (3). Then u is the unique global classical solution of the wave equation $\square u = F(u, \nabla u)$.*

Note that this method gives a possibility to obtain similar result in the case, when F satisfies the hypothesis **(H2)**. For initial data $f \in C^n(\mathbb{R}^n)$, $g \in C^{n-1}(\mathbb{R}^n)$, we have $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for $|\alpha|=n$, $\partial^\beta g \in L^2(\mathbb{R}^n)$ for $|\beta|=n-1$ and the following inequalities $|\partial^{\alpha+\gamma} f(x)| \leqq \varepsilon(1+|x|)^{-p+n-2-|\alpha|}$, $|\partial^{\beta+\gamma} g(x)| \leqq \varepsilon(1+|x|)^{-p+n-3-|\beta|}$ hold for any $x \in \mathbb{R}^n$, $0 \leqq |\alpha| \leqq (n-1)/2$, $0 \leqq |\beta| \leqq (n-3)/2$, $0 \leqq |\gamma| \leqq (n+1)/2$ and $n>3$ is odd.

In the case $n=3$, when F satisfies the hypothesis **(H3)**, we derive the following

Theorem 10. *Let u_0^- be the solution (1) with intial data $f \in C^5$ and $g \in C^4$. Assume the estimates*

$$|\partial^{\alpha+\gamma} f(x)| \leqq \varepsilon(1+|x|)^{-q/2-|\alpha|}, \quad |\partial^\gamma g(x)| \leqq \varepsilon(1+|x|)^{-q/2-1}$$

for any $x \in \mathbb{R}^n$, $|\alpha|=0,1$ and $|\gamma|=0, 1, 2, 3$ fulfilled. Moreover, $\partial^\alpha f(x) \in L^2(\mathbb{R}^n)$ for $|\alpha|=5$ and $\partial^\beta g(x) \in L^2(\mathbb{R}^n)$ for $|\beta|=4$. Then there exists a real number ε_0, so that for any $\varepsilon \in (0, \varepsilon_0]$ the integral unique equation (2) has a solution in X_δ for suitable chosen $\delta>0$, which is also a global classical solution to the equation (2).

For the proof of this theorem, we need to deal with the space W instead of the space V, where

$$W = \{u \in C^0(\mathbb{R}^n) / \|u\|_w = \|u\|_{1/2, q-1} < \infty\}.$$

The details of the proof are similar to Theorem 8.

Similar arguments give possibility to prove the following

Proposition 11. *The Cauchy problem*

$$\square u = F(u, \nabla u),$$
$$u(x, 0) = f,$$
$$u_t(x, 0) = g$$

has a unique global solution, if the initial data satisfy the assumptions in Theorem 8 (*in the case* $n=3$ *the assumptions Theorem* 10).

Introduce the energy norm $\|.\|_e$, defined for a function $v(x, t)$ by the equality:

$$\|v(x, t)\|_e = \|\nabla_{t,x} v(t)\|_{L^2(\mathbf{R}^n_x)} = \left(\|\nabla_x v(t)\|^2_{L^2(\mathbf{R}^n_x)} + \|u_t(t)\|^2_{L^2(\mathbf{R}^n_x)} \right)^{1/2}.$$

Lemma 12. *Suppose the assumptions of Theorem* 8 *fulfilled. Let* u *be the solution to* (2) *and let* u_0^- *be the solution to* (1). *Then*

$$\|(u-u_0^-)(t)\|_e \leqq (1+|t|)^{-p+n-1} \text{ if } t<0. \tag{10}$$

Proof. According to the basic energy inequality, we have

$$\|(u-u_0^-)(t)\|_e \leqq \int_{-\infty}^{t} \|F(u, \nabla u)(\tau)\|_{L^2(\mathbf{R}^n_x)} d\tau.$$

It is necessary to estimate the last integral. The domain of integration will be divided into five parts:

1)
$$I_1(x, t) = \int_{-\infty}^{t} \left(\int_{|x|\leqq|\tau|/2} |F(u, \nabla u)(x, \tau)|^2 dx \right)^{1/2} d\tau$$
$$\leqq C_1 \int_{-\infty}^{t} (1-\tau)^{-p(p-n+2)} \left(\int_{|x|\leqq|\tau|/2} dx \right)^{1/2} d\tau$$
$$\leqq C_2 (1+|t|)^{-p(p-n+2)+(n+1)/2} \leqq C(1+|t|)^{-p+n-1}.$$

2)
$$\int_{-\infty}^{t} \left(\int_{|\tau|/2\leqq|x|\leqq|\tau|-1} |F(u, \nabla u)(x, \tau)|^2 dx \right)^{1/2} d\tau$$
$$\leqq C_1 \int_{-\infty}^{t} (1+|\tau|)^{-p} \left(\int_{|\tau|/2}^{|\tau|-1} r^{n-1} (1+|r-|\tau||)^{-2p(p-n+1)} dr \right)^{1/2} d\tau$$
$$\leqq C_2 \int_{-\infty}^{t} |\tau|^{(n-1)/2} (1+|\tau|)^{-p} \left(\int_{|\tau|/2}^{|\tau|-1} (1+|r-|\tau||)^{-2p(p-n+1)} dr \right)^{1/2} d\tau$$
$$\leqq C_3 (1+|t|)^{-p(n+2)/2} \leqq C(1+|t|)^{-p+n-1}.$$

3)
$$\int_{-\infty}^{t} \left(\int_{|\tau|-1\leqq|x|\leqq|\tau|+1} |F(u, \nabla u)(x, \tau)|^2 dx \right)^{1/2} d\tau$$
$$\leqq C_1 \int_{-\infty}^{t} (1+|\tau|)^{-p} \left(\int_{|\tau|-1\leqq|x|\leqq|\tau|+1} dx \right)^{1/2} d\tau$$
$$\leqq C_2 (1+|t|)^{-p+(n+1)/2} \leqq C(1+|t|)^{-p+n-1}.$$

4) The case $|\tau|+1 \leqq |x| \leqq 2|\tau|$ can be treated as in 2).

5)
$$\int_{-\infty}^{t}\left(\int_{|x|\geq 2|\tau|}|F(u,\ \nabla u)(x,\ \tau)|^2\,dx\right)^{1/2}d\tau$$
$$\leqq C_1\int_{-\infty}^{t}\left(\int_{|x|\geq 2|\tau|}(1+|x|)^{-2p(p-n+2)}\,dx\right)^{1/2}d\tau$$
$$=C_1\int_{-\infty}^{t}\left(\int_{2|\tau|}^{+\infty}\sigma^{n-1}(1+\sigma)^{-2p(p-n+2)}\,d\sigma\right)^{1/2}d\tau$$
$$\leqq C_2\int_{-\infty}^{t}(1+|\tau|)^{-p(p-n+2)+(n-1)/2}\,d\tau\leqq C_3(1+|t|)^{-p(p-n+2)+(n+1)/2}$$
$$\leqq C(1+|t|)^{-p+n-1}.$$

We used the fact $u\in X_\delta$ and $t<0$. This completes the proof.∎

Analogously we obtain the following

Lemma 13. *Let assumptions of Theorem* 10 *hold. Then*

(11) $$\|(u-u_0^-)(t)\|_e\leqq(1+|t|)^{-(p-4)/2}\text{ for }t<0,$$

where u and u_0^- are defined as in previous lemma.

Finally, taking advantage of the fact $u\in X$ we define the function

$$u_0^+(x,\ t)=u(x,\ t)-\overline{LF}(u)(x,\ t),$$

where

$$\overline{LF}(u)(x,\ t)=\int_{t}^{+\infty}\sum_{\nu=0}^{(n-3)/2}(\nu+1)\,a_\nu(\tau-t)^{\nu+1}$$
$$\partial_t^\nu\left[\int_{|\xi|=1}F(u(x+(\tau-t)\xi,\ \tau),\ \nabla u(x+(\tau-t)\xi,\ \tau))\,dS_\xi\right]d\tau.$$

Then, obviously u_0^+ is a classical solution to the free wave equation and

(12) $$\|(u-u_0^+)(t)\|_e\leqq C(1+|t|)^{-B}\text{ for }t\geqq 0,$$

where $B=\begin{cases}p-n+1 & \text{if } n>3,\\ (p-4)/2 & \text{if } n=3.\end{cases}$

Denote by $W_{k,N}$, where k and N are nonnegative integer, the space of all functions $h\in C^N(\mathbf{R}^n)$, such that

$$|h|_{k,N}=\sum_{|\alpha|\leqq N}\sup_{x\in\mathbf{R}^n}[(1+|x|)^{k+|\alpha|}|\partial^\alpha h(x)|]<\infty.$$

As an immediate consequence of (10), (11) and (12) we obtain our main result:

Theorem 14. *There exists a sufficiently small neighborhood U of zero in $W_{p-n+2,n+2}\times W_{p-n+3,n+1}$ (respectively in the case $n=3$ $W_{q/2,5}\times W_{q/2+1,4}$) such that the scattering operator*

$$S:(f,\ g)\in U\to(u_0^+(x,\ 0),\ \partial_t u_0^+(x,\ 0))\in W_{p-n+2,(n+3)/2}\times W_{p-n+3,(n+1)/2}$$

(respectively $W_{q-1/2,3} \times W_{q-1/2+1,2}$) exists in U in the sense of the energy norm.

Note, that the approach of this work allows to obtain similar results in the case, when the coefficients of the function F and their derivatives up to order n are uniformly bounded.

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Laboratory of Differential Equations
Institute of Mathematics of
Bulgarian Academy of Sciences
1090 Sofia, P.O. Box 373,
BULGARIA

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