Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 2

On Submanifolds of Sasakian Manifolds

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Any k-dimensional, $k \geq 4$, totally-umbilical submanifold tangent to the contact vector of a Sasakian space-form is totally-geodesic. A complete simply-connected extrinsic sphere tangent to the contact vector of a Sasakian manifold and having a flat normal connection is isometric to a standard sphere. Any submanifold tangent to the contact vector of a Sasakian space-form and having a parallel second fundamental form is either invariant (and thus totally-geodesic) or anti-invariant.

1. Introduction

Totally-umbilical submanifolds of complex space-forms have been classified by B. Y. Chen & K. Ogiue, cf. Th. 1 in [6], p. 225. Their classification theorem is based on the earlier (cf. prop. 3.1. in [7], p. 260) observation that submanifolds (of a complex space-form) invariant under the curvature transformation are either holomorphic or totally-real. We extend these ideas to submanifolds of Sasakian space-forms and obtain the following

Theorem 1. *Let M^{2m+1} be an odd dimensional totally-umbilical submanifold of a Sasakian space-form $M^{2n+1}(c)$, $1<m<n$. If M^{2m+1} is tangent to the contact vector ξ of $M^{2n+1}(c)$ then M^{2m+1} is a Sasakian space-form immersed in $M^{2n+1}(c)$ as a totally-geodesic submanifold.*

In contrast with prop. 3.1. in [7], p. 260, an invariant (in the sense of K. Ogiue [13], p. 389) submanifold M^{2m+1} of a Sasakian space-form is always φ-invariant (in the sense of K. Yano & M. Kon [14], p. 48) provided that $m>1$ and M^{2m+1} is tangent to ξ. Consequently, case b) in Th. 1 of [6], p. 225, has no analogue in contact geometry. In particular, cf. our Th. 1, there do not exist extrinsic spheres M^{2m+1}, $m>1$, tangent to the structure vector ξ of a Sasakian space-form.

As to extrinsic spheres (i.e. totally-umbilical submanifolds whose non-zero mean curvature vector H is parallel in the normal bundle) we obtain the following

Theorem 2. *Let M^{2m+1} be a complete simply-connected extrinsic sphere tangent to the contact vector ξ of a Sasakian manifold M^{2n+1}, $1 \leq m<n$. If M^{2m+1}*

has a flat normal connection then M^{2m+1} is isometric to the standard sphere $S^{2m+1}(\frac{1}{c})$ of radius $\frac{1}{c}$, where $c=\|H\|$.

Cf. B. Y. Chen [3], p. 327, a Riemannian manifold M^k, $\dim(M^k)=k$, is sufficiently curved if for any $x\in M^k$ the maximal linear subspace V of the tangent space $T_x(M^k)$ with $\bar{R}_x(u, v)=0$ for all $u, v\in V$, (here $\bar{R}$ denotes the curvature of M^k), has real algebraic dimension less than $k-2$. We show that if a submanifold M^{2m+1} bearing the hypothesis of our Th. 2 is considered, then M^{2n+1} is a subject to

$$\bar{R}(X, Y)=0, \tag{1.1}$$

for all tangent vector fields X, Y on M^{2m+1}, i.e. if M^{2m+1} has codimension two (i.e. $m=n-1$) then the ambient space M^{2n+1} is not sufficiently curved. Note that Sasakian space-forms $M^{2n+1}(c)$ do not verify (1.1).

Theorem 3. *Let M^m be a real m-dimensional submanifold of the Sasakian manifold M^{2n+1}. If M^m is tangent to the contact vector of M^{2n+1} and has a parallel second fundamental form then M^m is a contact Cauchy-Riemann submanifold.*

In particular, any extrinsic sphere of M^{2n+1} is a contact C.R. submanifold. Also our th. 3 leads to the following

Corollary. *Let M^m be a real m-dimensional submanifold of the Sasakian space-form $M^{2n+1}(c)$. Suppose M^m is tangent to ξ and $\nabla h=0$. Then either M^m is invariant (and thus totally geodesic) or M^m is anti-invariant.*

2. Notations, conventions and basic formulae

Let M^{2n+1} be a real $(2n+1)$-dimensional differentiable manifold. An almost contact metrical (a.ct.m.) structure $(\varphi, \xi, \bar{\eta}, \bar{g})$ on M^{2n+1} consists of a (1,1)-tensor field φ, a tangent vector field ξ, a differential 1-form $\bar{\eta}$, and a Riemannian metric $\bar{g}$ such that the following relations hold:

$$\begin{gathered}\varphi^2=-I+\bar{\eta}\otimes\xi\\ \bar{\eta}\circ\varphi=0,\quad \varphi(\xi)=0\\ \bar{\eta}(\xi)=1,\quad \xi=\#\,\bar{\eta}\\ \bar{g}(\varphi X, \varphi Y)=\bar{g}(X, Y)-\bar{\eta}(X)\bar{\eta}(Y)\end{gathered} \tag{2.1}$$

for any $X, Y\in\mathfrak{X}(M^{2n+1})$. We denote by $C^\infty(M^{2n+1})$ the ring of all IR-valued differentiable functions on M^{2n+1} and by $\mathfrak{X}(M^{2n+1})$ the $C^\infty(M^{2n+1})$-module of all tangent vector fields on M^{2n+1}. Also $\#$ indicates raising of indices by $\bar{g}$. An a.ct.m. structure is normal if $N^{(1)}=0$, where $N^{(1)}=[\varphi, \varphi]+2(d\bar{\eta})\otimes\xi$, while $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ. See D. E. Blair [1], p. 48. The fundamental 2-form $\bar{\Phi}$ of an a.ct.m. manifold M^{2n+1} is given by $\bar{\Phi}(X, Y)=\bar{g}(X, \varphi Y)$. A manifold M^{2n+1} carrying a normal a.ct.m. structure is termed

Sasakian if $\bar{\Phi}=d\bar{\eta}$. Let $\bar{\nabla}$ be the Riemannian connection of $\bar{g}$. If M^{2n+1} is a Sasakian manifold then

$$(\bar{\nabla}_X \varphi)Y=-\bar{g}(X, Y)\xi+\bar{\eta}(Y)X \tag{2.2}$$

for all X, $Y\in\mathfrak{X}(M^{2n+1})$. Cf. e. g. [14], p. 10. The curvature $\bar{R}$ of a Sasakian space-form $M^{2n+1}(c)$, (cf. [1], p. 98, for definitions), is given by:

$$\begin{aligned}\bar{R}(X, Y)Z=&\frac{1}{4}(c+3)(\bar{g}(Y, Z)X-\bar{g}(X, Z)Y)\\ &-\frac{1}{4}(c-1)(\bar{\eta}(Y)\bar{\eta}(Z)X-\bar{\eta}(X)\bar{\eta}(Z)Y+\bar{g}(Y, Z)\bar{\eta}(X)\xi-\bar{g}(X, Z)\bar{\eta}(Y)\xi\\ &-\bar{g}(\varphi Y, Z)\varphi X+\bar{g}(\varphi X, Z)\varphi Y+2\bar{g}(\varphi X, \varphi Y)\varphi Z).\end{aligned} \tag{2.3}$$

Let M^k be a submanifold of the Sasakian manifold M^{2n+1}. Let $g=i^*\bar{g}$ be the induced metric, where $i: M^k \to M^{2n+1}$ denotes the canonical inclusion. We recall the Gauss and Weingarten formulae:

$$\begin{aligned}\bar{\nabla}_X Y&=\nabla_X Y+h(X, Y)\\ \bar{\nabla}_X N&=-A_N X+\nabla^{\perp}_X N\end{aligned} \tag{2.4}$$

for any X, $Y\in\mathfrak{X}(M^k)$, respectively any cross-section N in the normal bundle of i. Here ∇, h, A_N and $\nabla^{\perp}$ denote respectively the induced connection, the second fundamental form, the Weingarten operator (associated with the normal section N), and the normal connection. Let $H=\frac{1}{k}\mathrm{Trace}(h)$ be the mean curvature vector of M^k in M^{2n+1}. In §3 to §5 we are concerned with totally-umbilical (i. e. $h=g\otimes H$) submanifolds of Sasakian manifolds. A submanifold M^k of a Sasakian manifold is called φ-invariant if

$$\varphi_x(T_x(M^k))\subseteq T_x(M^k)$$

for any $x\in M^k$. Also M^k is said to be invariant (under the curvature transformation) if for any $x\in M^k$ and any u, $v\in T_x(M^k)$ the tangent space $T_x(M^k)$ is invariant under the transformation $\bar{R}_x(u, v): T_x(M^{2n+1})\to T_x(M^{2n+1})$, i. e.

$$\bar{R}_x(u, v)(T_x(M^k))\subseteq T_x(M^k).$$

Cf. K. Ogiue [13]. A submanifold M^k is termed extrinsic sphere if $h=g\otimes H$, $H\neq 0$, $\nabla^{\perp}H=0$.

3. Classifying the totally-umbilical submanifolds in Sasakian space-forms

Let M^k be a totally-umbilical submanifold of the Sasakian space form $M^{2n+1}(c)$. Let $E\to M^k$ be the normal bundle of $i: M^k\to M^{2n+1}(c)$. We denote by $\tan_x$, nor_x the natural projections associated with the direct sum decomposition:

$$T_x(M^{2n+1}(c)) = T_x(M^k) \oplus E_x, \quad x \in M^k.$$

Since $h = g \otimes H$, the covariant derivative of h is given by

$$(\nabla_X h)(Y, Z) = g(Y, Z)\nabla^{\perp}_X H \tag{3.1}$$

such that the Codazzi equation, i.e. eq. (2.9) in [2], p. 46 becomes

$$\operatorname{nor}(\bar{R}(X, Y)Z) = g(Y, Z)\nabla^{\perp}_X H - g(X, Z)\nabla^{\perp}_Y H \tag{3.2}$$

for any $X, Y, Z \in \mathfrak{X}(M^k)$. Let $X \in \mathfrak{X}(M^k)$ be arbitrary. Suppose from now on that $k \geqq 3$. If this is the case, then we may choose $Y \in \mathfrak{X}(M^k)$ such that $\|Y\| = 1$, $g(X, Y) = 0$, $\bar{g}(\varphi X, Y) = 0$. Therefore, by our (3.2), it follows:

$$\operatorname{nor}(\bar{R}(X, Y)Y) = \nabla^{\perp}_X H. \tag{3.3}$$

Let us put $\xi = \tan(\bar{\xi})$, $\xi^{\perp} = \operatorname{nor}(\bar{\xi})$. Also we define a vector valued differential 1-form F on M^k by setting $FX = \operatorname{nor}(\varphi X)$, $X \in \mathfrak{X}(M^k)$. As a consequence of (2.3) one has

$$\operatorname{nor}(\bar{R}(X, Y)Z) = \frac{1}{4}(c-1)(2\eta(X)\eta(Y)FY - \eta(X)\xi^{\perp}), \tag{3.4}$$

where $\eta = i^* \bar{\eta}$. One obtains the following partial result:

Proposition 3.1. *A totally-umbilical submanifold M^k tangent to the contact vector $\bar{\xi}$ of a Sasakian space-form is either totally-geodesic or an extrinsic sphere, provided that $k \geqq 4$.*

Proof. If $k \geqq 4$ we may choose Y from the very beginning to be orthogonal on ξ. Moreover (3.3) – (3.4) and $\xi^{\perp} = 0$ lead to

$$\nabla^{\perp}_X H = 0. \tag{3.5}$$

Now the two situations in our prop. 3.1. correspond to the cases $H = 0$ and $H \neq 0$, respectively. Q. E. D.

To prove our Th. 1, we shall show that the second case in our Prop. 3.1. actually does not occur.

4. Proof of Theorem 1

Combining (3.2), (3.5) we obtain

$$\operatorname{nor}(\bar{R}(X, Y)Z) = 0 \tag{4.1}$$

for any $X, Y, Z \in \mathfrak{X}(M^k)$, i.e. M^k follows to be invariant (under the curvature transformation). We need the following:

Proposition 4.1. *Let M^k be an invariant submanifold, $k \geqq 2$, of a Sasakian space-form $M^{2n+1}(c)$. Then M^k is φ-invariant.*

Proof. Using (2.3) and (4.1), (i.e. the invariance assumption) one shows that $\bar{R}(X, Y)X \in \mathfrak{X}(M^k)$, i.e. $(2\bar{g}(\varphi X, \varphi Y) - \Phi(X, Y))\varphi X$ is tangent to M^k. Here

$\Phi = i^*\bar{\Phi}$. Let $x \in M^k$ be a fixed point of the submanifold. Then $(2\langle \varphi_x u, \varphi_x v\rangle - \Phi_x(u, v))\varphi_x u$ belongs to the tangent space $T_x(M^k)$, for any $u, v \in T_x(M^k)$. Here $\bar{g}_x = \langle , \rangle$. The rest of the proof is by contradiction. Suppose that M^k is not φ-invariant. Then there exist $x_0 \in M^k$ and $u_0 \in T_{x_0}(M^k)$ such that $\varphi_{x_0} u_0$ does not stay in $T_{x_0}(M^k)$. For arbitrary $v \in T_{x_0}(M^k)$, let $a_v = 2\langle \varphi_{x_0} u_0, \varphi_{x_0} v\rangle - \Phi_{x_0}(u_0, v)$, $a_v \in \mathrm{IR}$. Since $a_v \varphi_{x_0} u_0 \in T_{x_0}(M^k)$, we must have $a_v = 0$; in particular, for $v = u_0$ one obtains $\|\varphi_{x_0} u_0\| = 0$, i.e. $\varphi_{x_0} u_0 = 0$, a contradiction. Q.E.D.

Therefore, by prop. 3.1. and prop. 4.1., a totally-umbilical submanifold M^k, $k \geqq 4$, tangent to the structure vector ξ of a Sasakian space-form is φ-invariant, (and k must be odd, i.e. $k = 2m+1$, $m > 1$). Now, *if M^{2m+1} is φ-invariant and $\xi^\perp = 0$ then M^{2m+1} must be minimal.* Indeed, by the Gauss formula (cf. our (2.4)) and by (2.2) one has

$$h(X, \varphi Y) = \varphi h(X, Y) \tag{4.2}$$

for any $X, Y \in \mathfrak{X}(M^{2m+1})$. Then by choosing an orthonormal tangential frame of the form $(\xi, X_i, \varphi X_i)_{1 \leqq i \leqq m}$ and using (4.2) one shows that $(2m+1)H = h(\xi, \xi) = 0$. Our Th. 1 is completely proved.

5. Extrinsic spheres

Let M^{2m+1} be an extrinsic sphere of the Sasakian manifold M^{2n+1}. Let $R^\perp$ be the curvature of the normal connection. Since M^{2m+1} is taken to be simply-connected, the assumption $R^\perp = 0$ is equivalent to the existence of a frame (in the normal bundle) consisting of mutually orthogonal parallel (in the normal bundle) unit vector fields. Let $c = \|H\|$. Then $c = \text{const.} > 0$. Let $N = \frac{1}{c} H$. Then N is a parallel unit normal vector field. We may choose an orthonormal frame $(N_a)_1 \leqq a \leqq 2n-2m$ in the normal bundle such that $N_1 = N$, and

$$\nabla^\perp N_a = 0, \; a \geqq 1.$$

Let us construct the functions $f_a \in C^\infty(M^{2m+1})$ by setting $f_a = \bar{g}(\varphi N, N_a)$, $2 \leqq a \leqq 2n-2m$. At this point we may prove our Th. 2. This is done in several steps, following the ideas in [3].

S t e p 1. *The following relation holds*

$$X(f_a) = c\,\bar{\Phi}(X, N_a), \; a > 1 \tag{5.1}$$

for any $X \in \mathfrak{X}(M^{2m+1})$.

P r o o f. As $A_{N_a} X = \bar{g}(N_a, H)X = 0$, we have $X(f_a) = X(\bar{g}(\varphi N, N_a)) = \bar{g}(\bar{\nabla}_X \varphi N, N_a) + \bar{g}(\varphi N, \bar{\nabla}_X N_a) = \bar{g}(\varphi \bar{\nabla}_X N, N_a) + \bar{\eta}(N)\bar{g}(X, N_a) - \bar{g}(\varphi N, A_{N_a} X) = -\bar{g}(\varphi A_N X, N_a)$, as a consequence of (2.2). Q.E.D.

Step 2. *The functions f_a are subject to*:

$$(\nabla_X df_a)Y = c(\bar{\eta}(N_a) - cf_a)g(X, Y) \tag{5.2}$$

for any $X, Y \in \mathfrak{X}(M^{2m+1})$.

Proof. Indeed, by (5.1) one has $X(Y(f_a)) = cX(\bar{\Phi}(Y, N_a)) = c\bar{g}(\bar{\nabla}_X Y, \varphi N_a) + c\bar{g}(Y, \bar{\nabla}_X \varphi N_a) = c\bar{g}(h(X, Y), \varphi N_a) + c\bar{g}(\nabla_X Y, \varphi N_a) + c\bar{g}(Y, \varphi\bar{\nabla}_X N_a) + c\bar{g}(Y, X)\bar{\eta}(N_a) = -c^2 g(X, Y) f_a + (\nabla_X Y)(f_a) + cg(X, Y)\bar{\eta}(N_a)$ and Step 2 is completely proved.

Step 3. *If $\xi^\perp = 0$ then there exists $2 \leqq a \leqq 2n-2m$ such that f_a is non-constant.*

Proof. The proof is by contradiction. Suppose that all f_a are constant. Then $0 = X(f_a) = c\bar{g}(X, \varphi N_a)$, by Step 1. Thus φN_a, $2 \leqq a \leqq 2n-2m$, are still normal. Let W_x be the linear subspace (of the normal space E_x) spanned by $N_{a,x}$, $\varphi_x N_{a,x}$, $2 \leqq a \leqq 2n-2m$, for $x \in M^{2m+1}$ fixed. Since M^{2m+1} is tangent to ξ, the contact 1-form vanishes on normal vectors. Thus, by (2.1), φ_x is a complex structure on W_x and consequently

$$\dim_{\mathrm{IR}} W_x > 2n-2m-1.$$

Therefore $W_x = E_x$ such that E_x follows to be φ_x-invariant. Consequently, M^{2m+1} is a φ-invariant submanifold (tangent to ξ) and thus M^{2m+1} is minimal, a contradiction.

Step 4. $(M^{2m+1}, g) \approx S^{2m+1}\left(\frac{1}{c}\right)$ *(an isometry)*.

Proof. By Step 2 and Step 3, the differential equation $\nabla_X \operatorname{grad}(f) = -c^2 f X$, where $\operatorname{grad}(f) = \#(df)$, (raising of indices is understood with respect to g), $f \in C^\infty(M^{2m+1})$, admits some non-constant solution f_a, provided that $\xi^\perp = 0$. Since (M^{2m+1}, g) was assumed to be complete, we may apply Th. A of M. Obata, [12], p. 334, such as to conclude that (M^{2m+1}, g) and the standard sphere of radius $\frac{1}{c}$ in IR^{2m+2} are isometric, Q. E. D.

Let M^{2n+1} be a Sasakian manifold and suppose that there exists an extrinsic sphere M^{2m+1} in M^{2n+1} bearing the hypothesis of our Th. 2. The proof that (1.1) holds is similar to the one in [3], p. 329. Indeed, since M^{2m+1} has a flat normal connection, one has $\tan(\bar{R}(X, Y)Z) = 0$ as a consequence of the Gauss equation (i. e. eq. (2.6) of [2], p. 45), of umbilicity and of $g(R(X, Y)Z, W) = c^2(g(X, W)g(Y, Z) - g(X, Z)g(Y, W))$, $X, Y, Z, W \in \mathfrak{X}(M^{2m+1})$, by our Th. 2. Also $\operatorname{nor}(\bar{R}(X, Y)Z) = 0$ (by $\nabla^\perp H = 0$, $h = g \otimes H$ and the Codazzi equation. On the other hand, as A_N is proportional (for each normal section N) to the identity, the Weingarten operators commute. Thus $\operatorname{nor}(\bar{R}(X, Y)N) = 0$, by the Ricci equation, i. e. eq. (2.11) in [2], p. 46. Finally, as $\operatorname{nor}(\bar{R}(X, Y)Z) = 0$ and as the Riemann-Christoffel tensor is skew-symmetric in the last two indices, one has $\tan(\bar{R}(X, Y)N) = 0$. Q. E. D.

6. Submanifolds with parallel second fundamental form

Let M^{2n+1} be a Sasakian manifold and M^m a submanifold of M^{2n+1}. Then M^m is a contact Cauchy-Riemann (C. R.) submanifold if it carries a pair of distributions D, $D^\perp$ such that i) $D^\perp$ is the g-orthogonal complement of D, ii) D is φ-invariant, i.e. $\varphi_x D_x \subseteq D_x$, for any $x \in M^m$, iii) $D^\perp$ is φ-anti-invariant, i.e. $\varphi_x D_x^\perp \subseteq E_x$, for any $x \in M^m$. See [14]. Let $PX = \tan(\varphi X)$, $X \in \mathfrak{X}(M^m)$. Suppose from now on that M^m is tangent to the contact vector of M^{2n+1}. Besides (2.2) let us recall (cf. [14], p. 10) the following formula:

$$\bar{\nabla}_X \xi = \varphi X \tag{6.1}$$

for any $X \in \mathfrak{X}(M^{2n+1})$. At this point, the Gauss and Weingarten formulae (2.4) furnish

$$\nabla_X \xi = PX \tag{6.2}$$

$$h(X, \xi) = FX \tag{6.3}$$

for any $X \in \mathfrak{X}(M^m)$, as a consequence of (6.1). Analogously (2.2) leads to

$$(\nabla_X P)Y = A_{FY} X + t\, h(X, Y) - g(X, Y)\xi + \eta(Y)X \tag{6.4}$$

$$(\nabla_X F)Y = -h(X, PY) + f h(X, Y). \tag{6.5}$$

Here $tN = \tan(\varphi N)$, $fN = \mathrm{nor}(\varphi N)$, for any normal section N on M^m. Suppose from now on that M^m has a parallel second fundamental form, i.e. $\nabla h = 0$. Using (6.3) and (6.2) we may perform the following computation:

$$0 = (\nabla_X h)(Y, \xi) = \nabla_X^\perp h(Y, \xi) - h(\nabla_X Y, \xi) - h(Y, \nabla_X \xi)$$
$$= \nabla_X^\perp FY - F\nabla_X Y - h(Y, PX) = (\nabla_X F)Y - h(Y, PX).$$

At this point we may substitute from (6.5) such as to yield

$$h(X, PY) + h(Y, PX) = f h(X, Y). \tag{6.6}$$

By (2.1), $\varphi\xi = 0$. Thus $P\xi = 0$, $F\xi = 0$. Let us set $Y = \xi$ in (6.6). Then, using again the identity (6.3), we obtain

$$FP = fF. \tag{6.7}$$

On the other hand, applying once more φ to the identity $\varphi X = PX + FX$ and using (2.1) and the uniqueness of the direct sum decomposition we obtain

$$P^2 + tF = -I + \eta \otimes \xi \tag{6.8}$$

$$FP + fF = 0. \tag{6.9}$$

See also [14], p. 44. Finally (6.7), (6.9) lead to

$$FP = 0. \tag{6.10}$$

This ends the proof of our Th. 3; indeed, by a result of K. Yano & M. Kon, (see

Th. 2.1. of [14], p. 55) a submanifold M^m tangent to the structure vector ξ of a Sasakian manifold M^{2n+1} is a contact C. R. submanifold if and only if (6.10) holds.

To prove the corollary, let M^m be a contact C. R. submanifold of the Sasakian space form $M^{2n+1}(c)$. By (2.3) the Codazzi equation of M^m in M^{2n+1} reads

$$(6.11)\ (\nabla_X h)(Y, Z)-(\nabla_Y h)(Y, Z)=\frac{c-1}{4}(g(PY, Z)FX-g(PX, Z)FY$$
$$+2g(X, PY)FZ).$$

As h is assumed to be parallel, (6.11) turns into

$$(6.12)\qquad g(PY, Z)FX-g(PX, Z)FY+2g(X, PY)FZ=0$$

for any $X, Y, Z\in\mathfrak{X}(M^m)$. Let D be the φ-invariant distribution of M^m, as a contact C. R. submanifold. Clearly the normal bundle valued l-form F on M^m vanishes on D. Let us use (6.12) for $X, Y\in D$, and Z arbitrary. We obtain

$$(6.13)\qquad g(X, PY)FZ=0.$$

Cf. [14], p. 53, $j=-P^2+\eta\otimes\xi$ and $j^\perp=I-j$ are the projectors of the (complementary) distributions D and $D^\perp$ of M^m. Note that $j\xi=\xi$, i.e. $\xi\in D$ and thus $D\neq(0)$. We distinguish two possibilities: either $D^\perp=(0)$ and then M^m is φ-invariant (and in this case we may apply prop. 1.4. of [14], p. 49, such as to conclude that M^m is totally-geodesic), or $D^\perp\neq(0)$. If this situation occurs, we shall end the proof of the Corollary by establishing the following

Lemma. *The invariant distribution is spanned by the contact vector.*

Indeed, it would follow that M^m is φ-anti-invariant (and by (6.3) the submanifold cannot be totally-geodesic).

To prove the lemma, let $x\in M^m$ and consider $u\in D_x$, $u\neq 0$. Consequently $F_x u\neq 0$. Therefore our (6.13) furnishes

$$(6.14)\qquad \langle v, P_x w\rangle=0$$

for any $v, w\in D_x$. The rest of the proof is by contradiction. Suppose that $\dim_{\mathbb{R}} D_x\geqq 2$; we may consider then $v\in D_x$ such that $v\neq 0$ and $\langle v,\xi_x\rangle=0$. Let $w=P_x v$. Clearly $w\neq 0$, and $w\in D_x$ (as P is D-valued). Finally we may use (6.14) and (6.8) to perform the following computation:

$$0=\langle v, P_x^2 v\rangle=-\|v\|^2+\eta(v)^2=-\|v\|^2$$

a contradiction.

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Received 28.06.1989