

Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

The attached copy is furnished for non-commercial research and education use only. Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on Mathematica Balkanica visit the website of the journal
<http://www.mathbalkanica.info>

or contact:

Mathematica Balkanica - Editorial Office;
Acad. G. Bonchev str., Bl. 25A, 1113 Sofia, Bulgaria
Phone: +359-2-979-6311, Fax: +359-2-870-7273,
E-mail: balmat@bas.bg

An Extension of a Result of J. E. Pečarić

Horst Alzer

Presented by V. Popov

In this note we prove:

Let $p=(p_1, \dots, p_n)$, $x=(x_1, \dots, x_n)$ and $a=(a_1, \dots, a_n)$ be positive n -tuples of real numbers with

$$x_1 \leq \dots \leq x_n, \quad a_1 \leq \dots \leq a_n \quad \text{and} \quad x_1/a_1 \leq \dots \leq x_n/a_n,$$

and let $M_r(x; p)$ be the weighted power mean of order r .
If r and s are real numbers with $r < s$, then the function

$$f(t) = M_r(ta + (1-t)x; p) / M_s(ta + (1-t)x; p)$$

is increasing on $[0, 1]$.

In 1977 K.-M. Chong [1] published the following interesting refinement of the classical arithmetic mean-geometric mean inequality.

Theorem A. Let $p=(p_1, \dots, p_n)$ and $x=(x_1, \dots, x_n)$ be two nonnegative n -tuples of real numbers and let the function F be defined by

$$(1) \quad F(t) = \prod_{i=1}^n \left(\frac{t}{P_n} \sum_{j=1}^n p_j x_j + (1-t)x_i \right)^{p_i/P_n} \left(P_n = \sum_{i=1}^n p_i \right).$$

Then F is increasing on $[0, 1]$.

An immediate consequence of this result is

$$\prod_{i=1}^n x_i^{p_i/P_n} \leq F(t) \leq \frac{1}{P_n} \sum_{i=1}^n p_i x_i, \quad 0 \leq t \leq 1,$$

where F is defined by (1).

Recently, J. E. Pečarić [4] has discovered a generalization of Chong's Theorem. He proved

Theorem B. Let $p=(p_1, \dots, p_n)$ and $x=(x_1, \dots, x_n)$ be two nonnegative n -tuples of real numbers and let the function G be defined by

$$(2) \quad G(t) = G(t, a) = \prod_{i=1}^n (ta + (1-t)x_i)^{p_i/P_n} / \frac{1}{P_n} \sum_{i=1}^n p_i (ta + (1-t)x_i),$$

where a is a nonnegative real number.

Then G is increasing on $[0, 1]$.

Indeed, if we set $a = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$ in (2), then we get (1).

The aim of this paper is to establish an extension of Pečarić's result.

In what follows let $p = (p_1, \dots, p_n)$, $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n)$ be positive n -tuples of real numbers with

$$(3) \quad x_1 \leq \dots \leq x_n, \quad a_1 \leq \dots \leq a_n \quad \text{and} \quad x_1/a_1 \leq \dots \leq x_n/a_n.$$

Further we denote by $M_r(x; p)$ the well-known weighted power mean of order r which is defined by

$$M_r(x; p) = \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i^r \right)^{1/r}, \quad r \in \mathbb{R} - \{0\},$$

$$M_0(x; p) = \prod_{i=1}^n x_i^{p_i/P_n};$$

see [3, pp. 74-79].

Then the following proposition holds:

Theorem. *If r and s are real numbers with $r < s$, then the function*

$$(4) \quad f(t) = M_r(ta + (1-t)x; p) / M_s(ta + (1-t)x; p)$$

is increasing on $[0, 1]$.

Proof. Since the ratio given in (4) is continuous in r it suffices to prove the monotonicity of f for r and s with $r < s$ and $r \neq 0$.

We designate by g_r the function

$$g_r(t) = \log M_r(ta + (1-t)x; p),$$

then we have

$$\log f(t) = g_r(t) - g_s(t).$$

Our aim is to verify the inequality

$$g'_r(t) - g'_s(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq 1.$$

Defining

$$h(r) = g'_r(t) = \frac{\sum_{i=1}^n p_i [ta_i + (1-t)x_i]^{r-1} (a_i - x_i)}{\sum_{i=1}^n p_i [ta_i + (1-t)x_i]^r},$$

it remains to show that h is decreasing on \mathbb{R} .

Next we set

$$b_i = ta_i + (1-t)x_i \quad \text{and} \quad c_i = (a_i - x_i) / (ta_i + (1-t)x_i) \quad (1 \leq i \leq n).$$

Then we get for h the representation

$$h(r) = \frac{\sum_{i=1}^n p_i c_i b_i^r}{\sum_{i=1}^n p_i b_i^r}.$$

Differentiation leads to

$$(5) \quad \left(\sum_{i=1}^n p_i b_i^r \right)^2 h'(r) = \sum_{i=1}^n q_i c_i \log(b_i) \sum_{i=1}^n q_i - \sum_{i=1}^n q_i c_i \sum_{i=1}^n q_i \log(b_i)$$

with $q_i = p_i b_i^r$ ($1 \leq i \leq n$).

A simple calculation yields for $i = 1, \dots, n-1$:

$$b_{i+1} - b_i = t(a_{i+1} - a_i) + (1-t)(x_{i+1} - x_i)$$

and

$$c_{i+1} - c_i = \frac{x_i a_{i+1} - x_{i+1} a_i}{[ta_i + (1-t)x_i][ta_{i+1} + (1-t)x_{i+1}]}$$

From (3) we obtain immediately

$$b_i \leq b_{i+1} \quad \text{and} \quad c_i \geq c_{i+1} \quad (1 \leq i \leq n-1),$$

and from Tchebyschef's inequality [2, pp. 43-44] we conclude that the right-hand side of (5) is nonpositive.

This completes the proof.

Remarks:

1. The Theorem also holds if we replace (3) by

$$x_1 \geq \dots \geq x_n, \quad a_1 \geq \dots \geq a_n \quad \text{and} \quad x_1/a_1 \geq \dots \geq x_n/a_n.$$

2. If we set $a_1 = \dots = a_n$, $r=0$ and $s=1$ in (4), then the Theorem implies Pečarić's Theorem B.

References

1. K.-M. Chong. On the arithmetic mean-geometric mean inequality. *Nanta Math.*, **10**, 1977, 26-27.
2. G. H. Hardy, J.E. Littlewood, G. Pólya. *Inequalities*. Cambridge, 1952.
3. D. S. Mitrinović. *Analytic Inequalities*. New York, 1970.
4. J. E. Pečarić. On a generalization of the Chong Kong Ming theorem. *Mat. Bilten (Skopje)*, **7-8**, 1983-1984, 1-3.

Morsbacher Str. 10
5220 Waldbröl
FEDERAL REPUBLIC OF GERMANY

Received 29.06.89