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An Extension of a Result of J. E. Pečarić

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In this note we prove:
Let $p=(p_1,\dots, p_n)$, $x=(x_1,\dots, x_n)$ and $a=(a_1,\dots, a_n)$ be positive n-tuples of real numbers with

$$x_1 \leqq \dots \leqq x_n,\ a_1 \leqq \dots \leqq a_n \text{ and } x_1/a_1 \leqq \dots \leqq x_n/a_n,$$

and let $M_r(x;\ p)$ be the weighted power mean of order r.
If r and s are real numbers with $r<s$, then the function

$$f(t)=M_r(ta+(1-t)x;\ p)/M_s(ta+(1-t)x;\ p)$$

is increasing on [0, 1].

In 1977 K.-M. Chong [1] published the following interesting refinement of the classical arithmetic mean-geometric mean inequality:

Theorem A. *Let $p=(p_1,\dots, p_n)$ and $x=(x_1,\dots, x_n)$ be two nonnegative n-tuples of real numbers and let the function F be defined by*

$$(1)\qquad F(t)=\prod_{i=1}^{n}\left(\frac{t}{P_n}\sum_{j=1}^{n} p_j x_j+(1-t)x_i\right)^{p_i/P_n}\left(P_n=\sum_{i=1}^{n} p_i\right).$$

Then F is increasing on [0, 1].

An immediate consequence of this result is

$$\prod_{i=1}^{n} x_i^{p_i/P_n}\leqq F(t)\leqq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i,\quad 0\leqq t\leqq 1,$$

where F is defined by (1).

Recently, J. E. Pečarić [4] has discovered a generalization of Chong's Theorem. He proved

Theorem B. *Let $p=(p_1,\dots, p_n)$ and $x=(x_1,\dots, x_n)$ be two nonnegative n-tuples of real numbers and let the function G be defined by*

$$(2)\qquad G(t)=G(t,\ a)=\prod_{i=1}^{n}(ta+(1-t)x_i)^{p_i/P_n}\Big/\frac{1}{P_n}\sum_{i=1}^{n} p_i(ta+(1-t)x_i),$$

where a is a nonnegative real number.

Then G is increasing on [0, 1].

Indeed, if we set $a=\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ in (2), then we get (1).

The aim of this paper is to establish an extension of Pečarić's result.

In what follows let $p=(p_1,\ldots, p_n)$, $x=(x_1,\ldots, x_n)$ and $a=(a_1,\ldots, a_n)$ be positive n-tuples of real numbers with

(3) $$x_1 \leqq \ldots \leqq x_n,\ a_1 \leqq \ldots \leqq a_n \text{ and } x_1/a_1 \leqq \ldots \leqq x_n/a_n.$$

Further we denote by $M_r(x;\ p)$ the well-known weighted power mean of order r which is defined by

$$M_r(x;\ p)=\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^r\right)^{1/r},\quad r\in \mathrm{R}-\{0\},$$

$$M_0(x;\ p)=\prod_{i=1}^{n} x_i^{p_i/P_n};$$

see [3, pp. 74-79].

Then the following proposition holds:

Theorem. *If r and s are real numbers with $r<s$, then the function*

(4) $$f(t)=M_r(ta+(1-t)x;\ p)/M_s(ta+(1-t)x;\ p)$$

is increasing on [0, 1].

Proof. Since the ratio given in (4) is continuous in r it suffices to prove the monotonicity of f for r and s with $r<s$ and $r\neq 0$.

We designate by g_r the function

$$g_r(t)=\log M_r(ta+(1-t)x;\ p),$$

then we have

$$\log f(t)=g_r(t)-g_s(t).$$

Our aim is to verify the inequality

$$g_r'(t)-g_s'(t)\geqq 0 \text{ for } 0\leqq t\leqq 1.$$

Defining

$$h(r)=g_r'(t)=\frac{\sum_{i=1}^{n} p_i[ta_i+(1-t)x_i]^{r-1}(a_i-x_i)}{\sum_{i=1}^{n} p_i[ta_i+(1-t)x_i]^r},$$

it remains to show that h is decreasing on R.

Next we set

$$b_i=ta_i+(1-t)x_i \text{ and } c_i=(a_i-x_i)/(ta_i+(1-t)x_i)(1\leqq i\leqq n).$$

Then we get for h the representation

$$h(r)=\sum_{i=1}^{n} p_i c_i b_i^r \Big/ \sum_{i=1}^{n} p_i b_i^r .$$

Differentiation leads to

$$(5)\qquad \left(\sum_{i=1}^{n} p_i b_i^r\right)^2 h'(r)=\sum_{i=1}^{n} q_i c_i \log(b_i) \sum_{i=1}^{n} q_i-\sum_{i=1}^{n} q_i c_i \sum_{i=1}^{n} q_i \log(b_i)$$

with $q_i=p_i b_i^r\,(1\leqq i\leqq n)$.

A simple calculation yields for $i=1,\ldots, n-1$:

$$b_{i+1}-b_i=t(a_{i+1}-a_i)+(1-t)(x_{i+1}-x_i)$$

and

$$c_{i+1}-c_i=\frac{x_i a_{i+1}-x_{i+1} a_i}{[ta_i+(1-t)x_i][ta_{i+1}+(1-t)x_{i+1}]}.$$

From (3) we obtain immediately

$$b_i\leqq b_{i+1} \text{ and } c_i\geqq c_{i+1}\quad (1\leqq i\leqq n-1),$$

and from Tchebyschef's inequality [2, pp. 43-44] we conclude that the right-hand side of (5) is nonpositive.

This completes the proof.

Remarks:

1. The Theorem also holds if we replace (3) by

$$x_1\geqq\ldots\geqq x_n,\ a_1\geqq\ldots\geqq a_n \text{ and } x_1/a_1\geqq\ldots\geqq x_n/a_n .$$

2. If we set $a_1=\ldots=a_n$, $r=0$ and $s=1$ in (4), then the Theorem implies Pečarić's Theorem B.

References

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