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Uniform Semi-Classical Asymptotics of the Spectral Function for Schrödinger Operators and Periodic Bicharacteristics

Georgi E. Karadzhov

Presented by V. Popov

One considers the spectral function $e_h(\lambda, x, y)$ of the Schrödinger operator $A_h = -\frac{h^2}{2}\Delta + V$ with a C^∞ positive potential V, under some conditions on the periodic bicharacteristics of the hamiltonian $p(x, \xi) = \xi^2/2 + V(x)$. An asymptotics of the function $e_h(\lambda, x, x)$ as $h \to +0$ is obtained, which is locally uniform with respect to the parameters (λ, x). Near the caustic points $V(x) = \lambda$ this asymptotics is expressed in terms of the Airy functions.

1. Introduction and statement of the results

Let $A_h = -\frac{h^2}{2}\Delta + V$, $h > 0$ be the Schrödinger operator with a C^∞ positive potential V. This operator is essentially self-adjoint in $L^2(\mathrm{R}^n)$ and $A_h = \int \lambda dE_\lambda$. The kernel $e_h(\lambda, x, y)$ of the orthogonal projection E_λ is called the spectral function of the operator A_h.

The purpose of this paper is to find the asymptotics of the function $e_h(\lambda, x, x)$ as $h \to 0$, which is locally uniform with respect to the parameters (λ, x). One obtains the main term and an estimate of the rest. This estimate is improved when the periodic bicharacteristics of the hamiltonian $p(x, \xi) = \xi^2/2 + V(x)$ are not too much. More precisely, we consider two hypotheses – (H_1) and (H_2). Let $n \geqq 2$ and let $\Phi^t(y, \eta) = (x(t, y, \eta), \xi(t, y, \eta))$ be the hamiltonian flow of p, lying on the energy level $p(y, \eta) = \lambda$.

(H_1) We say that the point (λ, y) satisfies the hypothesis (H_1) if $\lambda - V(y) \geqq \delta > 0$ and if the measure of the set

$$S(\lambda, y) = \{\eta \in \mathrm{R}^n : p(y, \eta) = \lambda,\ x(t, y, \eta) = y \text{ for some } t \neq 0\}$$

is zero.

(H_2) We say that the point (λ, y) satisfies the hypothesis (H_2) if $V(y) = \lambda$ and the bicharacteristic $\Phi^t(y, 0)$ is not periodic.

It is not hard to see that the set of the points $\{(\lambda, y)\}$, satisfying (H_2) is open, provided that λ is not critical value of V.

Example 1. Let $V(x)=\sum_{j=1}^{n}\alpha_j^2 x_j^2$. If α_i/α_j is not rational number for some $i\neq j$, then the hypothesis (H_1) is satisfied for every point (λ, x) such that $\lambda-V(x)\geqq\delta>0$.

Example 2. For the potential $x_1^2+2x_2^2$ the points (λ, x), where $x_1^2+2x_2^2=\lambda$, $x_1\neq 0$, $x_2\neq 0$, satisfy the hypothesis (H_2).

We prove the following asymptotics and estimates of the spectral function $e_h(\lambda, x, x)$, which are locally uniform with respect to the parameters (λ, x). If the hypotheses (H_1) or (H_2) are not satisfied, then the quantity $o(1)$ in all estimates as $h\to 0$ must be replaced by $O(1)$.

Theorem 1. *(The case $V(x)\leqq\lambda-\delta$, $\delta>0$.) Let the points (λ, x) satisfy the hypothesis (H_1). Then*

$$(1)\qquad e_h(\lambda, x, x)=(2\pi)^{-n}V_n[2(\lambda-V(x))]^{n/2}h^{-n}+O(h^{-n+1}),\ h\to 0,$$

where V_h is the volume of the unit ball in $\mathbf{R}^n$.

Theorem 2. *(The case $\lambda-\delta\leqq V(x)\leqq\lambda$.) Let $V'(x)\neq 0$ and let the point (λ, x_0) satisfies the hypothesis (H_2). Then there exist a positive number δ and a neighborhood U of x_0 such that for every (λ, x) with $\lambda-\delta\leqq V(x)\leqq\lambda$, $x\in U$ we have*

$$(2)\ e_h(\lambda, x, x)=a_n(h, \lambda, x)h^{-2n/3}+b_n(h, \lambda, x)\,O(h^{-2n/3+1/3}),\ h\to 0,$$

where

$$(3)\qquad a_n(h, \lambda, x)=(2\pi)^{-n}V_n\left(\frac{2(\lambda-V(x))}{B(\lambda, x)}\right)^{n/2}f_n(-B(\lambda, x)h^{-2/3}),$$

$$(4)\qquad b_n(h, \lambda, x)=f_{n-2}(-B(\lambda, x)h^{-2/3}),$$

$$(5)\qquad f_n(s)=\int_0^\infty Ai(\sigma+s)\sigma^{n/2}\,d\sigma,\ Ai(\sigma)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i(\sigma s+s^3/3)}\,ds$$

being the Airy function and

$$(6)\qquad B=B(\lambda, x)=\left(\frac{3}{2}\psi(t(\lambda, x), \xi(\lambda, x), \lambda, x)\right)^{2/3},$$

$$(7)\qquad \psi(t, \xi, \lambda, x)=\lambda t+\varphi(t, \xi, x)-\xi x,$$

$$(8)\qquad \partial_t\varphi+\frac{1}{2}(\partial_x\varphi)^2+V(x)=0,\ \varphi(0, \xi, x)=\xi x.$$

Finally, $(t(\lambda, x), \xi(\lambda, x))$ is the critical point of ψ for which

$$(9)\qquad \lambda-V(x)-\frac{1}{8}(V'(x))^2t^2=O(t^4),\ \xi=\frac{1}{2}V'(x)t+O(t^3),\ t\to 0.$$

Remark 1. The coefficient $B(\lambda, x)$ has the asymptotics

(10) $$B(\lambda, x)=B_0(\lambda, x)+O((\lambda-V(x))^2) \text{ as } \lambda-V(x)\to 0,$$

where

(11) $$B_0=B_0(\lambda, x)=2|V'(x)|^{-2/3}(\lambda-V(x)).$$

Remark 2. The functions f_n are positive and satisfy the recurrence relations

(12) $$f_n(s)=-sf_{n-2}(s)+f''_{n-2}(s),\ n\geqq 2;\ f_0(s)=\int_s^\infty Ai(\sigma)d\sigma;$$

(13) $$f_1(s)=\pi 2^{1/3}\{-4^{-1/3}s(Ai(4^{-1/3}s))^2+(Ai'(4^{-1/3}s))^2\}.$$

Moreover, they decrease if $n\geqq 1$ and

(14) $$f_n(s)=(-s)^{n/2}+R_n(s),\ s\to-\infty,\ n\geqq 0,$$

where

(15) $$R_0(s)=O(|s|^{-3/4}),\ R_1(s)=O(|s|^{-1}),$$
$$R_{2k}(s)=O(|s|^{k-9/4}),\ R_{2k+1}(s)=O(|s|^{k-5/2}), k\geqq 1.$$

Theorem 3. *(The case $\lambda\leqq V(x)\leqq\lambda+\text{const } h^{2/3}$.) Under the conditions of Theorem 2*

(16) $$e_h(\lambda, x, x)=a_n(h, \lambda, x)h^{-2n/3}+o(h^{-2n/3+1/3}),\ h\to 0,$$

where $x\in U$,

(17) $$a_n(h, \lambda, x)=(2\pi)^{-n}V_n|V'(x)|^{n/3}f_n(-B_0h^{-2/3})$$

and B_0 is given by (11).

Corollary. *(The case $|\lambda-V(x)|\leqq\text{const } h$.) If $V'(x)\neq 0$ then*

(18) $$e_h(\lambda, x, x)=(2\pi)^{-n}V_n|V'(x)|^{n/3}f_n(0)h^{-2n/3}+O(h^{-2n/3+1/3}),\ h\to 0.$$

Theorem 4. *(The case $\lambda+h^{2/3-\varepsilon}\leqq V(x)$.) If $V'(x)\neq 0$ and $\varepsilon>0$ then*

(19) $$e_h(\lambda, x, x)=O(h^\infty),\ h\to 0.$$

Theorem 5. *(The case $V(x)\geqq\lambda+h^{1/2-\varepsilon}$.) In this case the uniform estimate* (19) *is valid, assuming only the condition $\varepsilon>0$.*

2. Method of the proofs

Using appropriate tauberian arguments, we reduce the asymptotics of the function $e_h(\lambda, x, x)$ to its averages

$$e_{h,\rho}(\lambda, x)=\int\rho_h(\lambda-\mu)e_h(\mu, x, x)d\mu,\quad e'_{h,\rho}(\lambda, x)=\int\rho\left(\frac{\lambda-\mu}{h}\right)de_h(\mu, x, x),$$

where $\rho_h(\lambda)=\frac{1}{h}\rho(\frac{\lambda}{h})$, ρ is smooth, even and rapidly decreasing function on R, which Fourier transform $\hat{\rho}(t)=\int e^{-i\lambda t}\rho(\lambda)d\lambda$ has a compact support and $\hat{\rho}(0)=1$.

Tauberian theorem. *Let the function $\lambda \to E_h(\lambda, x)$, depending on the parameter x, satisfy the conditions*:

(*i*) $|E_h(\lambda, x)| \leqq \mathrm{const}\, h^{-\alpha}(1+|\lambda|)^{\beta}$, $\alpha>0$, $\beta>0$, $0<h<h_0$, $\lambda \in \mathrm{R}$, *locally uniformly over* x;

(*ii*) $|E_h(\lambda+\sigma Th, x)-E_h(\lambda, x)| \leqq \mathrm{const}\, b(h, \lambda, x)h^{-\gamma}(T+C(T)O(1))$, $\gamma>0$, $T>0$, $|\sigma|\leqq 1$, $h\to 0$, *locally uniformly with respect to* (λ, x) *where*

(*iii*) $$b(h, \lambda, x)\geqq C_0>0.$$

If there exists a function $d(\lambda, x)$ *with the property*

(*iv*) $$b(h, \lambda+\sigma, x)\leqq \mathrm{const}\, b(h, \lambda, x) \text{ for } |\sigma|\leqq d(\lambda, x),$$

then

(*v*) $|E_h(\lambda, x)-E_{h,\rho_T}(\lambda, x)| \leqq \mathrm{const}\, b(h, \lambda, x)h^{-\gamma}(T+C(T)o(1))$, $h\to 0$,

locally uniformly in the region $d(\lambda, x)\geqq Ch^{\varepsilon}$, $0<\varepsilon<1$, $c>0$.

Further we use the relation

$$e'_{h,\rho}(\lambda, x)=\frac{1}{2\pi}\int e^{i\lambda h^{-1}t}\hat{\rho}(t)U_h(t, x, x)dt,$$

where $U_h(t,x,y)$ is the kernel of the operator $U_h(t)=e^{-ih^{-1}tA_h}$ and a h-approximation of this operator in the form

(20) $$Q_h(t)u(x)=(2\pi h)^{-n}\int e^{ih^{-1}\varphi(t,\xi,x)}q(t, \xi, x, h)\hat{u}(\xi)d\xi, \quad u\in C_0^{\infty},$$

where t varies on a compact interval.

Such a parametrix can be constructed by the methods from [2], [6], [11], [13], locally relative to the variables $x\in \mathrm{R}^n$ and $t\in \mathrm{R}$. In particular, for a fixed compact $K\subset \mathrm{R}^n$ the phase function φ is a solution of the problem (8) for $x\in K$ and if t is near zero. The amplitude function q has a form $q(t, \xi, x, h)=\Sigma_{j=0}^{N} h^j q_j(t, \xi, x)$, N-large enough, and

(21) $$\partial_t q_0+\partial_x\varphi\partial_x q_0+\frac{1}{2}\partial_x^2\varphi q_0=0, \quad q_0(0, \xi, x)=1.$$

If $|t|\geqq\delta>0$ then locally over (t, x) the parametrix $Q_h(t)$ is of the same form (20) with another phase function φ, satisfying the Hamilton-Jacobi equation

(22) $$\partial_t\varphi+\frac{1}{2}(\partial_x\varphi)^2+V(x)=0.$$

Let $\varkappa\in C_0^{\infty}(\mathrm{R})$ be an even function, $\varkappa(t)=1$ near $t=0$. As in [10] we have

(23) $$e'_{h,\rho_T}(\lambda, x)=h^{-n}\int e^{ih^{-1}\psi}r_1 dt d\xi+J_h(\lambda, x),$$

where ψ is given by (7) and

$$r_1(t, \xi, x, h)=(2\pi)^{-n-1}\hat{\rho}_T(t)\varkappa(t)q(t, \xi, x, h),$$

$$J_h(\lambda, x)\sim h^{-n}\int e^{ih^{-1}\psi}r_2 dt d\xi, \tag{24}$$

$$r_2(t, \xi, x, h)=(2\pi)^{-n-1}\hat{\rho}_T(t)(1-\varkappa(t))q(t, \xi, x, h).$$

The equivalence "$A_h(\lambda, x) \sim B_h(\lambda, x)$" means that $A_h(\lambda, x)-B_h(\lambda, x)=O(h^\infty)$ locally uniformly in (λ, x). Moreover, the support of the functions r_1 and r_2 can be supposed to be compact over (t, ξ).

The hypotheses (H_1) or (H_2) allow us to prove that $J_h(\lambda, x)=o(h^{-n+1})$ or $O(h^\infty)$ respectively. Thus the problem is to find the asymptotics of the integrals of the form (24) as $h\to 0$, which must be locally uniform with respect to the parameters (λ, x). In the case $V(x)\leqq\lambda-\delta$, $\delta>0$ the critical points of the phase function $(t, \sigma)\to\psi(t, \sigma\omega, \lambda, x)$, $|\omega|=1$ are nondegenerate and the method of the stationary phase is applicable. If $V(x)$ is close to λ, then the critical points of ψ may degenerate. In this case we apply the theory of the versal deformations [1], [12] and prove that if $V'(x)\neq 0$ then the function $\psi(t, \xi, \lambda, x)$ admits a normal form $-B(\lambda, x)t+t\xi^2+t^3/3$. After we use polar coordinates $\xi=\sigma\omega$ and the Malgrange preparation theorem.

3. Proof of the Tauberian theorem

Since

$$E_{h,\rho_T}(\lambda, x)-E_h(\lambda, x)=\int[E_h(\lambda-\mu Th, x)-E_h(\lambda, x)]\rho(\mu)\,d\mu, \tag{25}$$

it is sufficient to estimate the difference $\Delta E_h=E_h(\lambda-\mu Th, x)-E_h(\lambda, x)$ locally uniformly with respect to (λ, x).

1st case: $|\mu Th|\leqq d(\lambda, x)$. Then the conditions (ii), (iv) imply the bound

$$|\Delta E_h|\leqq \text{const}\, b(h, \lambda, x)h^{-\gamma}(T+C(T)o(1))(1+|\mu|),\ h\to 0. \tag{26}$$

2nd case: $|\mu Th|>d(\lambda, x)$. Now the conditions (i), (iii) show that

$$|\Delta E_h|\leqq \text{const}\, b(h, \lambda, x)h^{-\gamma}o(1)(1+|\mu|)^{\alpha+\beta},\ h\to 0 \tag{27}$$

in the region $d(\lambda, x)\geqq ch^\varepsilon$, $c>0$, $0<\varepsilon<1$.

Evidently the estimate (v) follows from (25) – (27).

Remark 3. The Tauberian theorem will be used at the proof of Theorem 2 in the region $B(\lambda, x)\geqq Ch^{2/3}$, $C>0$. In all other cases we shall use the more simple variant of the Tauberian theorem: the conditions (i), (ii) with $b=1$ imply (v) with $b=1$.

4. Proof of Theorem 1

We start from the formula (23). In view of (22) the critical points of the phase function ψ satisfy the relations

(28) $$\partial_\xi \varphi = x, \quad p(x, \partial_x \varphi) = \lambda$$

are taking into account the property $\Phi^t(\partial_\xi \varphi, \xi) = (x, \partial_x \varphi)$ we conclude that

(29) $$\Phi^t(x, \xi) = (x, \partial_x \varphi), \quad p(x, \partial_x \varphi) = \lambda, \quad p(x, \xi) = \lambda.$$

Using the expression (10.13) from [2] of the phase function φ it is not hard to see that the range of the Hessian φ'' in the critical points is equal to 2. Therefore the method of the stationary phase and the hypothesis (H_1) lead to the estimate

(30) $$J_h(\lambda, x) = o(h^{-n+1}), \quad h \to 0.$$

Further, integrating by parts in the integral (23) over each coordinate ξ_j and summing, we get from (23), (30) the formula

(31) $$e_{h, \rho_T}(\lambda, x) = I_h(\lambda, x) + o(h^{-n+1}), \quad h \to 0,$$

where

(32) $$I_h(\lambda, x) = h^{-n} \int e^{ih^{-1}\psi} r \, dt \, d\xi,$$

$$r(t, \xi, x, h) = \frac{(2\pi)^{-n-1}}{n} \frac{\hat{\rho}_T(t)}{t} [-h^{-1} \xi \partial_\xi \psi q + i \xi \partial_\xi q] \varkappa(t).$$

It is convenient to use polar coordinates $\xi = \sigma\omega$ in the integral (32). From (29) it follows that the point $t = 0$, $\sigma^2/2 = \lambda - V(x)$ is critical for the phase function $(t, \sigma) \to \psi(t, \sigma\omega, \lambda, x)$. Since φ satisfies (8), (28) we see that for the other critical points the estimate $\sigma \leqq C|t|$ is fulfilled, hence on the support of the integrand r there are not other critical points. Using the method of the stationary phase we find the asymptotics of the integral $I_h(\lambda, x)$, and taking into account (31), we obtain

(33) $$e_{h, \rho_T}(\lambda, x) = (2\pi)^{-n} V_n [2(\lambda - V(x))]^{n/2} h^{-n} + o(h^{-n+1}), \quad h \to 0.$$

Analogously,

(34) $$e'_{h, \rho_T}(\lambda, x) = a_n(\lambda, x) h^{-n+1} + O(h^{-n+2}), \quad h \to 0.$$

On the other hand it is known that [10]

(35) $$e_h(\lambda, x, x) \leqq \text{const}\, h^{-\alpha}(1 + |\lambda|)^\beta, \quad 0 < h < h_0, \quad \lambda \in \mathbb{R},$$

locally uniformly over $x \in \mathbb{R}^n$, for some $\alpha > 0$, $\beta > 0$.

Finally, (33) — (35) and remark 3 give the asymptotics (1).

5. Proof of Theorem 2

From the hypothesis (H_2) it follows that there exists a positive number δ such that if $|V(x) - \lambda| \leqq \delta$ then the critical points of the phase function ψ are outside of the support of the function $(t, \xi) \to r_2(t, \xi, x, h)$. Therefore the integral (24) satisfies the estimate $J_h(\lambda, x) = 0(h^\infty)$, $h \to 0$, locally uniformly in (λ, x) if $|V(x) - \lambda| \leqq \delta$. Thus we obtain analogously to (31),

(36) $$e_{h,\rho_T}(\lambda, x)=I_h(\lambda, x)+O(h^\infty),\ h\to 0.$$

To evaluate the integral $I_h(\lambda, x)$ we note first that the critical points of the phase function ψ satisfy the equations $\partial_t\psi=0$, $\partial_\xi\psi=0$ and

(37)
$$\partial_t\psi=\lambda-V(x)-\frac{\xi^2}{2}+V'(x)\xi t-\frac{t^2}{2}((V'(x))^2+\langle V''(x)\xi,\ \xi\rangle)+O(t^3),$$
$$\partial_\xi\psi=t\left[-\xi+\frac{1}{2}V'(x)t-V''(x)\xi\frac{t^2}{3}\right]+O(t^4),\ t\to 0.$$

Because of (7), (8) the function $(t, \xi)\to\psi(t, \xi, \lambda, x)$ is odd, therefore only the points $\{(0, \xi):\xi^2/2=\lambda-V(x)\}$ and $\{(\pm t(\lambda, x), \pm\xi(\lambda, x))\}$ are critical, where $(t(\lambda, x), \xi(\lambda, x))$ satisfies (9) and $p(x, \xi(\lambda, x))=\lambda$. In particular, $(t, \xi)\to 0$ if $V(x)\to\lambda$.

Let $V(x)=\lambda$. Then

$$\psi(t, \xi, \lambda, x)=-\frac{t}{2}\left[\left(\xi-\frac{tV'(x)}{2}\right)^2+\frac{t^2}{3}\left(\frac{(V'(x))^2}{4}+\langle V''(x)\xi,\ \xi\rangle\right)\right]+O(t^4),$$

hence there exists an odd change of variables $t=\tau p_1(\tau, \eta, \lambda, x)$, $\xi=\xi_1(\tau, \eta, \lambda, x)$ such that

(38) $$\psi(t, \xi, \lambda, x)=\tau\eta^2+\tau^3/3 \text{ if } V(x)=\lambda,\ V'(x)\neq 0.$$

From the theory of the versal deformations [1], [12] it follows that the family $ct+t\xi^2+t^3/3$ is a versal deformation of the function $t\xi^2+t^3/3$ in the class D of all smooth functions $g(t, \xi)$, defined in a neighborhood of the origin, with the properties: $g(-t, -\xi)=-g(t, \xi)$, $g(0, \xi)=0$, which class is invariant under the local diffeomorphisms $(\tau, \eta)=v(t, \xi):\mathrm{R}\times\mathrm{R}^n\to\mathrm{R}\times\mathrm{R}^n$, such that $v(-t, -\xi)=-v(t, \xi)$, $v(0, \xi)=(0, \eta)$.

Consequently, since the function $\psi\in D$ and satisfies (38), there exists an odd change of variables $(t, \xi)\to(\tau, \eta)$ such that

(39) $$t=\tau p(\tau, \eta, \lambda, x),\ \xi=\xi(\tau, \eta, \lambda, x),$$

(40) $$\psi(t, \xi, \lambda, x)=-B(\lambda, x)\tau+\tau\eta^2+\tau^3/3$$

if $V'(x)\neq 0$ and $|V(x)-\lambda|\leq\delta$, δ being sufficiently small. In addition, the coefficient $B(\lambda, x)$ satisfies (6) and the asymptotics (10), (11) if $V(x)\leq\lambda$.

Therefore by the principle of the stationary phase we obtain

(41) $$I_h(\lambda, x)\sim\int e^{ih^{-1}(-B\tau+\tau\eta^2+\tau^3/3)}g(\tau, \eta, x, h)\,d\tau\,d\eta,$$

where $g=h^{-1}\tilde{g}_1+\tilde{g}_0$,

(42) $$\tilde{g}_1(\tau, \eta, x)=-\frac{(2\pi)^{-n-1}}{n}\frac{\hat{\rho}_T(t)}{t}\varkappa(t)\xi\partial_\xi\psi q_0(t, \xi, x)X(\xi)J(\tau, \eta),$$

$$\tilde{g}_0(\tau, \eta, x, h)=\frac{(2\pi)^{-n-1}}{n}\frac{\hat{\rho}_T(t)}{t}\varkappa(t)\left[i\xi\partial_\xi q-\sum_{j=1}^{N}h^{j-1}q_j\xi\partial_\xi\psi\right]X(\xi)J(\tau, \eta),$$

$$J(\tau, \eta)=\left|\det\frac{D(t, \xi)}{D(\tau, \eta)}\right|$$

and $X \in C_0^\infty(\mathbb{R}^n)$ is a cutoff, even function. Note in view of (21), (8) that the function $(\tau, \eta) \to \tilde{g}_1(\tau, \eta, x)$ is even.

In polar coordinates $\eta = \sigma\omega$ the integral (41) becomes

$$I_h \sim I_{h,1} + I_{h,0}, \tag{43}$$

where

$$I_{h,j}(\lambda, x) = h^{-j-n} \int_0^\infty \int e^{ih^{-1}(-B\tau + \tau\sigma^2 + \tau^3/3)} \sigma^{n-1} g_j(\tau, \sigma, h)\, d\tau\, d\sigma \tag{44}$$

and

$$g_j(\tau, \sigma, h) = \int_{|\omega|=1} \tilde{g}_j(\tau, \sigma\omega, x, h)\, d\omega, \tag{45}$$

in particular $g_1(\tau, \sigma) = G_1(\tau, \sigma^2)$, $g_0(\tau, \sigma, h) = G_0(\tau, \sigma^2, h)$, $G_j \in C^\infty$. Finally, the function $\tau \to G_j(\tau, \sigma, h)$ is even.

Using the Malgrange preparation theorem, we can write

$$G_1(\tau, \sigma) = a_0 + a_2\sigma + (\tau^2 + \sigma - B)F_1 + \tau\sigma F_2, \quad F_j \in C^\infty,$$

whence

$$g_1(\tau, \sigma) = a_0 + a_2\sigma^2 + (\tau^2 + \sigma^2 - B)f_1 + \tau\sigma^2 f_2, \quad f_j \in C^\infty \tag{46}$$

and $f_j(\tau, \sigma) = F_j(\tau, \sigma^2)$. In addition, the coefficients a_0, a_2 satisfy the formulas: $a_0 = g_1(\sqrt{B}, 0)$, $a_2 = \frac{1}{B}(g_1(0, \sqrt{B}) - a_0)$. Since the critical point $(t(\lambda, x), \xi(\lambda, x))$ is image of the point $(\sqrt{B}, 0)$, it follows from (42) that

$$a_0 = 0, \quad a_2 = \frac{1}{B} g_1(0, \sqrt{B}). \tag{47}$$

Integrating by parts in the integral $I_{h,1}(\lambda, x)$ with the help of (46), (47), we get

$$I_{h,1}(\lambda, x) = a_2 h^{-1-n} \int_0^\infty \int e^{ih^{-1}(-B\tau + \tau\sigma^2 + \tau^3/3)} \sigma^{n+1} d\tau\, d\sigma + J_{h,1}(\lambda, x), \tag{48}$$

where

$$J_{h,1}(\lambda, x) \sim ih^{-n} \int_0^\infty \int e^{ih^{-1}(-B\tau + \tau\sigma^2 + \tau^3/3)} \sigma^{n-1} u(\tau, \sigma)\, d\tau\, d\sigma + R_1, \tag{49}$$

$$u(\tau, \sigma) = \partial_\tau f_1 + \frac{n}{2} f_2 + \frac{\sigma}{2} \partial_\sigma f_2 \tag{50}$$

and $R_1 = R_1(h, \lambda, x)$ is an integral of the same form, but of lower order with respect to h.

Since the function $\tau \to u(\tau, \sigma)$ is odd we can write

$$u(\tau, \sigma) = a_1\tau + (\tau^2 + \sigma^2 - B)u_1 + \tau\sigma^2 u_2, \tag{51}$$

therefore (48) – (51), (5) give

(52) $$I_{h,1}(\lambda, x)=\pi h^{-2n/3}[a_2 f_n(-Bh^{-2/3})+a_1 f'_{n-2}(-Bh^{-2/3})h^{2/3}]+R_1.$$

Iterating the previous considerations, we obtain

(53) $$R_1=(\tilde{b}_n(h, \lambda, x)+O(1))O(h^{-2n/3+4/3}),$$

where

(54) $$\tilde{b}_n(h, \lambda, x)=f_{n-2}(-Bh^{-2/3})+h^{1/3}|f'_{n-2}(-Bh^{-2/3})|+h^{2/3}f_n(-Bh^{-2/3}).$$

Later on we shall prove that

(55) $$0<c\leqq\tilde{b}_n(h, \lambda, x)\leqq\text{const}\, b_n(h, \lambda, x)$$

and

(56) $$|f'_{n-2}(-Bh^{-2/3})|\leqq\text{const}\, b_n(h, \lambda, x).$$

Thus (52) – (56) yield

(57) $$I_{h,1}(\lambda, x)=\pi a_2 f_n(-Bh^{-2/3})h^{-2n/3}+b_n(h, \lambda, x)O(h^{-2n/3+2/3}).$$

Analogously,

(58) $$I_{h,0}(\lambda, x)=b_n(h, \lambda, x)O(h^{-2n/3+2/3}).$$

To compute the coefficient a_2, we note that the points $(0, \sqrt{B}\,\omega)$ are images of the critical points $(0, \xi)$, $\xi^2/2=\lambda-V(x)$, therefore (47), (44), (42) give

(59) $$a_2=\frac{(2\pi)^{-n-1}}{n}\frac{2(\lambda-V(x))}{B(\lambda, x)}\int_{|\omega|=1}J(0, \sqrt{B}\,\omega)\,d\omega.$$

On the other hand, from (39), (40) it follows that

(60) $$\left(\det\frac{\partial\xi}{\partial\eta}\right)^2=(-2)^n\left(\frac{\partial\tau}{\partial t}\right)^n$$

and $J^2\det\psi''=2^{n+1}(\tau^2-\eta^2)\tau^{n-1}$ in the critical points. Hence $J(0, \sqrt{B}\,\omega)=2\left(\frac{2(\lambda-V(x))}{B(\lambda, x)}\right)^{n/2-1}$ and (59) implies

(61) $$a_2=(2\pi)^{-n-1}\,2V_n\left(\frac{2(\lambda-V(x))}{B(\lambda, x)}\right)^{n/2}.$$

Now (36), (41), (43), (57), (58), (59) lead to

(62) $$e_{h,\rho_T}(\lambda, x)=a_n(h, \lambda, x)h^{-2n/3}+b_n(h, \lambda, x)O(h^{-2n/3+2/3}),$$

where the coefficients a_n and b_n are given by (3), (4).

Analogously,

(63) $$|e'_{h,\rho_T}(\lambda, x)|\leqq\text{const}\, b_n(h, \lambda, x)h^{-2n/3+1/3}(1+O(h)).$$

To use the Tauberian theorem, we have to verify the properties (iii), (iv) for the function $b_n(h, \lambda, x)$. First we prove the asymptotics (14), (15). From the asymptotics of the Airy function it follows that:

$$f_0(s)=1-\frac{1}{\sqrt{\pi}}(-s)^{-3/4}\cos\left(\frac{2}{3}(-s)^{3/2}+\frac{\pi}{4}\right)+O(|s|^{-9/4}),\ s\to-\infty$$

and

$$f''_0(s)=\frac{1}{\sqrt{\pi}}(-s)^{1/4}\cos\left(\frac{2}{3}(-s)^{3/2}+\frac{\pi}{4}\right)+O(|s|^{-5/4}) \text{ as } s\to-\infty.$$

Therefore (12) shows that $f_2(s)=-s+O(|s|^{-5/4})$, $s\to-\infty$. Analogously,

(64) $$f_{2k}(s)=(-s)^k+O(|s|^{k-9/4}),\ s\to-\infty \text{ if } k=2,\ 3.$$

Now the formula (64) follows inductively for every $k\geqq 1$ in view of (12). In the same way (13) and (12) imply

$$f_1(s)=(-s)^{1/2}+\frac{(-s)^{-1}}{2}\cos\frac{4}{3}(-s)^{3/2}+O(|s|^{-5/2}),\ s\to-\infty$$

and

$$f''_1(s)=-\frac{1}{2}\cos\frac{4}{3}(-s)^{3/2}+O(|s|^{-3/2}),\ s\to-\infty,$$

whence

(65) $$f_{2k+1}(s)=(-s)^{k+1/2}+O(|s|^{k-5/2}),\ s\to-\infty,\ k\geqq 1.$$

Further, we need the bound

(66) $$f_0(s)>0.$$

Indeed, it sufficies to see that

(67) $$I_k=\int_{s_{2k+2}}^{s_{2k}} Ai(\sigma)\,d\sigma>0,\ k\geqq 0;\ s_0=\infty,$$

where $0>s_1>s_2>\dots$ are the zeros of the Airy function $Ai(s)$, so $s_{n+1}-s_{n+2}<s_n-s_{n+1}$ and $Ai(\sigma)>0$ on the intervals (s_{2k+1}, s_{2k}), $k\geqq 0$. Since

$$I_k=\int_{s_{2k+1}}^{s_{2k}} Ai(\sigma)\,d\sigma+\int_{s_{2k}}^{r_k} Ai(2s_{2k+1}-\sigma)\,d\sigma,$$

where $r_k=2s_{2k+1}-s_{2k+2}$, then

(68) $$I_k\geqq\int_{s_{2k+1}}^{r_k}(Ai(\sigma)-f(\sigma))\,d\sigma,$$

where $f(\sigma)=-Ai(2s_{2k+1}-\sigma)$ if $\sigma\in(s_{2k+1}, r_k)$. To compare the functions $Ai(\sigma)$ and $f(\sigma)$ on the interval (s_{2k+1}, r_k), we observe that there

$$f''(\sigma)+(\sigma-2s_{2k+1})f(\sigma)=0,\ Ai''(\sigma)+(-\sigma)Ai(\sigma)=0,$$
$$-\sigma<\sigma-2s_{2k+1},\ \sigma-2s_{2k+1}>0,\ f(\sigma)>0,\ Ai(\sigma)>0,$$
$$f(s_{2k+1})=Ai(s_{2k+1})=0,\ f'(s_{2k+1})=Ai'(s_{2k+1}).$$

Hence $f(\sigma)<Ai(\sigma)$ on the same interval and (67) follows from (68).

We assert that

(69) $f_n(s)$ is positive and decreasing function if $n\geqq 1$.

Indeed, for $s\leqq 0$ it is evident. Since $f'_1(s)=-\pi 2^{-1/3}Ai^2(4^{-1/3}s)\leqq 0$ and $f'_2(s)=-f_0(s)<0$, we have (69) for $n=1,2$. Now we obtain (69) by induction, using the property $f'_n(s)=-\frac{n}{2}f_{n-2}(s)$, $n\geqq 2$.

Now the bounds (55), (56) follow from the asymptotics (64), (65) and the properties (66), (69). In particular, the function b_n satisfies (iii).

To verify (iv) with $d(\lambda, x)=\frac{1}{2}(\lambda-V(x))$, we derive from (10) the equivalence

(70) $\quad C_1B(\lambda, x)\leqq B(\lambda+\sigma, x)\leqq C_2B(\lambda, x),\ C_j>0$ if $|\sigma|\leqq d(\lambda, x)$.

Now the property (iv) for the function $b_n(h, \lambda, x)=f_{n-2}(-Bh^{-2/3})$ is a consequence of (70) and the asymptotics (64), (65) or the estimates (66), (69).

Finally, Theorem 2 follows from (62), (63), (35) by the Tauberian theorem and Remark 3.

6. Proof of Theorem 3

In the considered case, $V(x)\geqq\lambda$, there are not critical points of the phase function ψ if $V(x)>\lambda$. Therefore we can not compute the coefficient $B(\lambda, x)$ in the formula (40) as before. We shall prove only the asymptotics (10). Namely, (39), (40) and (37) show that

(71) $$(\lambda-V(x))\frac{\partial t}{\partial\tau}(0, 0, \lambda, x)=-B(\lambda, x).$$

From here and the asymptotics (10) we get

(72) $$\frac{\partial t}{\partial\tau}(0, 0, \lambda, x)=-2|V'(x)|^{-2/3}\ \text{if}\ \lambda=V(x).$$

Now (71), (72) and the Taylor formula give the asymptotics (10) as $V(x)-\lambda\to 0$.

Further, following the scheme of the proof of Theorem 2, we have to know the coefficients a_0, a_2 from (46). The Taylor formula, (61) and (10) give

(73) $\quad a_2(\lambda, x)=(2\pi)^{-n-1}2V_n|V'(x)|^{n/3}+O(V(x)-\lambda)$ as $V(x)-\lambda\to 0$.

From (46), (45), (42), (37) it follows that

(74) $$a_0=Bf_1(0, 0),\ \partial_\tau^2 g_1(0, 0)=2f_1(0, 0)+O(B).$$

According to (42), (44) we have: $g_1(\tau, 0)=a(\tau)b(\tau)$, where $a(\tau)=\frac{1}{t}\xi\partial_\xi\psi$, so (37) gives $a(0)=0$, $a'(0)=0$ and

$$a''(0)=2\frac{\partial\xi}{\partial\tau}\left[-\frac{\partial\xi}{\partial\tau}+\frac{1}{2}V'(x)\frac{\partial t}{\partial\tau}\right],\ \tau=0. \tag{75}$$

Further, from (37) we derive

$$\frac{\partial\xi}{\partial\eta}V'(x)\left(\frac{\partial t}{\partial\tau}\right)^2-2\frac{\partial\xi}{\partial\eta}\frac{\partial\xi}{\partial\tau}\frac{\partial t}{\partial\tau}=O(B)\ \text{if}\ \tau=0,\ \eta=0,$$

hence (60) and (75) show that $a''(0)=O(B)$ and (74) implies

$$a_0=O(B^2). \tag{76}$$

Note that the function $f_n(s)$ is bounded for $s\geqq 0$ and $B(\lambda, x)\leqq 0$ if $V(x)\geqq\lambda$. Therefore, instead of (57), (58) now we obtain respectively

$$I_{h,1}(\lambda, x)=\pi h^{-2n/3}[a_0f_{n-2}(-Bh^{-2/3})h^{-2/3}+a_2f_n(-Bh^{-2/3})+O(h^{2/3})], \tag{77}$$

$$I_{h,0}(\lambda, x)=O(h^{-2n/3+2/3}),\ h\to 0. \tag{78}$$

Since $|B|\leqq C.h^{2/3}$ it follows from (77), (76), (73) and (10), (11)

$$I_{h,1}(\lambda, x)=(2\pi)^{-n}V_n|V'(x)|^{n/3}f_n(-B_0h^{-2/3})h^{-2n/3}+O(h^{-2n/3+2/3}). \tag{79}$$

Thus (44), (43), (79), (78) imply

$$e_{h,\rho_T}(\lambda, x)=a_n(h, \lambda, x)h^{-2n/3}+O(h^{-2n/3+2/3}), \tag{80}$$

where the coefficient a_n is given by (13).

Analogously,

$$|e'_{h,\rho_T}(\lambda, x)|\leqq \text{const}\, h^{-2n/3+1/3}(1+O(h)),\ h\to 0. \tag{81}$$

Finally, Theorem 3 follows from (80), (81), (35) and Remark 3.

7. Proof of Theorems 4 and 5

If $V(x)\geqq\lambda+h^{2/3-\varepsilon}$, $\varepsilon>0$ then $f_n(-Bh^{-2/3})=O(h^\infty)$, therefore analogously to the proof of Theorem 3 we obtain.

$$e_{h,\rho}(\lambda, x)=O(h^\infty),\ e'_{h,\rho}(\lambda, x)=O(h^\infty), \tag{82}$$

whence the estimate (15) follows.

If $V(x)\geqq\lambda+h^{1/2-\varepsilon}$, $\varepsilon>0$, then we can integrate by parts in the integral (23), using the estimate $(\partial_t\psi)^2+(\partial_\xi\psi)^2\geqq C(V(x)-\lambda+\xi^2)^2$. Here we do not need the condition $V'(x)\neq 0$. Thus we have the estimates (82).

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Bulgarian Academy of Sciences
Institute of Mathematics with Computer Center
P.O. Box 373
1090 Sofia,
BULGARIA

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