# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 2

# On Certain $p$-Valent Functions Involving Bounded Boundary Rotation

*Mamoru Nunokawa*[+], *Shigeyoshi Owa*[++]

*Presented by Ž. Mijajlović*

The object of the present paper is to give some sufficient conditions for certain $p$-valent functions in the unit disk.

## I. Introduction

Let $\mathscr{A}(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n, \quad (p\in\mathscr{N}=\{1, 2,\ldots,\}), \tag{1.1}$$

which are analytic in the unit disk $\mathscr{U}=\{z: |z|<1\}$. A function $f(z)$ belonging to the class $\mathscr{A}(p)$ is said to be $p$-valently starlike in $\mathscr{U}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0, \quad (z\in\mathscr{U}). \tag{1.2}$$

A function $f(z)$ in $\mathscr{A}(p)$ is said to be $p$-valently close-to-convex in $\mathscr{U}$ if there exists a function $g(z)$ which is $p$-valently starlike in $\mathscr{U}$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>0, \quad (z\in\mathscr{U}). \tag{1.3}$$

Noting that $g(z)=z^p$ is $p$-valently starlike in $\mathscr{U}$, we see that a function $f(z)\in\mathscr{A}(p)$ satisfying

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>0, \quad (z\in\mathscr{U}) \tag{1.4}$$

is $p$-valently close-to-convex in $\mathscr{U}$ (cf. [4]).

Further, a function $f(z)$ in $\mathscr{A}(p)$ is said to be $p$-valently $\alpha$-spiral in $\mathscr{U}$ if it satisfies

$$\operatorname{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\}>0, \quad (z\in\mathscr{U}), \tag{1.5}$$

for some real $\alpha$ ($|\alpha|<\pi/2$). Spacek [3] has shown that $f(z)\in\mathscr{A}(1)$ when $p=1$ which is $\alpha$-spiral function in $\mathscr{U}$ is univalent in $\mathscr{U}$ (cf. [1]).
Therefore, we observe that all $p$-valently $\alpha$-spiral functions are $p$-valent in $\mathscr{U}$.

## 2. $p$-valently starlike functions

We begin with the statement and the proof of the following result.

**Theorem 1.** *If $f(z)\in\mathscr{A}(p)$ satisfies*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\right|d\theta<k\pi \tag{2.1}$$

*for $z=re^{i\theta}$, $0<|z|<1$, then*

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{k}{2}\pi \quad (z\in\mathscr{U}), \tag{2.2}$$

*where $k$ is real.*

Proof. Define the function $g(z)$ by

$$g(z)=\frac{zf'(z)}{f(z)}. \tag{2.3}$$

Then we have $g(0)=p$ and

$$\begin{aligned}\int_0^{2\pi}\left|\frac{d}{d\theta}\arg g(z)\right|d\theta&=\int_0^{2\pi}\left|\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right)\right|d\theta\\&=\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\right|d\theta<k\pi.\end{aligned} \tag{2.4}$$

It follows from (2.4) that

$$|\arg g(z)|=\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{k}{2}\pi. \tag{2.5}$$

**Corollary 1.** *If $f(z)\in\mathscr{A}(p)$ satisfies*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\right|d\theta<\pi \tag{2.6}$$

*for $z=re^{i\theta}$, $0<|z|<1$, then $f(z)$ is $p$-valently starlike in $\mathscr{U}$.*

**Theorem 2.** *If $f(z)\in\mathscr{A}(p)$ satisfies*

$$\int_0^{2\pi}\left|\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)-p\right|d\theta<k\pi \tag{2.7}$$

*for $z=re^{i\theta}$, $0<|z|<1$, then*

$$\left|\arg\frac{f(z)}{z^p}\right|<\frac{k}{2}\pi \quad (z\in\mathcal{U}), \tag{2.8}$$

*where k is real.*

Proof. Letting $g(z)=f(z)/z^p$, we see that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)=\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)-p. \tag{2.9}$$

Therefore, spending the same manner of Theorem 1, we complete the proof of the theorem.

## 3. $p$-valently close-to-convex functions

Next, we derive

**Theorem 3.** *If $f(z)\in\mathcal{A}(p)$ satisfies*

$$\int_0^{2\pi}\left|1+\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)\right|d\theta<k\pi \tag{3.1}$$

*for $z=re^{i\theta}$, $0<|z|<1$, and, for $g(z)$ which is p-valently starlike in $\mathcal{U}$, then*

$$\left|\arg\frac{zf'(z)}{g(z)}\right|<\frac{k}{2}\pi \quad (z\in\mathcal{U}), \tag{3.2}$$

*where k is real.*

Proof. We define the function $h(z)$ by

$$h(z)=\frac{zf'(z)}{g(z)}. \tag{3.3}$$

Then $h(0)=p$ and

$$\begin{aligned}\int_0^{2\pi}\left|\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right)\right|d\theta&=\int_0^{2\pi}\left|\frac{d}{d\theta}\arg h(z)\right|d\theta\\&=\int_0^{2\pi}\left|1+\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)\right|d\theta<k\pi.\end{aligned} \tag{3.4}$$

This implies that

$$|\arg h(z)|=\left|\arg\frac{zf'(z)}{g(z)}\right|<\frac{k}{2}\pi. \tag{3.5}$$

**Corollary 2.** *If $f(z)\in\mathcal{A}(p)$ satisfies*

$$\int_0^{2\pi}\left|1+\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)\right|d\theta<\pi \tag{3.6}$$

*for* $z=re^{i\theta}$, $0<|z|<1$, *and, for* $g(z)$ *which is p-valently starlike in* $\mathscr{U}$, *then* $f(z)$ *is p-valently close-to-convex in* $\mathscr{U}$.

Further, taking $g(z)=z^p$ in Theorem 3, we have

**Corollary 3.** *If* $f(z)\in\mathscr{A}(p)$ *satisfies*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)-p\right|d\theta<k\pi \tag{3.7}$$

*for* $z=re^{i\theta}$, $0<|z|<1$, *then*

$$\left|\arg\frac{f'(z)}{z^{p-1}}\right|<\frac{k}{2}\pi\ \ (z\in\mathscr{U}), \tag{3.8}$$

*where* $k$ *is real.*

## 4. p-valently α-spiral functions

Finally, we consider for $p$-valently $\alpha$-spiral functions in the unit disk. Our final result is given as follows.

**Theorem 4.** *If* $f(z)\in\mathscr{A}(p)$ *satisfies*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\right|d\theta<2\pi \tag{4.1}$$

*for* $z=re^{i\theta}$, $0<|z|<1$, *then there exists a real number* $\alpha$ ($|\alpha|<\pi/2$) *such that*

$$\mathrm{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\}>0,\ \ (z\in\mathscr{U}). \tag{4.2}$$

*Therefore,* $f(z)$ *is p-valently α-spiral function in* $\mathscr{U}$.

Proof. Noting that

$$\left(\frac{zf'(z)}{f(z)}\right)_{z=0}=p \tag{4.3}$$

for $f(z)$ satisfying (4.1), we see that there exists a real number $\alpha$ ($|\alpha|<\pi/2$) which satisfies (4.2). It follows from [2, Corollary 3] that $f(z)$ is $p$-valent in $\mathscr{U}$. Thus we complete the assertion of Theorem 3.

## References

1. A. W. Goodman. Univalent Functions. Vol. 1, Mariner Publ. Comp., Tampa, Florida, 1983.
2. S. Ogawa. On some criteria for p-valence. *J. Math. Soc., Japan*, **13**, 1961, 431-441.
3. L. Spacek. Contribution à la théorie des fonctions univalentes. *Casopis Pest. Mat.*, **61**, 1932, 12-19.
4. T. Umezawa. Multivalently close-to-convex functions. *Proc. Amer. Math. Soc.*, **8**, 1957, 869-874.

+ *Department of Mathematics*
*Gunma University*
*Maebashi, Gunma 371*
*JAPAN*

++ *Department of Mathematics*
*Kinki University*
*Higashi-Osaka, Osaka 577*
*JAPAN*

*Received 10.07.1989*