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# Some Aspects of Nonstationarity. II

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## I. Introduction

In the last decade a growing interest in so-called completion problems appeared in connection with dilations and classical extrapolation theory. Motivated by dilation theory is the following contractive completion problem.

"Given operators $A, B, C$ it is required to find conditions for the existence of an operator $D$ such that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a contraction and when these conditions are fulfilled, a parametrization of all the solutions is of interest."

For the solution of this problem see [2], [25], [6], [15], [18].

Motivated by classical extrapolation theory (Carathéodory-Féjèr problem, trigonometric moment problem) appears to be the following.

"Given a banded block-matrix $M=(S_{ij})_{i,j=1}^{n}$ (banded with bandwidth $k$ means that $S_{ij}=0$ for $|i-j|>k$) it is required to find conditions for the existence of positive completions of $M$ and to find all of these when the existence conditions are fulfilled."

This problem is solved in [14] and a more general question concerning the characterisation of all patterns admitting positive completions is solved in [16]. This kind of problems generated a large amount of work — for recent reports on some of this work, see for instance [13], [17].

It turns out that conditions for the solvability of the above mentioned problems can be obtained by general results in dilation theory and so, the parametrization of the solutions appears to be the most important thing.

Our purpose in this paper is to point out a general framework exactly for this aspect of parametrizing solutions of completion problems. Actually, we consider the following problem. Fix $N\in Z\cup\{\infty\}$, $M\in Z\cup\{-\infty\}$, $M\leq N$, where Z denotes the set of integers; I is the set of integers between M-1 and N+1.

"Given $\mathscr{E}=\{\mathscr{E}_n\}_{n\in I}$, $\mathscr{F}=\{\mathscr{F}_n\}_{n\in I}$, $\mathscr{H}=\{\mathscr{H}_n\}_{n\in I}$, families of Hilbert spaces such that $\mathscr{E}_n$ and $\mathscr{F}_n$ are subspaces of $\mathscr{H}_n$ for every $n\in I$ and a family of unitary operators $w_n \in \mathscr{L}(\mathscr{E}_{n+1}, \mathscr{F}_n)$, it is required to find all the classes of minimal unitary extensions $W_n \in \mathscr{L}(\mathscr{K}_{n+1}, \mathscr{K}_n)$ of the operators $w_n$."

This problem may be viewed as a nonstationary variant of a one which runs throughout most of the classical work of M. G. Krein in extrapolation theory — see [21], [20].

In this paper we give a solution of this problem (in Section III), we give a corresponding variant of the Krein formula for generalised resolvents (in Section IV) which produces another solution of the problem, and based on the ideas emerging from the classical work of I. Schur [28], we connect both of them in Section V. Our main tool is a certain structure of the positive block-matrices and we recall this in Section II. The last section contains the applications which cover positive and contractive completions. We discuss the band extension problem, a Sz.-Nagy-Foias model for representations of the algebra of lower triangular matrices and some variants of the lifting theorem of Sarason-Sz.-Nagy-Foias.

## II. Preliminaries

In this section we briefly recall a certain structure of positive-definite kernels on Z and two consequences regarding the structure of the Kolmogorov decomposition and a stationary embedding. Most of the notation for Hilbert space operators is taken from [29]. Thus, for two Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$, $\mathscr{L}(\mathscr{H}, \mathscr{H}')$ is the set of the linear bounded operators from $\mathscr{H}$ into $\mathscr{H}'$. $O_{\mathscr{H}}(I_{\mathscr{H}})$ is the zero (identity) operator in the underlying space. For a contraction $T \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ (i.e. $\|T\| \leq 1$), $D_T = (I - T^* T)^{1/2}$ and $\mathscr{D}_T = \overline{D_T(\mathscr{H})}$ are the defect operator and the defect space of $T$. The unitary operator

$$J(T): \mathscr{H} \oplus \mathscr{D}_{T^*} \to \mathscr{H}' \oplus \mathscr{D}_T$$

$$J(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}$$

is the elementary rotation of $T$. Finally, for a closed subspace $\mathscr{L}$ of $\mathscr{H}$, $P^{\mathscr{H}}_{\mathscr{L}}$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{L}$.

**1.** We are concerned here with the following object: for a family $\{\mathscr{H}_n\}_{n \in Z}$ of Hilbert spaces is given an application $\varphi$ on $Z \times Z$ such that $\varphi(i, j) = S_{ij} \in \mathscr{L}(\mathscr{H}_j, \mathscr{H}_i)$ and the operators

$$M_{ij}(\varphi) = M_{ij}: \bigoplus_{k=i}^{j} \mathscr{H}_k \to \bigoplus_{k=i}^{j} \mathscr{H}_k \tag{2.1}$$

$$M_{ij} = (S_{mn})_{i \leq m, n \leq j}$$

are positive for $i, j \in Z$, $i \leq j$.

The aim is to give a description of $\varphi$ by means of a family of contractions $\zeta=\{G_{ij}\}_{\substack{i,j\in Z\\ i\leq j}}$ where $C_{ii}=0_{\overline{S_{ii}\mathscr{H}_i}}$ for $i\in Z$ and for $i, j\in Z$, $i<j$, $G_{ij}:\mathscr{D}_{G_{i+1,j}}\to\mathscr{D}_{G^*_{i,j-1}}$ (and of course, when $S_{ii}$ are not supposed to be $I_{\mathscr{H}_i}$ for $i\in Z$, the family $\{S_{ii}\}_{i\in Z}$ appears also as a parameter). This problem has its roots in the work of I. Schur [28] concerning the structure of analytic contractive functions on the unit disc D and in the Szegö's theory of orthogonal polynomials on the unit circle. The contractions $G_{ij}$ are called in these contexts Schur parameters, Szegö parameters, reflection coefficients, in the intertwining dilation theory they are referred to as choice parameters, so that, from now on, we simply call them parameters. Some developments of the classical results appear in [8], [1], [11], [5], [23], [24], [12].

Here, we follow [9] in order to state the one-to-one correspondence

(2.2)
$$S_{i,i+1}=S_{ii}^{1/2}\,G_{i,i+1}\,S_{i+1,i+1}^{1/2}$$

for $i\in Z$ and for $i, j\in Z$, $j>i+1$,

(2.3)
$$S_{ij}=S_{ii}^{1/2}(R_{i,j-1}\,U_{i+1,j-1}\,C_{i+1,j}$$
$$+D_{G^*_{i,i+1}}\dots D_{G^*_{i,j-1}}\,G_{ij}\,D_{G_{i+1,j}}\dots D_{G_{j-1,j}})\,S_{jj}^{1/2}$$

between $\varphi$ and $\zeta$. Let us explain the notation. For a fixed $i\in Z$, the family $\{G_{ik}\}_{i<k}$ defines a row contraction

(2.4)
$$R_i:\bigoplus_{k\geq i+1}\mathscr{D}_{G_{i+1,k}}\to\mathscr{H}_i$$
$$R_i=(G_{i,i+1},\ D_{G^*_{i,i+1}}\,G_{i,i+2},\dots)$$

and when $j>i$, $R_{ij}$ is the restriction of $R_i$ to $\bigoplus_{k=i+1}^{j}\mathscr{D}_{G_{i+1,\cdot}}$. By an obvious duality there are defined the column contractions $C_j$ and $C_{ij}$.

The unitary operators $U_{ij}$ are given by:

(2.5)
$$U_{ii}=I_{\mathscr{H}_i}$$

for $i\in Z$ and for $j>i$,

(2.6)
$$U_{ij}:\bigoplus_{k=-j}^{-i}\mathscr{D}_{G_{-k,j}}\to\bigoplus_{k=i}^{j}\mathscr{D}_{G_{ik}}$$
$$U_{ij}=J_j(G_{i,i+1})\,J_j(G_{i,i+2})\dots J_j(G_{ij})(U_{i+1,j}\oplus I_{\mathscr{D}_{G^*_{ij}}})$$

where the subscript $j$ at $J(G_{i,i+k})$ means that the elementary rotation of $G_{i,i+k}$ was extended with identity on corresponding spaces. The formulas (2.2) and (2.3) show that without loss of generality we can suppose $S_{ii}=I_{\mathscr{H}_i}$. Moreover, they give the structure of any positive block-matrix $M_{ij}$.

2. A first by-product of the above structure of $\varphi$ is an explicit description of its Kolmogorov decomposition. The construction goes through the following steps and comes from the remark that it is essentially given by the elementary rotations of $R_i$, $i\in\mathbf{Z}$. So that, the first step is to give an adequate identification of the defect spaces of a contraction of the type of $R_i$, i.e.

$$T=(T_1,\ D_{T_1^*}T_2,\ldots):\mathscr{H}\left(=\bigoplus_{k=1}^{\infty}\mathscr{H}_k\right)\to\mathscr{H}'$$

with $T_1$ a contraction in $\mathscr{L}(\mathscr{H}_1,\mathscr{H}')$ and $T_k$ contractions in $\mathscr{L}(\mathscr{H}_k,\mathscr{D}_{T_{k-1}^*})$. For $k\geqq 1$, we define the operators:

$$D_k(T):\bigoplus_{j=1}^{k}\mathscr{H}_j(=\mathscr{H}^{[k]})\to\bigoplus_{j=1}^{k}\mathscr{D}_{T_j}$$

$$D_k(T)=\begin{bmatrix} D_{T_1} & -T_1^*T_2 & \ldots & -T_1^*D_{T_2^*}\ldots D_{T_{k-1}^*}T_k \\ 0 & D_{T_2} & \ldots & -T_2^*D_{T_3^*}\ldots D_{T_{k-1}^*}T_k \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & D_{T_k} \end{bmatrix} \tag{2.7}$$

and

$$D_\infty(T):\mathscr{H}\to\bigoplus_{j=1}^{\infty}\mathscr{D}_{T_j}=\mathscr{D}(T) \tag{2.8}$$
$$D_\infty(T)=s\text{-}\lim_{k\to\infty}D_k(T)P^{\mathscr{H}}_{\mathscr{H}^{[k]}}$$

where $s\text{-}\lim_{k\to\infty}$ means the strong operatorial limit. The operator

$$\alpha(T):\mathscr{D}_T\to\mathscr{D}(T) \tag{2.9}$$
$$\alpha(T)D_T=D_\infty(T)$$

is a unitary one and gives an explicit description of the defect space $\mathscr{D}_T$. For identifying the space $\mathscr{D}_{T^*}$, we define

$$H_\infty(T):\mathscr{H}'\to\mathscr{H}' \tag{2.10}$$
$$H_\infty(T)=(s\text{-}\lim_{k\to\infty}D_{T_1^*}\ldots D^2_{T_k^*}\ldots D_{T_1^*})^{1/2}\,.$$

and the operator

$$(2.11)\qquad \begin{gathered}\beta(T):\mathscr{D}_{T^*}\to\overline{H_\infty(T)(\mathscr{H})}=\mathscr{D}_*(T)\\ \beta(T)D_{T^*}=H_\infty(T)\end{gathered}$$

is a unitary one. The next step consists in considering

$$(2.12)\qquad \begin{gathered}W(T):\mathscr{D}_*(T)\oplus\mathscr{H}\to\mathscr{H}\oplus\mathscr{D}(T)\\ W(T)=\begin{bmatrix} I & 0\\ 0 & \alpha(T)\end{bmatrix}J(T)\begin{bmatrix}0 & I\\ \beta(T) & 0\end{bmatrix}\end{gathered}$$

and taking into account this operator for the row contractions $R_i$ given by (2.4). Consequently, we define the spaces

$$(2.13)\qquad \mathscr{K}_i=\bigoplus_{j=-\infty}^{i-1}\mathscr{D}_*(R_j)\oplus\mathscr{H}_i\oplus\mathscr{D}(R_i)$$

and the unitary operators

$$(2.14)\qquad \begin{gathered}W_i:\mathscr{K}_{i+1}\to\mathscr{K}_i\\ W_i=I\oplus W(R_i)\end{gathered}$$

with respect to obvious decompositions of the spaces.

Finally, the Kolmogorov decomposition of $\varphi$ is given by $\mathscr{U}(\varphi)=\mathscr{U}=\{V(n)\}_{n\in Z}$,

$$V(n):\mathscr{H}_n\to\mathscr{K}_0$$

$$(2.15)\qquad V(n)=\begin{cases} W^*_{-1}W^*_{-2}\dots W^*_n/\mathscr{H}_n & n<0\\ P^{\mathscr{K}_0}_{\mathscr{H}_0}/\mathscr{H}_0 & n=0\\ W_0W_1\dots W_{n-1}/\mathscr{H}_n & n>0\end{cases}$$

in the sense that $S_{ij}=V(i)^*V(j)$ for $i, j\in Z$ and $\mathscr{K}_0=\bigvee_{n\in Z}V(n)\mathscr{H}_n$ — for the above construction see [9].

Of course, the last minimality condition yealds a natural unicity in the sense that if $W'_n\in\mathscr{L}(\mathscr{K}'_{n+1},\mathscr{K}'_n)$ is another family of unitary operators satisfying: $\mathscr{H}_n\subset\mathscr{K}'_n$, $S_{ij}=V'(i)^*V'(j)$ $i, j\in Z$ and $\mathscr{K}'_0=\bigvee_{n\in Z}V'(n)\mathscr{H}_n$, then there are unitary operators $\varphi_n:\mathscr{K}_n\to\mathscr{K}'_n$ such that $\varphi_n/\mathscr{H}_n=I_{\mathscr{H}_n}$ and $W'_n\varphi_{n+1}=\varphi_nW_n$.

3. Another by-product of the structure of $\varphi$ given by (2.2) and (2.3) is a certain stationary embedding. The first remark is that when the parameters $G_{ij}$ satisfy $G_{ij}=G_{i+k,\,j+k}$ for $k\in\mathbf{Z}$ then the associated kernel is Toeplitz. So that, defining $\tilde{\mathscr{H}}=\bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n$ $\tilde{G}_0=O_{\tilde{\mathscr{H}}}$ and for $n\geqq 1$,

$$\text{(2.16)}\qquad \begin{gathered}\tilde{G}_n:\mathscr{D}_{\tilde{G}_{n-1}}\to\mathscr{D}_{\tilde{G}^*_{n-1}}\\ (\tilde{G}_n)_{ij}=\begin{cases}G_{ij} & j=i+n,\quad n\in\mathbf{Z}\\ 0 & \text{in rest}\end{cases}\end{gathered}$$

the sequence $\{\tilde{G}_n\}_{n\geq 0}$ will produce by formulas (2.2) and (2.3) a Toeplitz kernel $\tilde{\varphi}$. If $\tilde{W}\in\mathscr{L}(\tilde{\mathscr{K}})$ is the unitary operator obtained by (2.14) (i.e. the Naimark dilation of $\tilde{\varphi}$) then

$$\text{(2.17)}\qquad V(n)=P^{\tilde{\mathscr{K}}}_{\mathscr{K}_0}\tilde{W}^n/\mathscr{H}_n$$

(with natural embeddings of $\mathscr{H}_n$ and $\mathscr{K}_0$ into $\tilde{\mathscr{K}}$). This construction is referred to as the stationary embedding of the kernel $\varphi$.

In view of this stationary embedding we have a suggestion for defining for non-Toeplitz kernels objects which are usually associated with Toeplitz kernels. More than that, the possibility to deduce the structure of a positive kernel from the structure for positive Toeplitz kernels appears. As it is easy to see, this is indeed the case and let us discuss some details in a simple case. That is, take two operators $S_{12}$ and $S_{23}$ and ask for an $S_{13}$ such that

$$\begin{bmatrix} I & S_{12} & S_{13}\\ S^*_{12} & I & S_{23}\\ S^*_{13} & S^*_{23} & I\end{bmatrix}$$

is positive — this being a band extension problem. The stationary embedding suggests us to take

$$S_1=\begin{bmatrix} 0 & S_{12} & 0\\ 0 & 0 & S_{23}\\ 0 & 0 & 0\end{bmatrix}$$

which is also a contraction when $S_{12}$ and $S_{23}$ are supposed, as necessary, to be contractions. We obtain the band extension problem for the band matrix

$$\begin{bmatrix} I & S_1 & \\ S_1^* & I & S_1 \\ & S_1^* & I \end{bmatrix}$$

By the trigonometric moment theory such an extension does exist, but it is necessary to support a certain structure. This can be obtained by taking into account the formula (2.3) and the special structure of $S_1$. Roughly speaking, the nonstationary case is the stationary one plus a "marking analysis". This convince us to keep the stationary embedding as a suggestion and to make the computations directly.

## III. A solution of the main problem

In this section we indicate a solution of the third problem in Introduction. First, let us specify the notions involved in the statement of this problem. So, fix $N \in \mathsf{Z} \cup \{\infty\}$, $M \in \mathsf{Z} \cup \{-\infty\}$, $M \leqq N$.

Define $\mathsf{I} = \{n \in \mathsf{Z} / M-1 < n < N+1\}$, $\mathsf{J} = \{n \in \mathsf{Z} / M-1 < n < N\}$, $\mathsf{K} = \{n \in \mathsf{Z} / M < n < N+1\}$ and without loss of generality, we suppose $0 \in J$. Fix the families of Hilbert spaces $\mathscr{E} = \{\mathscr{E}_n\}_{n \in \mathsf{I}}$, $\mathscr{F} = \{\mathscr{T}_n\}_{n \in \mathsf{I}}$ and $\mathscr{H} = \{\mathscr{H}_n\}_{n \in \mathsf{I}}$ such that $\mathscr{E}_n$ and $\mathscr{F}_n$ are subspaces of $\mathscr{H}_n$ for every $n \in \mathsf{I}$ and $w = \{w_n\}_{n \in \mathsf{J}}$ is a family of unitary operators, $w_n \in \mathscr{L}(\mathscr{E}_{n+1}, \mathscr{F}_n)$ for $n \in \mathsf{J}$. Consider now a family $W = \{W_n\}_{n \in \mathsf{J}}$ of unitary operators, $W_n \in \mathscr{L}(\mathscr{K}_{n+1}, \mathscr{K}_n)$, $\mathscr{K}_n \supset \mathscr{H}_n$, $W_n$ extending $w_n$ for every $n \in \mathsf{J}$.

For $n \in \mathsf{I}$, we define

$$V(n) : \mathscr{H}_n \to \mathscr{K}_0$$

$$(3.1) \qquad V(n) = \begin{cases} W_{-1}^* W_{-2}^* \dots W_n^* / \mathscr{H}_n, & M-1 < n < 0 \\ P_{\mathscr{H}_0}^{\mathscr{K}_0} / \mathscr{H}_0 \quad , & n = 0 \\ W_0 W_1 \dots W_{n-1} / \mathscr{H}_n, & N+1 > n > 0 \end{cases}$$

and the family $W = \{W_n\}_{n \in \mathsf{J}}$ is called minimal if

$$(3.2) \qquad \bigvee_{n \in \mathsf{I}} V(n) \mathscr{H}_n = \mathscr{K}_0 .$$

Then, the minimal unitary extensions $W = \{W_n\}_{n \in \mathsf{J}}$ and $W' = \{W'_n\}_{n \in \mathsf{J}}$ of the family $w = \{w_n\}_{n \in \mathsf{J}}$ are equivalent if there exists a family of unitary operators $\{\varphi_n\}_{n \in \mathsf{I}}$, $\varphi_n \in \mathscr{L}(\mathscr{K}_n, \mathscr{K}'_n)$ such that $\varphi_n / \mathscr{H}_n = \mathrm{I}_{\mathscr{H}_n}$ and $W'_n \varphi_{n+1} = \varphi_n W_n$ for $n \in \mathsf{J}$. We denote by $E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$ the classes of equivalent minimal unitary extensions of $w$, and this is the set we are interested in.

Define $\mathscr{P}_n = \mathscr{H}_n \ominus \mathscr{E}_n$, $\mathscr{R}_n = \mathscr{H}_n \ominus \mathscr{F}_n$, $n \in \mathsf{I}$. We can prove now the main result of this section.

**3.1. Theorem.** *There exists a one-to-one correspondence between $E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$ and the set $\pi$ of parameters $\zeta$ with $G_{n,n+1} \in \mathscr{L}(\mathscr{P}_{n+1}, \mathscr{R}_n)$ for $n \in \mathrm{J}$.*

Proof. Consider $\zeta$ with $G_{n,n+1} \in \mathscr{L}(\mathscr{P}_{n+1}, \mathscr{R}_n)$ for $n \in \mathrm{J}$. Then, for any $n \in \mathrm{J}$ we define the row contractions $R_n$ associated with $\{G_{nk}\}_{k \in \mathrm{K}}$ by formula (2.4); then, for $n \in \mathrm{K}$

$$\mathscr{H}_n^1 = \mathscr{D}_*(R_M) \oplus \ldots \oplus \mathscr{D}_*(R_{n-1}) \oplus \mathscr{P}_n \oplus \mathscr{D}_{G_{n,n+1}} \oplus \ldots \oplus \mathscr{D}_{G_{nN}} \tag{3.3}$$

(for finte $N$ and $n=N$, there is no space on the right of $\mathscr{P}_N$) and for $n \in \mathrm{J}$,

$$\mathscr{H}_n^2 = \mathscr{D}_*(R_M) \oplus \ldots \oplus \mathscr{D}_*(R_{n-1}) \oplus \mathscr{R}_n \oplus \mathscr{D}_{G_{n,n+1}} \oplus \ldots \oplus \mathscr{D}_{G_{nN}} \tag{3.4}$$

(for finte $M$ and $n=M$, there is no space on the left of $\mathscr{R}_M$).
Define for $n \in \mathrm{J}$ the unitary operators

$$\begin{gathered} W'_n : \mathscr{H}_{n+1}^1 \to \mathscr{H}_n^2 \\ W'_n = I \oplus \begin{bmatrix} I & 0 \\ 0 & \alpha(R_n) \end{bmatrix} J(R_n) \begin{bmatrix} 0 & I \\ \beta^*(R_n) & 0 \end{bmatrix}, \end{gathered} \tag{3.5}$$

where $\alpha(R_n)$ and $\beta(R_n)$ are given by (2.9) and (2.11).
Finally, define for $n \in \mathrm{I}$

$$\mathscr{H}_n = \begin{cases} \mathscr{H}_n^1 \oplus \mathscr{E}_n & n \in \mathrm{K} \\ \mathscr{H}_n^2 \oplus \mathscr{F}_n & n \in \mathrm{J} \end{cases} \tag{3.6}$$

and for $n \in \mathrm{J}$,

$$\begin{gathered} W_n : \mathscr{H}_{n+1} \to \mathscr{R}_n \\ W_n = W'_n \oplus w_n. \end{gathered} \tag{3.7}$$

We readily verify that $W = \{W_n\}_{n \in \mathrm{J}}$ is in $E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$. So, we can define the map

$$\begin{gathered} \Phi : \pi \to E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w) \\ \Phi(\zeta) = \mathrm{class}(W) \end{gathered} \tag{3.8}$$

$W$ being given by (3.7).
Supposing that $\Phi(\zeta) = \Phi(\zeta')$, there exists a family $\{\varphi_n\}_{n \in \mathrm{I}}$, $\varphi_n \in \mathscr{L}(\mathscr{H}_n, \mathscr{H}'_n)$ of unitary operators with $\varphi_n / \mathscr{H}_n = I_{\mathscr{H}_n}$ and $W'_n \varphi_{n+1} = \varphi_n W_n$ for $n \in \mathrm{J}$. For $i \in \mathrm{J}$, $j \in \mathrm{K}$, $i<j$,

$$P^{\mathscr{H}_i}_{\mathscr{H}_i} W_i \ldots W_{j-1} / \mathscr{H}_j = P^{\mathscr{H}_i}_{\mathscr{H}_i} W_i \ldots W_{j-1} \varphi_j^* / \mathscr{H}_j$$

$$=P^{\mathcal{K}_i}_{\mathcal{K}_i}\varphi_i^* W'_i \dots W'_{j-1}/\mathcal{H}_j = P^{\mathcal{K}'_i}_{\mathcal{K}_i} W'_i \dots W'_{i-1}/\mathcal{H}_j.$$

In view of (2.14) and (2.2) and (2.3), we obtain $\zeta=\zeta'$.

Then, let $W\in E(\mathcal{E},\mathcal{F},\mathcal{H};w)$. Define for $n\in J$, $S_{n,n+1}=P^{\mathcal{K}_n}_{\mathcal{K}_n} W_n/\mathcal{H}_{n+1}$, then $S_{n,n+1}=w_n\oplus G_{n,n+1}$ where $G_{n,n+1}$ is a contraction in $\mathcal{L}(\mathcal{P}_{n+1},\mathcal{R}_n)$. Then, we define for $i\in J, j\in K, i<j$, $S_{ij}=P^{\mathcal{K}_i}_{\mathcal{K}_i} W_i\dots W_{j-1}/\mathcal{H}_j$ and if $S_{ii}=I_{\mathcal{K}_i}$, $i\in I$ then $\{S_{ij}\}_{i,j\in I}$ are the coefficients of a positive-definite kernel on I. Let $\zeta'=\{G'_{ij}\}_{i\in J, j\in K}$ be its parameter given by (2.2) and (2.3), then $G'_{n,n+1}=w_n\oplus G_{n,n+1}$ for $n\in J$ and

$$\mathcal{D}_{G'_{n,n+1}}=0\oplus\mathcal{D}_{G_{n,n+1}} \quad n\in J$$

$$\mathcal{D}_{G'^*_{n,n+1}}=0\oplus\mathcal{D}_{G^*_{n,n+1}} \qquad n\in J.$$

Consequently, $\zeta=\{G_{n,n+1}\}_{n\in J}\cup\{G'_{ij}\}_{\substack{i\in J, j\in K\\ i<j+1}}$ belongs to $\pi$ and $\Phi(\zeta)=W$. That is, $\Phi$ is a one-to-one correspondence between $\pi$ and $E(\mathcal{E},\mathcal{F},\mathcal{H};w)$. ■

**3.2. Corollary.** *$E(\mathcal{E},\mathcal{F},\mathcal{H};w)$ contains only one element if and only if $\mathcal{P}_n=0$ for all $n\in K$ or $\mathcal{R}_n=0$ for all $n\in J$.* ■

## IV. Krein type formula

An extension in $E(\mathcal{E},\mathcal{F},\mathcal{H};w)$ is quite an abstract object; on the other hand, its parameter is quite intrinsic. So that, another "observable" element must be added to the scheme. Usually, i. e. for the stationary variant of the problem under consideration, this object is the so-called generalized resolvent — see [21]. Equally useful is the so-called generalized coresolvent — see [22], because this function belongs to a Carathéodory class and so, it mediates between a trigonometric moment problem and a Schur problem — see [20].

In this section we also choose to associate to an extension in $E(\mathcal{E},\mathcal{F},\mathcal{H};w)$ an object resembling the generalized coresolvent.

We begin by introducing the marking operators. Let be given a family $\mathcal{L}=\{\mathcal{L}_n\}_{n\in I}$ of Hilbert spaces, for $i, j\in I$, $i\leq j$, we define

(4.1) $$\mathcal{L}_{[ij]}=\bigoplus_{k=i}^{j}\mathcal{L}_k.$$

For a family of operators $\{T_n\}_{n\in J}$, $T_n\in\mathcal{L}(\mathcal{L}_{n+1},\mathcal{L}_n)$, for $i\in J$, $j\in K$, $i\leq j$, we define

(4.2) $$T_{[ij]}:\mathcal{L}_{[i+1,j+1]}\to\mathcal{L}_{[ij]}$$

$$T_{[ij]}=\oplus_{k=i}^{j}T_k$$

Now, for $i\in J$, $j\in K$, $i<j$, the marking operators are defined by

$$m_{[ij]}=m_{[ij]}(\mathscr{L});\ \mathscr{L}_{[ij]}\to\mathscr{L}_{[i+1,j+1]} \tag{4.3}$$
$$m_{[ij]}(l_i,l_{i+1},\ldots,l_j)=(l_{i+1},\ldots,l_j,0)$$

Consider $W\in E(\mathscr{E},\mathscr{F},\mathscr{H};w)$, the generalized coresolvent of $W$ is defined for $i\in J$, $j\in K$, $i<j$, by the formula:

$$b_{ij}(W)=P^{\mathscr{X}}_{\mathscr{X}[ij]}(I+W_{[ij]}m_{[ij]})(I-W_{[ij]}m_{[ij]})^{-1}/\mathscr{H}_{[ij]}, \tag{4.4}$$

the invertibility of $(I-W_{[ij]}m_{[ij]})$ on $\mathscr{H}_{[ij]}$ being obvious.
We can decompose every $W_n$ as

$$W_n=\begin{bmatrix}\hat W_n & \hat C_n\\ \hat B_n & \hat A_n\end{bmatrix}:\mathscr{H}_{n+1}\oplus(\mathscr{K}_{n+1}\ominus\mathscr{H}_{n+1})\to\mathscr{H}_n\oplus(\mathscr{K}_n\ominus\mathscr{H}_n)$$

and we take into consideration the extension $W^0$ given by the parameter $\zeta^0=\{0_{ij}\}$.
We can prove the main result of this section.

**4.1. Proposition.** *For* $W\in E(\mathscr{E},\mathscr{F},\mathscr{H};w)$, $i\in J$, $j\in K$, $i<j$,

$$b_{[ij]}(W)=b_{[ij]}(W^0)+2(I-\hat W^0_{[ij]}m_{[ij]})^{-1}\hat E_{[ij]}m_{[ij]}$$
$$(I-(I-\hat W^0_{[ij]}m_{[ij]})^{-1}\hat E_{[ij]}m_{[ij]})^{-1}(I-W^0_{[ij]}m_{[ij]})^{-1}$$

*where for* $i\in J$, $j\in K$, $j>i$,

$$\hat E_{[ij]}=\hat D_{[ij]}+\hat C_{[ij]}m_{[ij]}(I-\hat A_{[ij]}m_{[ij]})^{-1}B_{[ij]}$$

*and for* $n\in J$,

$$\hat D_n=\hat W_n-\hat W^0_n.$$

Proof. For simplifying the notation we suppress the indices $i$, $j$.
That is, we can write:

$$b_{[ij]}(W)=-I_{\mathscr{H}[ij]}+2P^{\mathscr{X}}_{\mathscr{X}[ij]}(I-W_{[ij]}m_{[ij]})^{-1}/\mathscr{H}_{[ij]}$$
$$=-I+2P^{\mathscr{X}}_{\mathscr{X}}\begin{bmatrix}I-\hat Wm & -\hat Cm\\ -\hat Bm & I-\hat Am\end{bmatrix}^{-1}/\mathscr{H}=-I+2((I-\hat Wm)^{-1}$$
$$+(I-\hat Wm)^{-1}\hat Cm(I-\hat Am-\hat Bm(I-\hat Wm)^{-1}\hat Cm)^{-1}\hat Bm(I-\hat Wm)^{-1})=-I$$
$$+2((I-\hat Wm)^{-1}+(I-\hat Wm)^{-1}\hat Cm(I-\hat Am)^{-1}\hat Bm(I-(I-\hat Wm)^{-1}\hat Cm(I-\hat Am)^{-1}$$
$$\hat Bm)^{-1}(I-\hat Wm)^{-1}.$$

On the other hand,

$$b_{[ij]}(W^0) = -I_{\mathscr{X}_{[ij]}} + 2P^{\mathscr{X}_{[ij]}}_{\mathscr{X}_{[ij]}}(I - W^0_{[ij]} m_{[ij]})^{-1}/\mathscr{H}_{[ij]} = +I + 2(I - \hat{W}^0 m)^{-1}$$

such that

$$b_{[ij]}(W) - b_{[ij]}(W^0) = 2((I - \hat{W}m)^{-1} - (I - \hat{W}^0 m)^{-1}$$
$$+ (I - \hat{W}m)^{-1} \hat{C}m(I - \hat{A}m)^{-1} \hat{B}m(I - (I - \hat{W}m)^{-1} \hat{C}m(I - \hat{A}m)^{-1} \hat{B}m)^{-1}(I - \hat{W}m)^{-1}.$$

Now, with the notation

$$\hat{D}_n = \hat{W}_n - \hat{W}^0_n, \quad n \in \mathrm{J}$$

we have

$$(I - \hat{W}m)^{-1} - (I - \hat{W}^0 m)^{-1} = (I - \hat{W}^0 m)^{-1} \hat{D}m(I - \hat{W}m)^{-1}$$

which gives

$$(I - \hat{W}m)^{-1} = (I - (I - \hat{W}^0 m)^{-1} \hat{D}m)^{-1}(I - \hat{W}^0 m)^{-1} = (I - \hat{W}^0 m)^{-1}(I - \hat{D}m(I - \hat{W}^0 m)^{-1})^{-1}.$$

Then, we can continue to compute

$$b_{[ij]}(W) - b_{[ij]}(W^0) = (I - \hat{W}^0 m)^{-1}(\hat{D}m(I - (I - \hat{W}^0 m)^{-1} \hat{D}m)^{-1}$$
$$+ (I - \hat{D}m(I - \hat{W}^0 m)^{-1})^{-1} \hat{C}m(I - \hat{A}m)^{-1} \hat{B}m(I - ((I - \hat{W}^0 m)^{-1} \hat{D}m)^{-1}(I - \hat{W}^0 m)^{-1} \hat{C}m(I - \hat{A}m)^{-1} \hat{B}m)^{-1}$$
$$(I - (I - \hat{W}^0 m)^{-1} \hat{D}m)^{-1}(I - \hat{W}^0 m)^{-1} = (I - \hat{W}^0 m)^{-1}(I - \hat{D}m(I - \hat{W}^0 m)^{-1})^{-1}$$
$$(\hat{D}m + \hat{C}m(I - \hat{A}m)^{-1} \hat{B}m(I - (I - \hat{W}^0 m)^{-1} \hat{E}m)^{-1})(I - \hat{W}^0 m)^{-1}$$
$$= (I - \hat{W}^0 m)^{-1}(I - \hat{D}m(I - \hat{W}^0 m)^{-1})^{-1}$$
$$(\hat{D}m(I - (I - \hat{W}^0 m)^{-1} \hat{E}m) + \hat{C}m(I - \hat{A}m)^{-1} \hat{B}m)$$
$$(I - (I - \hat{W}^0 m)^{-1} \hat{E}m)^{-1}(I - \hat{W}^0 m)^{-1} = (I - \hat{W}^0 m)^{-1}(I - \hat{D}m(I - \hat{W}^0 m)^{-1})^{-1}$$
$$(\hat{E}m - \hat{D}m(I - \hat{W}^0 m)^{-1} \hat{E}m)(I - (I - \hat{W}^0 m)^{-1} \hat{E}m)^{-1}(I - \hat{W}^0 m)^{-1}$$
$$= (I - \hat{W}^0 m)^{-1} \hat{E}m(I - (I - \hat{W}^0 m)^{-1} \hat{E}m)^{-1}(I - \hat{W}^0 m)^{-1}.$$

Taking the indices into account this is exactly the formula to be proved. ■

Now, let $\zeta$ be the parameter of $W$. Define for $n \in \mathrm{J}$, (4.5) $$D_n = G_{n,n+1}$$

(4.6) $$B_n = (\dots 0, \dots, D_{G_{n,n+1}}, 0, \dots)^t$$

("t" meaning the matrix transpose),

(4.7) $$C_n = (\dots 0,\ H_\infty(R_n), D_{G^*_{n,n+1}} G_{n,n+2}, \dots)$$

(again, for $n = N-1$, there is no term on the right of $H_\infty(R_n)$),

(4.8) $A_n$ = what remains in $W'_n$ given by (3.5) after deleting

$D_n$, $B_n$ and $C_n$.

The remark now is that $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$, $\hat{D}_n$ are extensions of $A_n$, $B_n$, $C_n$, $D_n$ with 0 on corresponding spaces, and the formula in Proposition 4.1 becomes:

(4.9) $$b_{[i,j]}(W) = b_{[i,j]}(W^0)$$

$$+2(I - \hat{W}^0_{[i,j]} m_{[i,j]})^{-1} P_{\mathscr{R}_{[i,j]}} E_{[i,j]} m_{[i,j]} (I - X_{[i,j]} E_{[i,j]} m_{[i,j]})^{-1} P_{\mathscr{P}_{[i,j]}} (I - \hat{W}^0_{[i,j]} m_{[i,j]})^{-1},$$

where

(4.10)
$$E_{[i,j]} : \mathscr{P}_{[i+1,\,j+1]} \to \mathscr{R}_{[i,j]}$$
$$E_{[i,j]} = D_{[i,j]} + C_{[i,j]} m_{[i,j]} (I - A_{[i,j]} m_{[i,j]})^{-1} B_{[i,j]}$$

and

(4.11)
$$X_{[i,j]} : \mathscr{R}_{[i,j]} \to \mathscr{P}_{[i,j]}$$
$$X_{[i,j]} = P_{\mathscr{P}_{[i,j]}} (I - W^0_{[i,j]} m_{[i,j]})^{-1} / \mathscr{R}_{[i,j]}.$$

We will refer to the formula (4.9) as the generalized coresolvent formula.

## V. Schur analysis

In this section we analyse some properties of the operators $E_{[i,j]}$ given by (4.10). For this, we will need some other consequences of the structure of positive-definite kernels on Z given by (2.2) and (2.3).

**1.** Let us consider two families $\mathscr{H} = \{\mathscr{H}_n\}_{n\in K}$, $\mathscr{H}' = \{\mathscr{H}'_n\}_{n\in J}$ of Hilbert spaces and the upper triangular contraction

(5.1)
$$T : \mathscr{H}_K \left(= \bigoplus_{n\in K} \mathscr{H}_n\right) \to \mathscr{H}'_J \left(= \bigoplus_{n\in J} \mathscr{H}'_n\right)$$
$$T = (T_{ij})_{i\in J,\, j\in K}$$

with $T_{ij}=0$ for $i>j$.

Based on the remark that $T$ is a contraction if and only if $\begin{bmatrix} I & T \\ T^* & I \end{bmatrix}$ is positive and using (2.2) and (2.3) we obtain that there exists a one-to-one correspondence between the set of contractions of type (5.1) and the set $\pi$ of parameters $\zeta$ with $G_{n,n+1}\in\mathscr{L}(\mathscr{H}_{n+1},\mathscr{H}'_n)$ given by the formulas: for $n\in J$,

$$T_{nn}=G_{n,n+1} \tag{5.2}$$

and for $i\in J$, $j>i$,

$$T_{ij}=R_{i,j-1}Q_{i+1,j-1}C_{i+1,j} + D_{G^*_{i,i+1}}\dots D_{G^*_{i,j-1}}G_{ij}D_{G_{i+1,j}}\dots D_{G_{j-1,j}}, \tag{5.3}$$

where

$$Q_{nn}=0:\mathscr{H}'_n\to\mathscr{H}_n \quad n\in K$$

and for $i\in K$, $j>i$,

$$\begin{aligned} &Q_{ij}:\mathscr{H}'_j\oplus\mathscr{D}_{G_{j-1,j}}\oplus\dots\oplus\mathscr{D}_{G_{ij}} \\ &\to\mathscr{H}_i\oplus\mathscr{D}_{G^*_{i,i+1}}\oplus\dots\oplus\mathscr{D}_{G^*_{ij}} \end{aligned} \tag{5.4}$$

$$Q_{ij}=\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} J_j(G_{i,i+1})J_j(G_{i,i+2})\dots J_j(G_{ij})\begin{bmatrix} Q_{i+1,j} & 0 \\ 0 & I_{\mathscr{D}_{G^*_{ij}}} \end{bmatrix}$$

(for details see [4]). When the stationary case (i.e. $T$ is Toeplitz) is taken into account, the above relations (in the scalar case) follow from the algorithm of Schur in [28].

**2.** Consider the linear, time variant system:

$$\begin{cases} x_n=A_nx_{n+1}+B_nu_{n+1} \\ y_n=C_nx_{n+1}+D_nu_{n+1} \end{cases} \quad n\in J, \tag{5.5}$$

where $u_n\in\mathscr{H}_n$, $y_n\in\mathscr{H}'_n$, $x_n\in\mathscr{X}_n$, such that

$$\begin{aligned} &\Lambda_n:\mathscr{X}_{n+1}\oplus\mathscr{H}_{n+1}\to\mathscr{X}_n\oplus\mathscr{H}_n \\ &\Lambda_n=\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \end{aligned} \tag{5.6}$$

is unitary for every $n\in J$.

Its transfer operator is the upper triangular contraction

$$
\begin{aligned}
&T=(T_{ij}):\mathscr{H}_{\mathrm{K}}\to\mathscr{H}'_{\mathrm{J}}\\
&T_{ii}=D_i,\ i\in \mathrm{J}\\
(5.7)\qquad &T_{i,i+1}=C_iB_{i+1}, \qquad\qquad , i\in\mathrm{J}\\
&T_{ij}=C_iA_{i+1}\ldots A_{j-1}B_j,\quad i\in\mathrm{J},\ j>i+1\\
&T_{ij}=0, \qquad\qquad\qquad , i<j
\end{aligned}
$$

Conversely, every contraction $T$ of the form (5.1) can be realized in the form (5.7), i.e. it is the transfer operator of a unitary system of the form (5.5). Indeed, taking into account the parameter $\zeta$ of $T$ associated by (5.2) and (5.3), the realization is exactly given by the formulas (4.5), (4.6), (4.7) and (4.8). For details, see also [4]—for the stationary case see [19].

3. We can prove now the main result of this section.

**5.1. Theorem.** *There exists a one-to-one correspondence between $E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$ and the set $\tau$ of upper triangular contractions in $\mathscr{L}(\mathscr{P}_{\mathrm{K}}, \mathscr{R}_{\mathrm{J}})$, and this correspondence is given by the generalized coresolvent formula.*

Proof. By the generalized coresolvent formula, we have associated with an $W$ in $E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$ a unique element in $\tau$ given by (4.10). Conversely, for $E\in\tau$, let $\zeta$ be its parameter given by (5.2 and 5.3). Let $W\in E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$ associated to $\zeta$ by Theorem 3.1. In view of the above mentioned realization result, the generalized coresolvent formula for $W$ will produce exactly the given $E$.■

Let us consider now a positive kernel $\varphi$ on $\mathrm{I}\times\mathrm{I}$ (with $S_{ii}=I_{\mathscr{H}_i}$). Let $\zeta$ be its parameter and let $E$ be the upper triangular contraction associated by (5.2) and (5.3) to $\zeta$. Let $W=\{W_n\}_{n\in\mathrm{J}}$ be the Kolmogorov decomposition of $\varphi$ and choosing $\mathscr{E}_n=\mathscr{F}_n=0$, we can use the generalized coresolvent formula and Theorem 5.1 to get for $i\in\mathrm{J}$, $j\in\mathrm{K}$, $i<j$:

$$(5.8)\qquad b_{[ij]}(W)=(I+E_{[ij]}m_{[ij]})(I-E_{[ij]}m_{[ij]})^{-1}.$$

Moreover, for $M_{ij}(\varphi)$ given by (2.1), we have

$$(5.9)\qquad M_{ij}(\varphi)=\frac{1}{2}(b_{[ij]}(W)+b_{[ij]}(W)^*)$$

$$=(I-m^*_{[ij]}E^*_{[ij]})^{-1}(I-m^*_{[ij]}E^*_{[ij]}E_{[ij]}m_{[ij]})(I-E_{[ij]}m_{[ij]})^{-1}.$$

Another consequence is a nonstationary variant of the Schur algorithm. Here, the stationary embedding is also quite useful, because there is no marking analysis and it produces almost immediately: for $i\in\mathrm{J}$, $j\in\mathrm{K}$, $i<j$,

$$(5.10)\qquad E_{[ij]}=G_{[ij]}+D_{G^*_{[ij]}}E^{(1)}_{[ij]}m_{[ij]}(I-G^*_{[ij]}E^{(1)}_{[ij]}m_{[ij]})^{-1}D_{G_{[ij]}},$$

where $E^{(1)}$ is the upper triangular contraction given by the parameter $\varphi^{(1)}=\{G_{ij}/i\in \mathsf{J},\ j\in \mathsf{K},\ j>i+1\}$.

## VI. Applications

In this section we will discuss several applications regarding positive and contractive completions.

### Positive completions

1. We begin with the band extension problem, i.e. the second problem mentioned in Introduction. Let $\mathcal{M}_n$ denote the $n\times n$ complex matrices. For a $C^*$-algebra $\mathcal{A}$, $\mathcal{M}_n(\mathcal{A})=\mathcal{M}_n\otimes\mathcal{A}$, and for $k\leqq n$, $\mathcal{BM}_{n,k}$ denotes the subset of $\mathcal{M}_n$ formed by the matrices with bandwidth $k$. $B\mathcal{M}_{n,k}$ is an operator system of $\mathcal{M}_n$, i.e. it is a selfadjoint subspace of $\mathcal{M}_n$ containing the identity—see [26].

In view of a result of M. D. Choi (Proposition 3.12 in [26]), to give a positive block-matrix $M=(S_{ij})_{i,j=0}^{n-1}$ in $\mathcal{M}_n(\mathcal{L}(\mathcal{H}))$is equivalent to give a completely positive map $\Phi:\mathcal{M}_n\to\mathcal{L}(\mathcal{H})$.

In the case of $\mathcal{BM}_{n,k}$, a slight modification of this result gives us that the following assertions are equivalent:
(1) $\Phi:\mathcal{BM}_{n,k}\to\mathcal{L}(\mathcal{H})$ is completely positive
(2) $\Phi$ is $k$-positive
(3) $(\Phi\ (E_{i+p,\,j+p}))_{i,j=0}^{n-1}$ are positive matrices for all $0\leqq p\leqq n-k$, where $(E_{ij})_{i,j=0}^{n-1}$ is the cannonical basis of $\mathcal{M}_n$—for the standard definitions see for instance [26].

So that, an application of Arveson's extension theorem gives the conditions for the solvability of the band extension problem.

Regarding the parametrization of the solutions, we suppose that this conditions are fulfilled, i.e. we give a banded matrix $M_0=(S_{ij})_{i,j=0}^{n-1}$, $S_{ij}\in\mathcal{L}(\mathcal{L}_j,\mathcal{L}_i)$ such that $(S_{i+p,\,j+p})_{i,j=0}^{k-1}$ are positive matrices for $0\leqq p\leqq n-k$ (and $S_{ii}=I_{\mathcal{L}_i}$ for $i=0,\dots,n-1$).

Define the spaces $\mathcal{H}_p$, $p\in\{0,\ 1,\dots,n-1\}=\mathsf{I}$ by renorming $\mathcal{L}_p\oplus\dots\oplus\mathcal{L}_{p+k-1}$ with the positive matrix $(S_{i+p,\,j+p})_{i,j=0}^{k-1}$ (in the above expressions we stop when the indices depass the admissible values). Then, for $0\leqq p\leqq n-1$,

$\mathcal{E}_p=$ the space generated by $\{(h_p,\dots,h_{k+p-2},0)/h_i\in\mathcal{L}_i\}$

$\mathcal{F}_p=$ the space generated by $\{(0,\ h_{p+1},\dots,h_{k+p-1})/h_i\in\mathcal{L}_i\}$ and for $0\leqq p\leqq n-1$

$$w_p:\mathcal{E}_{p+1}\to\mathcal{F}_p$$

$$w_p(h_{p+1},\dots,h_{k+p-1},0)=(0,\ h_{p+1},\dots,h_{k+p-1}).$$

$w=\{w_p\}_{0\leqq p<n-1}$ is a family of unitary operators and the set of the solutions of the band extension problem with the data $M_0$ is $E(\mathcal{E},\mathcal{F},\mathcal{H};w)$ with $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, and $w$ as given above.

2. In order to use the supplimentary structure involved by this problem, let us consider a positive extension $M$ of $M_0$ and let $\varphi$ be the parameter of $M$; the

elements $\{G_{ij}/j \leq k-1\}$ are imposed by $M_0$ and only $\{G_{ij}/j>k-1\}$ are the parameters of $M$ as element in $E(\mathscr{E}, \mathscr{F}, \mathscr{H}; w)$. Let $E$ be the upper triangular contraction associated with $\zeta$ and $E^{(k)}$ the upper triangular contraction associated with $\zeta^{(k)} = \{G_{ij}/j>k-1\}$. For expressing $E$ in terms of $E^{(k)}$ we have to use (5.10) and the Redheffer product. Remind first the definition of the Redheffer product. For a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and for $X$ we define the cascade transformation

$$C_{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]}(X) = A + BX(I-DX)^{-1}C$$

whenever the inverse of $(I-DX)$ exists. For two block-matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$, we define the Redheffer product by

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \times \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 + B_1 A(I-D_1 A)^{-1} C_1 & B_1 A(I-D_1 A)^{-1} D_1 B + B_1 B \\ C(I-D_1 A)^{-1} C_1 & C(I-D_1 A)^{-1} D_1 B + D \end{bmatrix}$$

with the property that $C_{\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]}(C_{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]}) = C_{\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right] \times \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]}$.

Now, we obtain the formula:

$$E_{[ij]} = a_{[ij]}^{(k-1)} + b_{[ij]}^{(k-1)} E_{[ij]}^{(k)} m_{[ij]}^{k} (I - d_{[ij]}^{(k-1)} E_{[ij]}^{(k)}) m_{[ij]}^{k})^{-1} c_{[ij]}^{(k-1)}),$$

where $a^{(k-1)}$, $b^{(k-1)}$, $c^{(k-1)}$, $d^{(k-1)}$ are upper triangular operators defined by the use of (5.10) and Redheffer product such that $\begin{bmatrix} a^{(k-1)} & b^{(k-1)} \\ c^{(k-1)} & d^{(k-1)} \end{bmatrix}$ is unitary; $m_{[ij]}^{k}$ is a notation for the composition of the involved marking operators. By (5.8),

$$\text{(6.1)} \qquad b_{[ij]}(W) = C_{\left[\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}\right] * \left[\begin{smallmatrix} a_{[ij]}^{(k-1)} m_{[ij]} & b_{[ij]}^{(k-1)} \\ c_{[ij]}^{(k-1)} m_{[ij]} & d_{[ij]}^{(k-1)} \end{smallmatrix}\right]} (E_{[ij]}^{(k)} m_{[ij]}^{k})$$

Moreover, by (5.9),

$$\text{(6.2)} \qquad M_{ij}(\varphi) = (I - m_{[ij]}^{*} E_{[ij]}^{*})^{-1} c_{[ij]}^{(k-1)*} (I - (E_{[ij]}^{(k)} m_{[ij]}^{k})^{*} d_{[ij]}^{(k-1)*}))^{-1}$$

$$(I - (E_{[ij]}^{(k)} m_{[ij]}^{k})^{*} E_{[ij]}^{(k)} m_{[ij]}^{k}) (I - d_{[ij]}^{(k-1)} E_{[ij]}^{(k)} m_{[ij]}^{k})^{-1} c_{[ij]}^{(k-1)} (I - E_{[ij]} m_{[ij]})^{-1}.$$

This formula is another variant of (10.11) in [13]—for the stationary case see [20]:

## Contractive completions

3. Let $l^2(\mathrm{I})$ be the space of sequences $\{x_n\}_{n \in \mathrm{I}}$ indexed with I and $x_n \in \mathbf{C}$, the complex numbers, with coordinatewise addition and scalar multiplication. It is

organized as a Hilbert space in an usual way, and $\mathcal{M}_I = \mathcal{L}(l^2(I))$. Moreover, denote by $\mathcal{U}_I$ the algebra of upper triangular matrices in $\mathcal{M}_I$ and $E_{ij}$ $i, j \in I$ is the set of standard matrix units for $\mathcal{M}_I$. By a result of Mc. Asey—Muhly (Proposition 6.10 in [26]), every (unital and contractive) representation of $\mathcal{U}_I$ in $\mathcal{L}(\mathcal{H})$ is completely contractive. Indeed, let $\rho$ be such a representation. Let $\mathcal{H}_i$ be the range of the projection $\rho(E_{ii})$ and $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$. The operator matrix of $\rho(E_{ij})$ in this decomposition is $T_{ij}$ on the $(i,j)$-th position and 0 elsewhere. Moreover, for $i<j$,

$$T_{ij} = T_{i,i+1} T_{i+1,i+2} \dots T_{j-1,j}$$

so that, to give the representation $\rho$ means to give the contractions $\{T_i = T_{i,i+1}\}_{i \in J}$. To show that $\rho$ is completely contractive it is sufficient to show that the kernel $\varphi$ on $I \times I$ given by $S_{ii} = I_{\mathcal{H}_i}$, $S_{ij} = T_{ij}$ for $i<j$ and $S_{ij} = S_{ji}^*$ for $i>j$, is positive, and in view of (2.2) and (2.3) this is indeed the case because the parameter of $\varphi$ is given by $G_{i,i+1} = T_{i,i+1}$, $i \in J$ and 0 elsewhere. We can prove now a decomposition for representations of $\mathcal{U}_I$ which is the analogue of the canonical decomposition of a contraction (Theorem 3.2 Ch. I, [29]). We call $\rho$ completely non-unitary on $\mathcal{H}$ if for no nonzero reducing subspace $\mathcal{L}$ for $\rho$ $\rho/\mathcal{L}$ extends to a unital *-homomorphism of $\mathcal{M}_I$ on $\mathcal{L}$.

**6.1. Proposition.** *Let $\rho$ be a representation of $\mathcal{U}_I$ on $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ then there exists a uniquely determined decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ reducing $\rho$ such that $\rho/\mathcal{H}_1$ is completely non-unitary and $\rho/\mathcal{H}_2$ extends to a unital *-homomorphism of $\mathcal{M}_I$* on $\mathcal{H}_2$.

*Moreover, $\mathcal{H}_2 = \bigoplus_{i \in I} \mathcal{H}_2^{(i)}$, where*

$$\mathcal{H}_2^{(i)} = \{h_i \in \mathcal{H}_i / \dots = \|T_{i-1} h_i\| = \|h_i\| = \|T_i^* h_i\| = \|T_{i+1}^* T_i^* h_i\| = \dots\}.$$

Proof. Based on the fact that for a contraction $T$, $\|Th\| = \|h\|$ implies $T^* Th = h$, we obtain that $\mathcal{H}_2$ reduces $\rho$. Moreover, $\rho(E_{ij})/\mathcal{H}_2$ are unitary operators for $i<j$ and we simply extend $\rho/\mathcal{H}_2$ to $\mathcal{M}_I$ by taking $\rho(E_{ij}) = \rho^*(E_{ji})$, $i>j$ which defines a unital *-homomorphism of $\mathcal{M}_I$ on $\mathcal{H}_2$. The rest is plain. ■

From now on we follow [10] in order to obtain a model for the representations of $\mathcal{U}_I$ along the line of the Sz.—Nagy—Foias model of a contraction. In [10] these facts were formulated as a model for time variant linear systems, when $I = \mathbf{Z}$.

Let us consider the Kolmogorov decomposition of the kernel $\varphi$ associated to $\rho$ (i.e. $S_{ij} = T_{ij}$ for $i \leq j$ and $S_{ij} = T_{ji}^*$, $i>j$) and let $\zeta$ be the parameter of $\varphi$ given by (2.2) and (2.3) (i.e. $G_{n,n+1} = T_n$, $n \in J$ and $G_{ij} = 0$ elsewhere). That is, in view of (2.12) and (2.13), we define:

$$(6.3) \qquad \mathcal{K}_n = \dots \oplus \mathcal{D}_{T_{n-2}^*} \oplus \mathcal{D}_{T_{n-1}^*} \oplus \mathcal{H}_n \oplus \mathcal{D}_{T_n} \oplus \mathcal{D}_{T_{n+1}} \oplus \dots$$

and

$$W_n : \mathcal{K}_{n+1} \to \mathcal{K}_n$$

(6.4)
$$W_n(\ldots, d_{*,n}, h_{n+1}, d_{n+1}, \ldots)$$
$$= (\ldots, d_{*,n-1}, D_{T_n^*} d_{*,n} + T_n h_{n+1}, -T_n^* d_{*,n} + D_{T_n} h_{n+1}, d_{n+1}, d_{n+2}, \ldots).$$

We continue by defining the spaces

(6.5)
$$\mathcal{K}_n^+ = \mathcal{H}_n \oplus \mathcal{D}_{T_n} \oplus \mathcal{D}_{T_{n+1}} \oplus \cdots$$

and the isometries

(6.6)
$$W_n^+ : \mathcal{K}_{n+1}^+ \to \mathcal{K}_n^+$$
$$W_n^+ = W_n / \mathcal{K}_{n+1}^+.$$

Then we take into consideration the Wold decomposition of the family $\{W_k^+\}_{k \geq n}$, $n \in \mathrm{J}$ and we denote

$$\mathcal{L}_n^+ = \mathcal{K}_n^+ \ominus W_n^+ \mathcal{K}_{n+1}^+ = W_n(\ldots \oplus O \oplus \mathcal{D}_{T_n^*} \oplus O \mathcal{H}_{n+1} \oplus O \oplus \ldots) = W_n \mathcal{D}_{T_n^*}^{(-1)}$$

for $n \in \mathrm{J}$ ($\mathcal{D}_{T_n^*}^{(-1)}$ being a notation for the space $\ldots \oplus O \oplus \mathcal{D}_{T_n^*} \oplus O_{\mathcal{H}_{n+1}} \oplus O \oplus \ldots$). We have

(6.7)
$$\mathcal{K}_n^+ = (\mathcal{L}_n^+ \oplus \bigoplus_{p=1}^{N-n-1} W_n^+ \ldots W_{n+p-1}^+ \mathcal{L}_{n+p}^+) \oplus \mathcal{R}_n^+$$

(when $N$ is finite, $\mathcal{L}_N^+ = \mathcal{K}_N^+$ and $\mathcal{R}_n^+ = W_n^+ \ldots W_{N-1}^+ \mathcal{L}_N^+$ and when $N = \infty$ $\mathcal{R}_n^+ = \bigcap_{p=0}^{\infty} W_n^+ \ldots W_{n+p}^+ \mathcal{K}_{n+p+1}^+$).

In a similar way, we define the spaces

(6.8)
$$\mathcal{K}_n^- = \ldots \oplus \mathcal{D}_{T_{n-2}^*} \oplus \mathcal{D}_{T_{n-1}^*} \oplus \mathcal{H}_n$$

and the isometries

(6.9)
$$W_n^- : \mathcal{K}_n^- \to \mathcal{K}_{n+1}^-$$
$$W_n^- = W_n^* / \mathcal{K}_n^- \qquad \text{for} \qquad n \in \mathrm{J}$$

We use again the Wold decomposition for the family of isometries $\{W_k^-\}_{k \leq n-1}$ and define for $n \in \mathrm{K}$

$$\mathcal{L}_n^- = \mathcal{K}_n^- \ominus W_{n-1}^- \mathcal{K}_{n-1}^- = W_{n-1}^*(\ldots O \oplus O_{\mathcal{H}_{n-1}} \oplus \mathcal{D}_{T_{n-1}} \oplus \ldots) = W_{n-1}^* \mathcal{D}_{T_{n-1}}^{(1)}.$$

We get

$$(6.10)\qquad \mathcal{K}_n^- = (\mathcal{L}_n^- \oplus \bigoplus_{p=1}^{n-M-1} W_{n-1}^- \dots W_{n-p}^- \mathcal{L}_{n-p}) \oplus \mathcal{R}_n^-$$

(when $M$ is finite, $\mathcal{L}_M^- = \mathcal{K}_M^-$ and $\mathcal{R}_n^- = W_{n-1}^- \dots W_M^- \mathcal{L}^{-M}$, and when $M = -\infty$, $\mathcal{R}_n^- = \bigcap_{p=1}^{\infty} W_{n-1}^* \dots W_{n-p}^* \mathcal{K}_{n-p}^-$).

Now, we can define the spaces

$$(6.11)\qquad \mathcal{K}_n^{\text{out}} = \bigoplus_{q=1}^{n-M-1} W_{n-1}^* \dots W_{n-q}^* \mathcal{D}_{T_{n-q-1}^*}^{(-1)}$$

$$\oplus \mathcal{D}_{T_{n-1}^*}^{(-1)} \oplus \bigoplus_{p=0}^{N-n-1} W_n \dots W_{n+p} \mathcal{D}_{T_{n+p}^*}^{(-1)}$$

with an obvious interpretation for $n = M$ and

$$(6.12)\, \mathcal{K}_n^{\text{inp}} = \bigoplus_{p=1}^{n-M} W_{n-1}^* \dots W_{n-p}^* \mathcal{D}_{T_{n-p}}^{(1)} \oplus \mathcal{D}_{T_n}^{(1)} \oplus \bigoplus_{q=1}^{N-n-1} W_n \dots W_{n+q-1} \mathcal{D}_{n+q}^{(1)}$$

again with a corresponding interpretation when $n = N$.

**6.2. Proposition.** *$\rho$ is completely non-unitary if and only if*

$$\mathcal{K}_n = \mathcal{K}_n^{\text{inp}} \vee \mathcal{K}_n^{\text{out}} \quad \textit{for } n \in \mathrm{I}.$$

Proof. Based on the explicit form of the Kolmogorov decomposition of $\varphi$, we compute that

$$\mathcal{K}_n \ominus (\mathcal{K}_n^{\text{inp}} \vee \mathcal{K}_n^{\text{out}}) = \mathcal{H}_2^{(n)},$$

where $\mathcal{H}_2^{(n)}$ are the spaces defined in the statement of Proposition 6.1. According to this proposition, we obtain that $\rho$ is completely non-unitary if and only if $\mathcal{K}_n = \mathcal{K}_n^{\text{inp}} \vee \mathcal{K}_n^{\text{out}}$, $n \in \mathrm{I}$. ■

Now, define $\mathcal{K} = \bigoplus_{n \in \mathrm{I}} \mathcal{K}_n$ and

$$\pi : \mathcal{M}_\mathrm{I} \to \mathcal{L}(\mathcal{K})$$

$$\pi(E_{ii}) = P_{\mathcal{K}_i}^{\mathcal{K}} \qquad \text{for} \qquad i \in \mathrm{I}$$

$$\text{for } i<j \qquad \pi(E_{ij}) = \begin{cases} W_i \dots W_{j-1} & \text{on the } (i,j)\text{-th position} \\ 0 & \text{elsewhere} \end{cases}$$

and for $i>j$, $\pi(E_{ij}) = \pi(E_{ji})^*$.

It is readily seen that $\pi$ is a unital $*$-homomorphism. Now we introduce the characteristic operator of $\rho$. Define the spaces

$$\mathcal{M}_+ = \bigoplus_{n\in J} \mathcal{D}_{T_n^*}, \qquad \mathcal{M} = \bigoplus_{n\in J} \mathcal{D}_{T_n}$$

and the lower triangular contraction

$$(6.13) \qquad \begin{aligned} &\Theta : \mathcal{M}_- \to \mathcal{M}_+ \\ &\Theta_{ii} = -T_i, && i\in J \\ &\Theta_{ij} = D_{T_i^*} T_{i+1}^* \dots T_{j-1}^* D_{T_j}, && i>j \\ &\Theta_{ij} = 0 \text{ elsewhere} \end{aligned}$$

is the characteristic operator of $\rho$.

Indeed, we define

$$(6.14) \qquad \begin{aligned} &Q_n : \mathcal{K}_n^{\text{inp}} \to \mathcal{K}_n^{\text{out}} \\ &Q_n = P^{\mathcal{K}_n}_{\mathcal{K}_n^{\text{out}}} / \mathcal{K}_n^{\text{inp}} \end{aligned}$$

$$(6.15) \qquad \begin{aligned} &\Phi_n^+ : \mathcal{K}_n^{\text{out}} \to \mathcal{M}_+ \\ &\Phi_n^+ (\dots, W_{n-1}^* d_{*,n-2}^{(-1)}, d_{*,n-l}^{(-l)}, W_n d_{*,n}^{(-1)}, \dots) = (\dots, d_{*,n-2}, d_{*,n-1}, d_{*,n}, \dots), \end{aligned}$$

where $d_{*,n}^{(-1)} = (\dots, 0, d_{*,n}, 0_{\mathcal{K}_{n+1}}, 0, \dots)$, $d_{*,n} \in \mathcal{D}_{T_n^*}$
and

$$(6.16) \qquad \begin{aligned} &\Phi_n^- : \mathcal{K}_n^{\text{inp}} \to \mathcal{M}_- \\ &\Phi_n^- (\dots, W_{n-1}^* d_{n-1}^{(1)}, d_n^{(1)}, W_n d_{n+1}^{(1)}, \dots) = (\dots d_{n-1}, d_n, d_{n+1}, \dots), \end{aligned}$$

where $d_n^{(1)} = (\dots 0, 0_{\mathcal{K}_n}, d_n, 0, \dots)$, $d_n \in \mathcal{D}_{T_n}$

We get the relation

$$(6.17) \qquad \Phi_n^+ Q_n (\Phi_n^-)^* = \Theta$$

and we can obtain a model of $\rho$ in terms of $\Theta$. In view of Proposition 6.2, it results that

$$(6.18) \qquad \mathcal{R}_n^+ = \overline{(I - Q_n)\mathcal{K}_n^{\text{inp}}}$$

and

$$(6.19) \qquad \begin{aligned} &\Phi_{\mathcal{R}_h^+} : \mathcal{R}_n^+ \to \mathcal{D}_\Theta \\ &\Phi_{\mathcal{R}_h^+} (I - Q_n) k = D_\Theta \Phi_n^- k, \qquad k \in \mathcal{K}_n^{\text{inp}} \end{aligned}$$

are unitary operators for $n \in I$. Finally,

$$\text{(6.20)}\qquad \begin{aligned}&\Psi_n:\mathscr{K}_n\to\mathscr{M}_+\oplus\mathscr{D}_\Theta\\ &\Psi_n=\Phi_n^+\oplus\Phi_{\mathscr{R}_n^+}\end{aligned}$$

are unitary operators identifying the spaces $\mathscr{K}_n$ in terms of the characteristic operator. Moreover, in view of Theorem 2.2 [10] (or directly) we get

$$\text{(6.21)}\qquad \hat{\mathscr{H}}_n=\psi_n\mathscr{H}_n=(\bigoplus_{k\geq n}\mathscr{D}_{T_k^*}\oplus\mathscr{D}_\Theta)\ominus\{\Theta v\oplus D_\Theta v/v\in\bigoplus_{k\geq n}\mathscr{D}_{T_k}\}$$

for $n\in$I. Defining the marking operators

$$\text{(6.22)}\qquad \begin{aligned}&m_n^+:\bigoplus_{k\geq n+1}\mathscr{D}_{T_k^*}\to\bigoplus_{k\geq n}\mathscr{D}_{T_k^*}\\ &m_n^+(d_{*,n+1},d_{*,n},\ldots)=(0,d_{*,n+1},\ldots)\end{aligned}$$

we obtain that

$$\text{(6.23)}\qquad \hat{T}_n=\Psi_n T_n\Psi_{n+1}^*=P_{\hat{\mathscr{K}}_n}(m_n^+u\oplus v),\ u\in\bigoplus_{k\geq n+1}\mathscr{D}_{T_k^*},\ v\in\mathscr{D}_\Theta.$$

Defining the representation

$$\text{(6.24)}\qquad \begin{aligned}&\hat{\rho}:\mathscr{U}_\text{I}\to\mathscr{L}(\hat{\mathscr{H}}),\quad \hat{\mathscr{H}}=\bigoplus_{n\in\text{I}}\hat{\mathscr{H}}_n\\ &\hat{\rho}(E_{i,i+1})=\begin{cases}\hat{T}_i \text{ on the } (i,\ i+1)\text{-th position}\\ 0 \text{ elsewhere}\end{cases}\end{aligned}$$

and the unitary operator

$$\text{(6.25)}\qquad \begin{aligned}&\Psi:\mathscr{H}\to\hat{\mathscr{H}}\\ &\Psi=\bigoplus_{n\in\text{I}}\Psi_n/\mathscr{H}_n\end{aligned}$$

we obtain the following result.

**6.3. Theorem.** *The representation $\rho:\mathscr{U}_\text{I}\to\mathscr{L}(\mathscr{H})$ is unitarely equivalent to its model $\hat{\rho}$ given by* (6.24), *in the sense that*

$$\Psi\rho(U)=\hat{\rho}(U)\Psi \quad \textit{for all } U\in\mathscr{U}_\text{I}. \qquad ■$$

The characteristic operator of $\rho$ has the distinguished property that $\|\Theta_{ii}h_i\|<\|h_i\|$ for $i\in$J and $h_i\in\mathscr{D}_{T_i^*}-\{0\}$, a situation when we say that $\Theta$ is purely contractive.

6.4. Remark. There is a simple consequence, for example of (5.3) that every lower triangular contraction has a uniquely determined decomposition into a purely contractive lower triangular part and a unitary diagonal part, which constitutes an analogue of Proposition 2.1, Ch. V [29]. ■

One more definition says that two lower triangular contractions $\Theta\in\mathscr{L}(\bigoplus_{n\in J}\mathscr{D}_{*,n}, \bigoplus_{n\in J}\mathscr{D}_n)$ and $\Theta'\in\mathscr{L}(\bigoplus_{n\in J}\mathscr{D}'_{*,n}, \bigoplus_{n\in J}\mathscr{D}'_n)$ coincide if there exist unitary operators $\tau_{*,n}$ from $\mathscr{D}_{*,n}$ onto $\mathscr{D}_{*,n}$ and unitary operators $\tau_n$ from $\mathscr{D}_n$ onto $\mathscr{D}'_n$ such that

$$\Theta' = \bigoplus_{n\in J}\tau_{*,n}\Theta\bigoplus_{n\in J}\tau_n.$$

We can complete Theorem 6.3 with the following result.

**6.5. Theorem.** *Two completely non-unitary representations of $\mathscr{U}_I$ are unitarely equivalent if and only if their characteristic operators coincide.*

Proof. Of course, we have at hand all necessary devices for paralleling the proofs of Theorem 3.1 and Theorem 3.4, Ch. VI [29]. ■

**6.6.** Remark. Models for representations of $\mathscr{U}_I$ along the line of Sz.—Nagy—Foias model for contractions are taken into account in [7] but the questions touched in Theorem 6.3 and Theorem 6.5 are left as unsettled in [7]. ■

4. Two remarks on a lifting theorem of Ball and Gogberg in [7] are to be discussed. This theorem asserts that in the case when card $I<\infty$ and dim $\mathscr{H}, \mathscr{H}'<\infty$, for two representations $\rho$, $\rho'$ of $\mathscr{U}_I$ on $\mathscr{H}$ and $\mathscr{H}'$ and an operator $X\in\mathscr{L}(\mathscr{H}, \mathscr{H}')$ such that

$$X\rho(U)=\rho'(U)X \quad \text{for} \quad U\in\mathscr{U}_I$$

there exists an $Y\in\mathscr{L}(\mathscr{K}, \mathscr{K}')$ such that

$$\text{(i)} \quad Y\sigma(U)=\sigma'(U)Y \quad U\in\mathscr{U}_I$$

$$\text{(ii)} \quad XP_{\mathscr{H}}=P_{\mathscr{H}'}Y \qquad ,$$

$$\text{(iii)} \quad \|X\|=\|Y\|$$

where $\sigma:\mathscr{U}_I\to\mathscr{L}(\mathscr{K})$ and $\sigma':\mathscr{U}_I\to\mathscr{L}(\mathscr{K}')$ are substar dilations of $\rho$ and respectively $\rho'$. For explaining the notion of substar dilation, we take $\pi$ the Stinespring dilation on $\tilde{\mathscr{K}}$ of the Kernel $\varphi$ associated with $\rho$. Let $\mathscr{M}$ and $\mathscr{U}$ be subspaces of $\tilde{\mathscr{K}}$, $\mathscr{U}\subset\mathscr{M}$ invariant for the algebra $\pi(\mathscr{U}_I)$ such that $\mathscr{H}=\mathscr{M}\ominus\mathscr{U}$. Then

$$\sigma:\mathscr{U}_I\to\mathscr{L}(\mathscr{M})$$

$$\sigma(U)=\pi(U)/\mathscr{M} \quad U\in\mathscr{U}_I$$

is a substar dilation of $\rho$.

Of course, we have to ask the standard minimality conditions, that is $\mathcal{K}=\overline{\pi(\mathcal{M}_I)\mathcal{H}}$ and $\mathcal{M}=\overline{\pi(\mathcal{U}_I)\mathcal{H}}$. In this case the substar dilations of $\rho$ are unitarely equivalent (because the only one completely positive extension of $\rho$ to $\mathcal{M}_I$ is given by the kernel $\varphi$—see [26] for a detailed discussion). In this way, the above theorem of Ball and Gohberg is equivalent with the following formulation in [10]: let $\{T_n\}_{n\in J}$, $T_n\in\mathcal{L}(\mathcal{H}_{n+1}, \mathcal{H}_n)$ determining $\rho$ and $\{T'_n\}_{n\in J}$, $T'_n\in\mathcal{L}(\mathcal{H}'_{n+1}, \mathcal{H}'_n)$ determining $\rho'$; then, to give $X\in\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $X\rho(U)=\rho'(U)X$, $U\in\mathcal{U}_I$ means to give a family of operators $\{A_n\}_{n\in I}$ such that $A_n\in\mathcal{L}(\mathcal{H}_n, \mathcal{H}'_n)$ and $T'_n A_{n+1}=A_n T_n$ for $n\in J$. We suppose that $X$ is a contraction. Let $\{W_n\}_{n\in J}$, $W_n\in\mathcal{L}(\mathcal{K}_{n+1}, \mathcal{K}_n)$ and $\{W'_n\}_{n\in J}$, $W'_n\in\mathcal{L}(\mathcal{K}'_{n+1}, \mathcal{K}'_n)$ be the Kolmogorov decompositions of $\{T_n\}_{n\in J}$ and respectively $\{T'_n\}_{n\in J}$. Recalling the definitions (6.5), (6.6) and (6.8), (6.9), we set

$\mathrm{CID}(\{A_n\}_{n\in I})=\{\{B_n\}_{n\in I}/B_n$ are contractions in $\mathcal{L}(\mathcal{K}_n^+, \mathcal{K}_n'^+)$, $W_n'^+ B_{n+1}=B_n W_n^+$ and $P_{\mathcal{H}'_n} B_n=A_n P_{\mathcal{H}_n}\}$ and the result is that $\mathrm{CID}(\{A_n\}_{n\in I})\neq\emptyset$. This assertion also holds for arbitrary I and is independent of the dimensions of $\mathcal{H}$ and $\mathcal{H}'$. Moreover, in this formulation we can obtain a parametrization of $\mathrm{CID}(\{A_n\}_{n\in I})$ (Theorem 4.2 in [10]) without any restriction on $X$ (as is the case in [7]).

We end by showing that $\mathrm{CID}(\{A_n\}_{n\in I})$ is an $E(\mathcal{E}, \mathcal{F}, \mathcal{H}; w)$.

In the stationary case (i.e. for the set CID of [5]—and also of [27] and [29]) this fact was pointed out in [3], based on a method originating in [1]. So that, in the following we have to show how to adapt the construction in [3] to our setting. Moreover, for some simplicity, we choose to restrict the analysis to $M=0$, $N=1$, which is the generic case. This case is equivalent to the first problem mentioned in Introduction—see [10]. Consider contractions $T_0\in\mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$, $T'_0\in\mathcal{L}(\mathcal{H}'_1, \mathcal{H}'_0)$ and contractions $A_0\in\mathcal{L}(\mathcal{H}_0, \mathcal{H}'_0)$, $A_1\in\mathcal{L}(\mathcal{H}_1, \mathcal{H}'_1)$ such that $T'_0 A_1=A_0 T_0$. Define the positive matrices

$$K_0=\begin{bmatrix} I & A_0 & A_0 T_0 \\ A_0^* & I & T_0 \\ T_0^* A_0^* & T_0^* & I \end{bmatrix} : \mathcal{H}'_0\oplus\mathcal{H}_0\oplus\mathcal{H}\to\mathcal{H}'_0\oplus\mathcal{H}_0\oplus\mathcal{H}_1$$

$$K_1=\begin{bmatrix} I & T'_0 & T'_0 A_1 \\ T_0'^* & I & A_1 \\ A_1^* T_0'^* & A_1^* & I \end{bmatrix} : \mathcal{H}'_0\oplus\mathcal{H}'_1\oplus\mathcal{H}_1\to\mathcal{H}'_0\oplus\mathcal{H}'_1\oplus\mathcal{H}_1$$

the positivity being a simple consequence of (2.3). Denote by $\mathcal{L}_0$ the space obtained by renorming $\mathcal{H}'_0\oplus\mathcal{H}_0\oplus\mathcal{H}_1$ with $K_0$ and $\mathcal{L}_1$ the space obtained by renorming $\mathcal{H}'_0\oplus\mathcal{H}'_1\oplus\mathcal{H}_1$ with $K_1$. Define the subspaces

$$\mathscr{E}_0=\{(0,l_0,l_1)/l_0\in\mathscr{H}_0,\quad l_1\in\mathscr{H}_1\}$$

$$\mathscr{F}_0=\{(l'_0,0,l_1)/l'_0\in\mathscr{H}'_0,\quad l_1\in\mathscr{H}_1\}$$

of $\mathscr{L}_0$ and the subspaces

$$\mathscr{E}_1=\{(l'_0,0,l_1)/l'_0\in\mathscr{H}'_0,\quad l_1\in\mathscr{H}_1\}$$

$$\mathscr{F}_1=\{(l'_0,l'_1,0)/l'_0\in\mathscr{H}'_0,\quad l'_1\in\mathscr{H}'_1\}$$

of $\mathscr{L}_1$ and the operator

$$w_0:\mathscr{E}_1\to\mathscr{F}_0$$

$$w_0(l'_0,0,l_1)=(l'_0,0,l_1)$$

is unitary in view of the relation $A_0T_0=T'_0A_1$.

Define $\mathscr{E}=\{\mathscr{E}_0,\mathscr{E}_1\}$, $\mathscr{F}=\{\mathscr{F}_0,\mathscr{F}_1\}$, $\mathscr{L}=\{\mathscr{L}_0,\mathscr{L}_1\}$, $w=\{w_0\}$, then, there exists a one-to-one correspondence between $E(\mathscr{E},\mathscr{F},\mathscr{L};w)$ and $\mathrm{CID}(\{A_0,A_1\})$. For proving this, we continue by embedding in an obvious way $\mathscr{H}_0$, $\mathscr{H}'_0$ in $\mathscr{L}_0$ and $\mathscr{H}_1$, $\mathscr{H}'_1$ in $\mathscr{L}_1$. We denote these embeddings by $\hat{\mathscr{H}}_0$, $\hat{\mathscr{H}}'_0$, $\hat{\mathscr{H}}_1$, $\hat{\mathscr{H}}'_1$. Take $W=\{W_0\}\in E(\mathscr{E},\mathscr{F},\mathscr{L};w)$ and define

$$\hat{\mathscr{K}}_1^+=\hat{\mathscr{H}}_1,\quad \hat{\mathscr{K}}_0^+=\hat{\mathscr{H}}_0\vee W\hat{\mathscr{H}}_1$$

$$\hat{\mathscr{K}}_1'^+=\hat{\mathscr{H}}'_1,\quad \hat{\mathscr{K}}_0'^+=\hat{\mathscr{H}}'_0\vee W\hat{\mathscr{H}}'_1$$

then

$$W_0^+=W_0/\hat{\mathscr{K}}_1^+:\hat{\mathscr{K}}_1^+\to\hat{\mathscr{K}}_0^+$$

$$W_0'^+=W_0/\hat{\mathscr{K}}_1'^+:\hat{\mathscr{K}}_1'^+\to\hat{\mathscr{K}}_0'^+$$

are the isometric operators given by (6.5) and (6.6). The final step is to define $B_0=P_{\mathscr{K}_0'^+}/\mathscr{K}_0^+$ and $B_1=P_{\mathscr{K}_1'^+}/\mathscr{K}_1^+$; it is a matter of simple computations to obtain that $\{B_0,B_1\}\in\mathrm{CID}(\{A_0,A_1\})$. For the converse, we can use Theorem 4.2 in [10] and Theorem 5.1.

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