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Realization of K-Functionals on Subsets and Constrained Approximation

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The aim of this article is the investigation of the cases when Peetre K-functional can be realized on some approximation subsets. Applications of this property are made for Zamansky type results for best approximation and constrained approximations.

1. Introduction

Let (X_0, X_1) be a couple of normed spaces with (semi-) norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, $X_1 \subset X_0$. The Peetre K-functional for this couple is given by

$$(1.1) \qquad K(f,t) := K(f,t; X_0, X_1) := \inf\{\|f-g\|_0 + t\|g\|_1 : g \in X_1\}$$

for any $f \in X_0$ and $t > 0$.

Let $\{G_n\}_1^\infty$ be a family of subsets of X_1. For this family we assume: i) $0 \in G_1$. ii) $G_n \subset G_{n+1}$, iii) $G_n = -G_n$, iv) the closure of $\{G_n\}_1^\infty$ in X_0 is X_0. The best approximation of $f \in X_0$ by the elements of G_n is given by

$$E_n(f)_0 := \inf\{\|f-g\|_0 : g \in G_n\}.$$

Thus assumption iv) simply means that $E_n(f)_0 \to 0$ when $n \to \infty$ for any $f \in X_0$ and ii) implies monotonicity for the sequence $\{E_n(f)_0\}_{n=1}^\infty$. Moreover we suppose that this family satisfies *inequalities of Jackson and Bernstein type*, that is, there are positive constants c_1, c_2, α such that for any n we have

$$(1.2) \qquad E_n(f)_0 \leqq c_1 n^{-\alpha} \|f\|_1 \qquad \forall f \in X_1;$$

$$(1.3) \qquad \|g_1 - g_2\|_1 \leqq c_2 n^\alpha \|g_1 - g_2\|_0 \quad \forall g_1, g_2 \in G_n.$$

Without loss of generality we may assume that c_1 and c_2 are not less than 1. Let us remark that (1.3) implies ($g_2 = 0$)

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(1.3a) $$\|g\|_1 \leqq c_2 n^{\alpha} \|g\|_0 \quad \forall g \in G_n,$$

which is usually called a Bernstein type inequality. We prefer the form of (1.3) because in general $g_1 - g_2$ may not belong to G_n. But in the case when G_n is a space (1.3a) readily implies (1.3).

From the semi-additivity of E_n and Jackson type inequality (1.2) we have for any $g \in X_1$

$$E_n(f)_0 \leqq E_n(f-g)_0 + E_n(g)_0 \leqq \|f-g\|_0 + c_1 n^{-\alpha} \|g\|_1 \leqq c_1 \{\|f-g\|_0 + n^{-\alpha} \|g\|_1\},$$

which by taking infimum on g implies the direct estimate for the best approximations to every $f \in X_0$

(1.4) $$E_n(f)_0 \leqq c_1 K(f, n^{-\alpha}; X_0, X_1).$$

On the other hand by the standard technique of the telescopic sums Bernstein type inequality (1.3) implies the converse estimate

(1.5) $$K(f, n^{-\alpha}; X_0, X_1) \leqq c(\alpha) n^{-\alpha} \sum_{k=1}^{n} k^{\alpha-1} E_k(f)_0$$

(provided $\|g\|_1 = 0$ for every $g \in G_1$, otherwise the term $2c_2 n^{-\alpha} \|f\|_0$ should be added to the right-hand side of (1.5)).

Properties (1.2) and (1.3) are usually required for the family $\{G_n\}_1^{\infty}$ in approximation theory. Here we shall impose one additional condition on the approximating family:

(1.6) $$K(f, n^{-\alpha}; X_0, G_n) := \inf \{\|f-g\|_0 + n^{-\alpha} \|g\|_1 : g \in G_n\}$$
$$\leqq c_3 K(f, n^{-\alpha}; X_0, X_1) \quad \forall f \in X_0$$

for some positive c_3. Because of $G_n \subset X_1$ we have

$$K(f, n^{-\alpha}; X_0, X_1) \leqq K(f, n^{-\alpha}; X_0, G_n)$$

and hence $c_3 \geqq 1$. In other words, for every $f \in X_0$ up to the multiplier c_3 the K-functional of f with argument $n^{-\alpha}$ *can be realized on the subset* G_n *of* X_1.

The following question arises: Is condition (1.6) independent of Jackson and Bernstein inequalities (1.2), (1.3) or implied by them?

We do not know the answer but we show in Theorem 2.2 that (1.6) will follow if (1.3) is replaced by a somewhat stronger condition. So we keep (1.6) as a separate from (1.2) and (1.3) property of the approximating family $\{G_n\}$.

In the next section we give some necessary and sufficient conditions for the validity of (1.6). In Section 3 different cases of importance for approximation theory in which (1.2), (1.3) and (1.6) hold are examined. Section 4 is devoted to applications in constrained approximation. In Section 5 we investigate in details the cases of onesided approximation from below and from above while shape-preserving types of approximation are considered in Section 6. Finally, some remarks and open problems are given in Section 7.

2. Realization of K-functional on subsets

Necessary and sufficient conditions for the validity of (1.6) are obtained in this section. After that we generalize the statement of the question in view of further applications for constrained approximations.

First we need the following

Lemma 2.1. *For every* $g, h \in G_n$ *and for every* $f \in X_0$ (1.3) *implies*

$$|\{\|f-g\|_0+n^{-\alpha}\|g\|_1\}-\{\|f-h\|_0+n^{-\alpha}\|h\|_1\}|\leqq(1+c_2)\|g-h\|_0.$$

Proof. Condition (1.3) gives

$$\begin{aligned}&|\{\|f-g\|_0+n^{-\alpha}\|g\|_1\}-\{\|f-h\|_0+n^{-\alpha}\|h\|_1\}|\\ &\leqq|\,\|f-g\|_0-\|f-h\|_0|+n^{-\alpha}|\,\|g\|_1-\|h\|_1|\\ &\leqq\|g-h\|_0+n^{-\alpha}\|g-h\|_1\leqq(1+c_2)\|g-h\|_0. \qquad \square\end{aligned}$$

Denote by $P_n(f)$ an element of best approximation from G_n to f (assuming it exists) and by $Q_n(f)$ an element in G_n approximating f with the order of the K-functional, i.e.

$$\|f-P_n(f)\|_0=E_n(f)_0, \quad \|f-Q_n(f)\|_0\leqq AK(f,n^{-\alpha};X_0,X_1),$$

where A is a fixed positive constant. In view of (1.4), for every element g of *near-best approximation* from G_n, that is $\|f-g\|_0\leqq A_1E_n(f)_0$, we have $g=Q_n(f)$. In view of Lemma 2.1, it will make no difference if in all of the following statements one replaces "for any $Q_n(f)$" with "there is $Q_n(f)$".

Theorem 2.2. *Let* (1.2) *and* (1.3) *hold. Then the following are equivalent*:

a) (1.6) *holds for* G_n;

b) $\|P_n(f)\|_1\leqq cn^{\alpha}K(f,n^{-\alpha};X_0,X_1)$ *for every* $f\in X_0$;

c) $\|Q_n(f)\|_1\leqq cn^{\alpha}K(f,n^{-\alpha};X_0,X_1)$ *for every* $f\in X_0$;

d) $\|g\|_1\leqq cn^{\alpha}K(g,n^{-\alpha};X_0,X_1)$ *for every* $g\in G_n$,

where constants c in b), c) and d) depend only on c_1, c_2, c_3 *and* A.

Proof. Assume b) holds. Then using (1.4) we get

$$\|f-P_n(f)\|_0+n^{-\alpha}\|P_n(f)\|_1=E_n(f)_0+n^{-\alpha}\|P_n(f)\|_1\leqq(c_1+c)K(f,n^{-\alpha};X_0,X_1),$$

which gives (1.6) with constant $c_3=c_1+c$. Similarly, c) implies a) with a constant $c_3=A+c$.

Assume a) holds. Then (1.6) is fulfilled for some $g\in G_n$ and now (1.4) and (1.6) implies

$$\|g-P_n(f)\|_0\leqq\|f-g\|_0+\|f-P_n(f)\|_0\leqq(c_3+c_1)K(f,n^{-\alpha};X_0,X_1).$$

Using this inequality in Lemma 2.1 together with (1.6) we get

$$\|f-P_n(f)\|_0+n^{-\alpha}\|P_n(f)\|_1\leqq((c_2+1)(c_3+c_1)+c_3)K(f,n^{-\alpha};X_0,X_1),$$

which implies b) with $c=(c_2+1)(c_3+c_1)+c_3$. c) follows from a) in a similar way with $c=(c_2+1)(c_3+A)+c_3$.

b) implies d) because of $g=P_n(g)$ for any $g\in G_n$. Finally, let d) hold. From (1.1) and (1.4) we get

$$K(P_n(f),n^{-\alpha};X_0,X_1)\leqq K(P_n(f)-f,n^{-\alpha};X_0,X_1)+K(f,n^{-\alpha};X_0,X_1)$$
$$\leqq \|P_n(f)-f\|_0+K(f,n^{-\alpha};X_0,X_1)\leqq(c_1+1)K(f,n^{-\alpha};X_0,X_1),$$

which together with d) gives b). □

From Theorem 2.2 we see that as far as the K-functional is realized (up to a multiplicative constant) on the set G_n it can be realized by the element of (near-) best approximation. Statements of the type of b) and c), in which information for the structural properties of the function (behaviour of the K-functional or moduli of smoothess) implies knowledge for the growth of some (semi-) norms of the polynomials of best approximation, are often called Zamansky type theorems (see Section 3.1).

Let us mention that (without any assumptions) b) implies d), which in turn implies (1.3a) because of $K(g,n^{-\alpha};X_0,X_1)\leqq\|g\|_0$.

Proposition 2.3. *Let* (1.2), (1.3) *and* (1.6) *hold. Assume that there is a subspace X_2 of X_0 with a seminorm $\|\cdot\|_2$ such that $\|g\|_2=0$ for every $g\in G_1$ and that the family $\{G_n\}$ satisfies a Bernstein inequality with parameter $\beta>\alpha$ with respect to the couple (X_0,X_2),* i.e.

$$\|g_1-g_2\|_2\leqq c_2' n^{\beta}\|g_1-g_2\|_0 \qquad \forall g_1,g_2\in G_n. \tag{2.1}$$

Then for every $f\in X_0$ we have

$$\|f-Q_n(f)\|_0+n^{-\alpha}\|Q_n(f)\|_1+n^{-\beta}\|Q_n(f)\|_2\leqq cK(f,n^{-\alpha};X_0,X_1). \tag{2.2}$$

Proof. Let $f\in X_0$ be fixed and for a given n let $h\in X_1$ be an element in X_1 realizing the K-functional:

$$\|f-h\|_0+n^{-\alpha}\|h\|_1=K(f,n^{-\alpha};X_0,X_1). \tag{2.3}$$

For $P_n(h)$, using (1.2) and (2.3), we get

$$\|f-P_n(h)\|_0\leqq\|f-h\|_0+\|h-P_n(h)\|_0 \tag{2.4}$$
$$\leqq\|f-h\|_0+c_1n^{-\alpha}\|h\|_1\leqq c_1K(f,n^{-\alpha};X_0,X_1).$$

Therefore we have $P_n(h)=Q_n(f)$.

From Theorem 2.2, a) => b), the definition of the K-functional and (2.3) we have

$$n^{-\alpha}\|P_n(h)\|_1\leqq cK(h,n^{-\alpha};X_0,X_1)\leqq cn^{-\alpha}\|h\|_1\leqq cK(f,n^{-\alpha};X_0,X_1). \tag{2.5}$$

Finally we shall establish

(2.6) $$n^{-\beta}\| P_n(h)\|_2 \leqq cK(f,\ n^{-\alpha};\ X_0,\ X_1),$$

which combined with (2.4) and (2.5) will complete the proof of the proposition. Let m be such natural that $2^m \leqq n < 2^{m+1}$. Using (2.1) and (1.2) we get

$$\begin{aligned}\| P_n(h)\|_2 &\leqq \| P_n(h) - P_{2^m}(h)\|_2 + \sum_{k=0}^{m-1} \| P_{2^{k+1}}(h) - P_{2^k}(h)\|_2 + \| P_1(h)\|_2 \\ &\leqq c_2' \{n^\beta \| P_n(h) - P_{2^m}(h)\|_0 + \sum_{k=0}^{m-1} (2^{k+1})^\beta \| P_{2^{k+1}}(h) - P_{2^k}(h)\|_0\} \\ &\leqq 2c_2' \sum_{k=0}^{m} (2^{k+1})^\beta E_{2^k}(h)_0 \\ &\leqq 2c_1c_2' \sum_{k=0}^{m} (2^{k+1})^\beta 2^{-k\alpha} \|h\|_1 \leqq 2^{1+2\beta-\alpha}(2^{\beta-\alpha}-1)^{-1} c_1 c_2' n^{\beta-\alpha} \|h\|_1,\end{aligned}$$

which combined with (2.3) implies (2.6). □

As a corollary of Proposition 2.3 we get that any Bernstein type inequality (2.1) with parameter $\beta > \alpha$ implies the equivalence of the K-functional (1.1) with the same functional augmented with a term for the space X_2, provided (1.2), (1.3) and (1.6) are valid:

Corollary 2.4. *Let* (1.2), (1.3) *and* (1.6) *hold. Then* (2.1) *with* $\beta > \alpha$ *implies*

$$\inf\{\|f-g\|_0 + n^{-\alpha}\|g\|_1 + n^{-\beta}\|g\|_2 : g \in X_1 \cap X_2\} \leqq c_4 K(f,\ n^{-\alpha};\ X_0,\ X_1).$$

Now we shall give another equivalent to (1.6) condition.

Theorem 2.5. *Let* (1.2) *and* (1.3) *hold. Assume that there is a subspace* X_2 *of* X_0 *with seminorm* $\|\cdot\|_2$ *such that* $\|g\|_2 = 0$ *for every* $g \in G_1$ *and family* $\{G_n\}$ *satisfies Bernstein inequality* (2.1) *and Jackson inequality with parameter* $\beta > \alpha$ *with respect to the couple* $(X_0,\ X_2)$, i.e.

(2.7) $$E_n(f)_0 \leqq c_1' n^{-\beta} \|f\|_2 \quad \forall f \in X_2;$$

Then (1.6) *is equivalent to the condition*: *for every* $f \in X_0$ *we have*

(2.8) $$\inf\{\|f-g\|_0 + n^{-\alpha}\|g\|_1 + n^{-\beta}\|g\|_2 : g \in X_1 \cap X_2\} \leqq c_4 K(f,\ n^{-\alpha};\ X_0,\ X_1).$$

Proof. The implication "(1.6) ⇒ (2.8)" is Corollary 2.4. Let (2.8) be fulfilled. For a given $f \in X_0$ let $g \in X_1 \cap X_2$ be an element for which

(2.9) $$\|f-g\|_0 + n^{-\alpha}\|g\|_1 + n^{-\beta}\|g\|_2 \leqq c_4 K(f,\ n^{-\alpha};\ X_0,\ X_1).$$

Consider the series

(2.10) $$\sum_{k=1}^{\infty} \{P_{n2^{k-1}}(g) - P_{n2^k}(g)\}.$$

In view of (1.2) it converges to $P_n(g) - g$ in X_0. Using (1.3), (2.7) we get

$$(2.11) \qquad \sum_{k=1}^{\infty} \| P_{n2^{k-1}}(g) - P_{n2^k}(g) \|_1$$

$$\leqq c_2 \sum_{k=1}^{\infty} (n2^k)^{\alpha} \| P_{n2^{k-1}}(g) - P_{n2^k}(g) \|_0$$

$$\leqq c_2 \sum_{k=1}^{\infty} (n2^k)^{\alpha} \{ \| P_{n2^{k-1}}(g) - g \|_0 + \| P_{n2^k}(g) - g \|_0 \}$$

$$\leqq 2c_2 \sum_{k=1}^{\infty} (n2^k)^{\alpha} E_{n2^{k-1}}(g)_0$$

$$\leqq 2c_1' c_2 \sum_{k=1}^{\infty} (n2^k)^{\alpha} (n2^{k-1})^{-\beta} \| g \|_2$$

$$= 2^{1+\alpha}(1-2^{\alpha-\beta})^{-1} c_1' c_2 n^{\alpha-\beta} \| g \|_2 =: c n^{\alpha-\beta} \| g \|_2 .$$

Therefore, series (2.10) converges in X_1 and the limit has to be $P_n(g) - g$.

Now (2.9) and (2.11) imply

$$(2.12) \qquad n^{-\alpha} \| P_n(g) \|_1 \leqq n^{-\alpha} \| g \|_1 + n^{-\alpha} \| P_n(g) - g \|_1$$

$$\leqq n^{-\alpha} \| g \|_1 + c n^{-\beta} \| g \|_2 \leqq c_4 c K(f, n^{-\alpha}; X_0, X_1).$$

Moreover (2.9) and (1.2) give

$$\| f - P_n(g) \|_0 \leqq \| f - g \|_0 + \| g - P_n(g) \|_0 \leqq \| f - g \|_0 + c_1 n^{-\alpha} \| g \|_1$$

$$\leqq c_4 c_1 K(f, n^{-\alpha}; X_0, X_1),$$

which combined with (2.12) gives (1.6). □

At the second part of this section we give one possible generalization of the problem described. The reason is that in some cases of constrained approximation one does need a K-functional more complicated than (1.1) (see e. g. Section 5 b), c)).

We replace the space X_1 by a finite set of spaces $X_{1,i}$ with (semi-)norms $\| \cdot \|_{1,i}$, $i = 1, 2, \ldots, j$, and change (1.1) to

$$(1.1') \qquad K(f, t_1, \ldots, t_j) := K(f, t_1, \ldots, t_j; X_0, X_{1,1}, \ldots, X_{1,j})$$

$$:= \inf \{ \| f - g \|_0 + \sum_{i=1}^{j} t_i \| g \|_{1,i} : g \in \bigcap_{i=1}^{j} X_{1,i} \}.$$

Jackson and Bernstein type inequalities (1.2) and (1.3) are to be replaced by

$$(1.2') \qquad E_n(f)_0 \leqq c_1 \sum_{i=1}^{j} n^{-\alpha_i} \| f \|_{1,i}, \quad \forall f \in \bigcap_{i=1}^{j} X_{1,i},$$

$$(1.3') \qquad \| g_1 - g_2 \|_{1,i} \leqq c_2 n^{\alpha_i} \| g_1 - g_2 \|_0, \quad \forall g_1, g_2 \in G_n, \quad i = 1, \ldots, j.$$

Then the direct and converse inequalities (1.4) and (1.5) remain true if one replaces $K(f, n^{-\alpha})$ by $K(f, n^{-\alpha_1}, \ldots, n^{-\alpha_j})$. Inequality (1.6) takes the form

$$(1.6') \qquad \inf\{\|f-g\|_0+\sum_{i=1}^{j} n^{-\alpha_i}\|g\|_{1,i}:g\in G_n\}$$

$$\leqq c_3 K(f,\ n^{-\alpha_1},\dots,n^{-\alpha_j};\ X_0,\ X_{1,i},\dots,X_{1,j})\quad \forall f\in X_0.$$

Now, going along the lines of the proofs of Theorems 2.2 and 2.5 we obtain

Theorem 2.6. *Let* (1.2′) *and* (1.3′) *hold. Then the following are equivalent:*
a) (1.6′) *holds for* G_n;
b) $\|P_n(f)\|_{1,i}\leqq cn^{\alpha_i}K(f,\ n^{-\alpha_1},\dots,n^{-\alpha_j})\quad \forall f\in X_0,\quad i=1,\dots,j;$
c) $\|Q_n(f)\|_{1,i}\leqq cn^{\alpha_i}K(f,\ n^{-\alpha_1},\dots,n^{-\alpha_j})\quad \forall f\in X_0,\quad i=1,\dots,j;$
d) $\|g\|_{1,i}\leqq cn^{\alpha_i}K(g,\ n^{-\alpha_1},\dots,n^{-\alpha_j})\quad \forall g\in G_n,\quad i=1,\dots,j,$
where constants c in b) *and* c) *depend only on* $c_1,\ c_2,\ c_3$ *and* A.

Theorem 2.7. *Let* (1.2′) *and* (1.3′) *hold. Assume that there are subspaces* $X_{2,i}$ *of* X_0 *with seminorms* $\|\cdot\|_{2,i}$, $i=1,2,\dots,j$, *such that* $\|g\|_{2,i}=0$ *for every* $g\in G_1$ *and family* $\{G_n\}$ *satisfies Jackson and Bernstein type inequalities with parameters* $\beta_i>\alpha_i$, i.e.

$$E_n(f)_0\leqq c_1'\sum_{i=1}^{j} n^{-\beta_i}\|f\|_{2,i},\quad \forall f\in \bigcap_{i=1}^{j} X_{2,i},$$

$$\|g_1-g_2\|_{2,i}\leqq c_2' n^{\beta_i}\|g_1-g_2\|_0,\quad \forall g_1,\ g_2\in G_n,\quad i=1,\dots,j.$$

Then (1.6′) *is equivalent to the condition: for every* $f\in X_0$ *we have*

$$\inf\{\|f-g\|_0+\sum_{i=1}^{j}[n^{-\alpha_i}\|g\|_{1,i}+n^{-\beta_i}\|g\|_{2,i}]:g\in \bigcap_{i=1}^{j}[X_{1,i}\cap X_{2,i}]\}$$

$$\leqq c_4 K(f,\ n^{-\alpha_1},\dots,n^{-\alpha_j};\ X_0,\ X_{1,1},\dots,X_{1,j}).$$

Comparing (1.2) and (1.2′) we see that the second inequality is weaker ((1.2) implies (1.2′)), while among (1.3) and (1.3′) the first inequality is weaker. Although (1.6) and (1.6′) look independent we can show that under some assumptions they are equivalent:

Theorem 2.8. *Let* (1.2) *and* (1.3′) *hold for the spaces* $X_0,\ X_1=X_{1,1},\ X_{1,2},\dots,X_{1,j}$, *let* $\alpha=\alpha_1$ *be less than* $\alpha_2,\dots,\alpha_j$ *and let* $\|g\|_{1,i}=0$ *for every* $g\in G_1$, $i=2,\dots,j$. *Then*

$$(2.13) \qquad K(f,\ n^{-\alpha},n^{-\alpha_2},\dots,n^{-\alpha_j})\leqq cK(f,\ n^{-\alpha})$$

and (1.6) *and* (1.6′) *are equivalent.*

Proof. Inequality (2.13) follows from Proposition 2.3 where we take X_2 and β to be $X_{1,i}$ and α_i, $i=2,\dots,j$. (2.13) shows that (1.6′) implies (1.6).

Now let (1.6) hold. Once more, from Proposition 2.3, $X_2=X_{1,i}$, $\beta=\alpha_i$, we get

$$\|Q_n(f)\|_{1,i}\leqq cn^{\alpha_i}K(f,\ n^{-\alpha_1},\dots,n^{-\alpha_j}),\quad i=1,\dots,j$$

and then Theorem 2.6 implies (1.6′). □

3. Approximations for which (1.6) holds

In this section we give several examples of approximation processes for which properties (1.2), (1.3) and (1.6) hold. By c we denote different positive numbers depending only on the parameters following in parentheses.

3.1. Approximation by trigonometric polynomials

Let r be natural, $1 \leqq p \leqq \infty$, $X_0 = L_p = L_p[0, 2\pi)$ — the space of all 2π-periodic functions from L_p with the norm

$$\|f\|_p = \{\int_0^{2\pi} |f(x)|^p dx\}^{1/p}$$

with the usual change to sup norm for $p = \infty$ and $X_1 = W_p^r = W_p^r[0, 2\pi)$ — the space of all 2π-periodic functions with r-th derivative in L_p equipped with the semi-norm $\|f^{(r)}\|_p$. The equivalence of the K-functional with the modulus of smoothness

$$\omega_r(f, t)_p = \sup\{\|\Delta_s^r f(\cdot)\|_p : |s| \leqq t\},$$

$$\Delta_s^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x+is),$$

is well known in this case: there are constants $C_1(r)$, $C_2(r)$ such that for every $f \in L_p[0, 2\pi)$ we have

$$C_1(r)\omega_r(f, t)_p \leqq K(f, t^r; L_p, W_p^r) \leqq C_2(r)\omega_r(f, t)_p. \tag{3.1}$$

Let G_n be the space of all trigonometric polynomials of degree $n-1$. It is well known (see e. g. [13, p. 275, 230]) that in this case Jackson and Bernstein inequalities (1.2) and (1.3) hold with $\alpha = r$:

$$E_n(f)_p \leqq c(r) n^{-r} \|f^{(r)}\|_p \quad \forall f \in W_p^r, \tag{3.2}$$

$$\|g^{(r)}\|_p \leqq n^r \|g\|_p \quad \forall g \in G_n. \tag{3.3}$$

In order to get (1.6) we utilize Theorem 2.5. We take W_p^{r+1} and $\|f^{(r+1)}\|_p$ for the space X_2 and the semi-norm $\|f\|_2$ respectively. Obviously the $r+1$-st derivative of any constant is zero and inequalities (3.2) and (3.3) with $r+1$ instead of r show that conditions (2.7) and (2.1) are fulfilled with parameter $\beta = r+1$. In order to prove (2.8) we make use of the modified Steklov function

$$g(x) = t^{-k} \int_0^t \cdots \int_0^t \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} f(x + \frac{i}{k}(u_1 + \ldots + u_k))\, du_1 \ldots du_k,$$

where g depends on f, t and k. From this definition we easily see that $g \in W_p^k$ (provided $f \in L_p$) and satisfies the inequalities

$$\|f-g\|_p \leqq \omega_k(f, t)_p, \tag{3.4}$$

$$\|g^{(m)}\|_p \leqq c(k) t^{-m} \omega_m(f, t)_p \text{ for every } m = 1, 2, \ldots, k. \tag{3.5}$$

Using the trivial property of the moduli $\omega_{r+1}(f, t)_p \leqq 2\omega_r(f, t)_p$, we get from (3.4) and (3.5) with $k=r+1$, $m=r$, $m=r+1$, $t=n^{-1}$ and from (3.1) with $t=n^{-1}$

$$\|f-g\|_p + n^{-r}\|g^{(r)}\|_p + n^{-r-1}\|g^{(r+1)}\|_p \leqq c(r)\omega_r(f, n^{-1})_p \leqq c(r)K(f, n^{-r}; L_p, W_p^r).$$

Thus condition (2.8) is also fulfilled and Theorem 2.5 gives

Proposition 3.1. *Let r be natural, $1 \leqq p \leqq \infty$, $X_0 = L_p$, $X_1 = W_p^r$ and let G_n be the space of all trigonometric polynomials of degree $n-1$. Then inequality* (1.6) *holds for any $f \in L_p$ with $\alpha = r$.*

From Proposition 3.1, (3.1) and Theorem 2.2 we get

Proposition 3.2. *Under the assumptions of Proposition* 3.1 *we have*

$$\| P_n(f)^{(r)} \|_p \leqq c(r) n^r \omega_r(g, n^{-1})_p.$$

for every $f \in L_p$, where $P_n(f)$ denotes the polynomial of best L_p approximation to f out of G_n. Moreover for every $g \in G_n$ we have

$$\|g^{(r)}\|_p \leqq c(r) n^r \omega_r(g, n^{-1})_p.$$

The above proposition gives $(0 \leqq \gamma \leqq r)$:
If $\omega_r(f, t)_p = O(t^\gamma)$ then $\| P_n(f)^{(r)} \|_p = O(n^{r-\gamma})$.

This statement represents the original result of M. Zamansky [16] (if we replace the condition on the r-th modulus of f with the equivalent for $r-1<\gamma<r$ condition on the first modulus of $f^{(r-1)}$). This is the reason why we call inequalities Theorem 2.2 b), Zamansky type results. As far as we know Proposition 3.2 for $p=\infty$ has been proved by S. B. Stechkin [17] deriving the first inequality from the second one.

3.2. Approximation by algebraic polynomials

Let r be natural, $\varphi(x) = \sqrt{1-x^2}$, $1 \leqq p \leqq \infty$, $X_0 = L_p = L_p[-1, 1]$ — the space of all functions from L_p with the norm

$$\|f\|_p = \{ \int_{-1}^{1} |f(x)|^p \, dx \}^{1/p}$$

and let $X_1 = W_p^r(\varphi) = W_p^r(\varphi; [-1, 1])$ — the space of all functions f defined in $[-1, 1]$ such that $\varphi^r f^{(r)} \in L_p$ equipped with the semi-norm $\| \varphi^r f^{(r)} \|_p$. A characterization of

$$K(f, t^r; L_p, W_p^r(\varphi)) := \inf \{ \|f-g\|_p + t^r \| \varphi^r g^{(r)} \|_p : g \in W_p^r(\varphi) \}$$

(similar to (3.1)) by two different type moduli of smoothness is given in [2, p. 11] and [7].

Let G_n be the space of all algebraic polynomials of degree $n-1$. The Jackson type inequality (1.2) for this case with $\alpha = r$

(3.6) $$E_n(f)_p \leqq c(r) n^{-r} \| \varphi^r f^{(r)} \|_p \quad \forall f \in W_p^r(\varphi),$$

has been proved in [2, p. 79] or [6]. What we call a Bernstein type inequality is now exactly the original Bernstein inequality (when $p=\infty$ and $r=1$, for the general case see e. g. [2, p. 92])

(3.7) $$\| \varphi^r g^{(r)} \|_p \leqq c(r) n^r \| g \|_p, \quad \forall g \in G_n.$$

One can repeat the way of proving (1.6) from Section 3.1, but here we shall utilize Theorem 2.2. A Zamansky type inequality

(3.8) $$n^{-r} \| \varphi^r P_n(f)^{(r)} \|_p \leqq c(r) K(f, n^{-r}; L_p, W_p^r(\varphi))$$

is proved in Theorem 7.3.1 in [2]. Therefore Theorem 2.2, b) => a), gives

Proposition 3.3. *Let r be natural, $1 \leqq p \leqq \infty$, $X_0 = L_p, X_1 = W_p^r(\varphi)$ and let G_n be the space of all algebraic polynomials of degree $n-1$. Then inequality* (1.6) *holds for any $f \in L_p$ with $\alpha = r$.*

From Markov inequality

(3.9) $$\| g^{(r)} \|_p \leqq c(r) n^{2r} \| g \|_p \quad \forall g \in G_n,$$

Proposition 3.3, Proposition 2.3 and Theorem 2.8 we get

Proposition 3.4. *For natural r, $1 \leqq p \leqq \infty$ and for any $f \in L_p[-1, 1]$ we have*

(3.10) $$K(f, n^{-r}; L_p, W_p^r(\varphi)) \leqq K(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$$

$$:= \inf \{ \| f-g \|_p + n^{-r} \| \varphi^r g^{(r)} \|_p + n^{-2r} \| g^{(r)} \|_p : g \in W_p^r \} \leqq c(r) K(f, n^{-r}; L_p, W_p^r(\varphi))$$

and condition (1.6′) *holds with $\alpha_1 = r$, $\alpha_2 = 2r$, $X_{1,1} = W_p^r(\varphi)$, $X_{1,2} = W_p^r$.*

Here we have deduced from inequalities (3.6) and (1.6) the equivalence of the functionals $K(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$ and $K(f, n^{-r}; L_p, W_p^r(\varphi))$ – a fact concerning only the spaces L_p, $W_p^r(\varphi)$ and W_p^r, but not the approximating family $\{G_n\}$ at all. In practice we have the reverse situation. First one gets the equivalence (3.10) (see [2, p. 24] or [7]), after that the Jackson type inequality

(3.11) $$E_n(f)_p \leqq c(r) \{ n^{-r} \| \varphi^r f^{(r)} \|_p + n^{-2r} \| f^{(r)} \|_p \} \quad \forall f \in W_p^r$$

(see [2, p. 80] or [6]) and finally (3.6) is deduced from (3.10) and (3.11). Note that inequalities of type (3.11) can be easier proved than inequalities of type (3.6).

From Proposition 3.3 and Theorem 2.2 we get

Proposition 3.5. *Under the assumption of Proposition* 3.3 *for every $g \in G_n$ we have*

$$\| \varphi^r g^{(r)} \|_p \leqq c n^r K(g, n^{-r}; L_p, W_p^r(\varphi)).$$

4. Approximation with constraints

In the general setting of Sections 1 and 2 assume that with every $f\in X_0^*$ ($X_0^*\subset X_0$) we associate a subset $R(f)$ of X_0. $R(f)$ can be thought as the restriction which we impose on the functions approximating f. Various type of constraints are given in Sections 5 and 6.

The only assumptions (necessary for the direct result — Theorem 4.1), made for the constraint are

$$R(g)\subset R(f),\quad \forall g\in R(f),\ \text{and}\ X_1\cap R(f)\subset X_0^*\quad \forall f\in X_0^*. \tag{4.1}$$

The constrained K-functional of $f\in X_0^*$ is given by

$$K^*(f,t;X_0,X_1):=\inf\{\|f-g\|_0+t\|g\|_1 : g\in X_1\cap R(f)\}$$

and the constrained best approximation of f by elements of G_n is

$$E_n^*(f)_0:=\inf\{\|f-g\|_0 : g\in G_n\cap R(f)\}.$$

If Jackson and Bernstein inequalities similar to (1.2) and (1.3) are proved for the constrained approximations then one can obtain direct and converse statements like (1.4) and (1.5). In our setting Bernstein inequality will not be connected with the type of the constraint, i.e. we shall use condition (1.3), but in general (1.3) may be fulfilled only for $g_1, g_2\in G_n\cap X_0^*$. The corresponding Jackson type inequality is

$$E_n^*(f)_0\leqq c_1''n^{-\alpha}\|f\|_1\quad \forall f\in X_1\cap X_0^*. \tag{4.2}$$

Theorem 4.1. *If* (4.1) *and* (4.2) *are fulfilled then for* $f\in X_0^*$ *we have*

$$E_n^*(f)_0\leqq c_1''K^*(f,n^{-\alpha};X_0,X_1).$$

P r o o f. Let $g\in X_1\cap R(f)$ be such that

$$\|f-g\|_0+n^{-\alpha}\|g\|_1=K^*(f,n^{-\alpha};X_0,X_1). \tag{4.3}$$

Then $g\in X_0^*$ because of (4.1) and from (4.2) we get $h\in G_n\cap R(g)$ for which

$$\|g-h\|_0\leqq c_1''n^{-\alpha}\|g\|_1. \tag{4.4}$$

Now (4.1) implies $h\in G_n\cap R(f)$ and using (4.3) and (4.4) we obtain $E_n^*(f)_0\leqq f-h\|_0\leqq\|f-g\|_0+\|g-h\|_0$

$$\leqq\|f-g\|_0+c_1''n^{-\alpha}\|g\|_1\leqq c_1''K^*(f,n^{-\alpha};X_0,X_1). \qquad \square$$

Theorem 4.2. *If* (1.3) *and* (1.6) *are fulfilled then for* $f\in X_0^*$ *we have*

$$K^*(f, n^{-\alpha}; X_0, X_1) \leqq (1+c_2) E_n^*(f)_0 + c_2 c_3 K(f, n^{-\alpha}; X_0, X_1).$$

Proof. Let $g \in G_n \cap R(f)$ be such that

$$\|f-g\|_0 = E_n^*(f)_0 \tag{4.5}$$

and (see (1.6)) let $h \in G_n$ be such that

$$\|f-h\|_0 + n^{-\alpha}\|h\|_1 \leqq c_3 K(f, n^{-\alpha}; X_0, X_1). \tag{4.6}$$

Then using (4.5), (1.3) and (4.6) we obtain

$$\begin{aligned} &K^*(f, n^{-\alpha}; X_0, X_1) \\ &\leqq \|f-g\|_0 + n^{-\alpha}\|g\|_1 \\ &\leqq E_n^*(f)_0 + n^{-\alpha}\|g-h\|_1 + n^{-\alpha}\|h\|_1 \\ &\leqq E_n^*(f)_0 + c_2\|g-h\|_0 + n^{-\alpha}\|h\|_1 \\ &\leqq E_n^*(f)_0 + c_2\|f-g\|_0 + c_2\{\|f-h\|_0 + n^{-\alpha}\|h\|_1\} \\ &\leqq (1+c_2)E_n^*(f)_0 + c_2 c_3 K(f, n^{-\alpha}; X_0, X_1). \end{aligned}$$ □

Theorem 4.2 looks as a strong-type inverse result. Indeed, if for some f and n we have that the constrained K-functional is several times bigger than the non-constrained one, e. g.

$$K^*(f, n^{-\alpha}) \geqq 2c_2 c_3 K(f, n^{-\alpha}) \tag{4.7}$$

then Theorem 4.2 implies

$$K^*(f, n^{-\alpha}; X_0, X_1) \leqq 2(1+c_2) E_n^*(f)_0.$$

But in general (4.7) may not be satisfied. Then from Theorem 4.2 and (1.5) using that $E_n(f)_p \leqq E_n^*(f)_p$ we get

$$K^*(f, n^{-\alpha}; X_0, X_1) \leqq c(\alpha) n^{-\alpha} \sum_{k=1}^{n} k^{\alpha-1} E_k^*(f)_0. \tag{4.8}$$

If we have converse theorems better than (1.5) (see next section) then, in view of Theorem 4.2, the corresponding inverse results are inherited by the constrained approximations.

Instead of proving an analog of Theorem 2.2 we shall show that (1.6) implies a similar inequality and Zamansky type inequalities in the constrained case.

Denote by $P_n^*(f)\in G_n\cap R(f)$ an element of best constrained approximation from G_n to $f\in X_0^*$ (assuming it exists) and by $Q_n^*(f)\in G_n\cap R(f)$ an element in G_n approximating f with the order of the K^*-functional, i.e.

$$\|f-P_n^*(f)\|_0=E_n^*(f)_0, \quad \|f-Q_n^*(f)\|_0\leqq AK^*(f,n^{-\alpha};X_0,X_1),$$

where A is a fixed positive constant.

Theorem 4.3. *Let* (1.2), (1.3), (1.6), (4.1) *and* (4.2) *hold. Then*

(4.9) $\inf\{\|f-g\|_0+n^{-\alpha}\|g\|_1 : g\in G_n\cap R(f)\}\leqq cK^*(f,n^{-\alpha};X_0,X_1)$;

$$n^{-\alpha}\|P_n^*(f)\|_1\leqq cK^*(f,n^{-\alpha};X_0,X_1); \tag{4.10}$$

$$n^{-\alpha}\|Q_n^*(f)\|_1\leqq cK^*(f,n^{-\alpha};X_0,X_1), \tag{4.11}$$

where constants c *depend only on* c_1'', c_2, c_3 *and* A.

Proof. From Theorem 2.2, a) => d), we have

$$\begin{aligned} n^{-\alpha}\|Q_n^*(f)\|_1 &\leqq cK(Q_n^*(f),n^{-\alpha};X_0,X_1)\\ &\leqq cK(Q_n^*(f)-f,n^{-\alpha};X_0,X_1)+cK(f,n^{-\alpha};X_0,X_1)\\ &\leqq c\|Q_n^*(f)-f\|_0+cK^*(f,n^{-\alpha};X_0,X_1)\leqq cK^*(f,n^{-\alpha};X_0,X_1), \end{aligned}$$

which is (4.11). From (4.11) and Theorem 4.1 we get (4.10). Finally (4.11) and the definition of $Q_n^*(f)$ give (4.9). □

In some situations (see e. g. Section 5) condition (4.2) may not be fulfilled but one has

$$E_n^*(f)_0\leqq c_1''\sum_{i=1}^{j} n^{-\alpha_i}\|f\|_{1,i} \quad \forall f\in \bigcap_{i=1}^{j} X_{1,i}\cap X_0^*. \tag{4.2'}$$

Repeating the above arguments we get

Theorem 4.4. *If* (4.1) *and* (4.2′) *are fulfilled then for* $f\in X_0^*$ *we have*

$$\begin{aligned} E_n^*(f)_0 &\leqq c_1''K^*(f,n^{-\alpha_1},\dots,n^{-\alpha_j};X_0,X_{1,1},\dots,X_{1,j}),\\ &:=c_1''\inf\{\|f-g\|_0+\sum_{i=1}^{j} n^{-\alpha_i}\|g\|_{1,i} : g\in \bigcap_{i=1}^{j} X_{1,i}\cap R(f)\}. \end{aligned}$$

Theorem 4.5. *If* (1.3′) *and* (1.6′) *are fulfilled then for* $f\in X_0^*$ *we have*

$$\begin{aligned} &K^*(f,n^{-\alpha_1},\dots,n^{-\alpha_j};X_0,X_{1,i},\dots,X_{1,j})\\ &\leqq(1+c_2)E_n^*(f)_0+c_2c_3K(f,n^{-\alpha_1},\dots,n^{-\alpha_j};X_0,X_{1,i},\dots,X_{1,j}). \end{aligned}$$

5. Onesided approximations

Here we apply the scheme from the previous section in the following three cases of constraints:

Let X_0 be a space of real-valued functions defined on a set Ω.

I. Approximation from below. Then X_0^* contains these functions from X_0 which are bounded from below and

$$R(f)=\{g\in X_0 : g(x)\leqq f(x) \quad \forall x\in\Omega\}=\{f\}-C_+,$$

where C_+ denote the cone of all non-negative functions, i.e.

$$C_+=\{g\in X_0 : g(x)\geqq 0 \quad \forall x\in\Omega\}.$$

The best onesided approximation from below with elements of G_n is given by

$$E_n^-(f)_0:=\inf\{\|f-g\|_0 : g\in G_n, g\leqq f\}.$$

II. Approximation from above. Similarly X_0^* contains these functions from X_0 which are bounded from above and

$$R(f)=\{g\in X_0 : g(x)\geqq f(x) \quad \forall x\in\Omega\}=\{f\}+C_+.$$

The best onesided approximation from above with elements of G_n is given by

$$E_n^+(f)_0:=\inf\{\|f-g\|_0 : g\in G_n, g\geqq f\}.$$

III. Onesided approximation. One approximates a function f simultaneously from above and from below. This case is a combination of I and II. The best onesided approximation with elements of G_n is given by

$$\tilde{E}_n(f)_0:=E_n^-(f)_0+E_n^+(f)_0.$$

In these cases the constraint is a cone with a vertex at f.

a) Trigonometric polynomials — univariate case

Let, as in Section 3.1, G_n denote the set of all trigonometric polynomials of degree $n-1$, $X_0=L_p[0,2\pi)$ and $X_1=W_p^r[0,2\pi)$ ($1\leqq p\leqq\infty$, natural r). Condition (4.1) is obviously satisfied because every function from X_1 is bounded. The constrained approximations corresponding to the three cases are denoted by $E_n^-(f)_p$, $E_n^+(f)_p$ and $\tilde{E}_n(f)_p$ respectively. $\tilde{E}_n(f)_p$ slightly differs from the usual definition $\bar{E}_n(f)_p=\inf\{\|g_1-g_2\|_p : g_1, g_2\in G_n;\ g_1\leqq f\leqq g_2\}$ but we have $2^{(1-p)p}\,\tilde{E}_n(f)_p\leqq\bar{E}_n(f)_p\leqq\tilde{E}_n(f)_p$.

The case of importance for onesided approximation is $p<\infty$. For $p=\infty$ we obviously have $E_n^-(f)_\infty=E_n^+(f)_\infty=2E_n(f)_\infty$.

The corresponding K-functionals are

$$K^-(f,t;L_p,W_p^r):=\inf\{\|f-g\|_p+t\|g^{(r)}\|_p: g\in W_p^r, g\leqq f\},$$

$$K^+(f,t;L_p,W_p^r):=\inf\{\|f-g\|_p+t\|g^{(r)}\|_p: g\in W_p^r,\ g\geqq f\},$$

$$\tilde{K}(f,t;L_p,W_p^r):=K^-(f,t;L_p,W_p^r)+K^+(f,t;L_p,W_p^r).$$

A characterization of the onesided K-functional is provided by the average moduli of smoothness (see e. g. [12])

$$\tau_r(f,t)_p:=\|\omega_r(f,\cdot,t)\|_p,$$

$$\omega_r(f,x,t):=\sup\{|\Delta_s^r f(y)|: y, y+rs\in[x-rt/2, x+rt/2]\}.$$

Similarly to (3.1) we have (see [9])

$$c(r)\tau_r(f,t)_p\leqq\tilde{K}(f,t^r;L_p,W_p^r)\leqq c(r)\tau_r(f,t)_p. \tag{5.1}$$

In forthcoming paper [5] we give a characterization of the onesided K-functionals from below and from above via moduli of smoothness for $r=1$ and $r=2$. The problem for characterizing these functionals for $r\geqq 3$ is open.

For the Jackson type inequality

$$\tilde{E}_n(f)_p\leqq c(r)n^{-r}\|f^{(r)}\|_p,\quad \forall f\in W_p^r \tag{5.2}$$

see [12, p. 166]. (5.2) implies similar inequalities for the onesided approximations from below and from above and hence Theorems 4.1 and 4.2 together with (3.3) and Proposition 3.1 imply

Theorem 5.1. *Let* $1\leqq p\leqq\infty$, *r be natural, f be bounded (from below, from above) and measurable. Then we have*

$$c(r)E_n^-(f)_p\leqq K^-(f,n^{-r};L_p,W_p^r)\leqq c(r)\{E_n^-(f)_p+K(f,n^{-r};L_p,W_p^r)\};$$

$$c(r)E_n^+(f)_p\leqq K^+(f,n^{-r};L_p,W_p^r)\leqq c(r)\{E_n^+(f)_p+K(f,n^{-r};L_p,W_p^r)\};$$

$$c(r)\tilde{E}_n(f)_p\leqq\tilde{K}(f,n^{-r};L_p,W_p^r)\leqq c(r)\{\tilde{E}_n(f)_p+K(f,n^{-r};L_p,W_p^r)\}.$$

Now (4.8) reads

$$\tilde{K}(f,n^{-r};L_p,W_p^r)\leqq c(r)n^{-r}\sum_{k=1}^{n}k^{r-1}\tilde{E}_k(f)_p \tag{5.3}$$

and similarly for the onesided approximations from below and from above. For $1<p<\infty$ (5.3) can be improved. Using (3.1), Theorem 5.1 and the inverse result of M. F. Timan [19] ($q=\min\{2,p\}$, $1<p<\infty$)

$$\omega_r(f,t)_p \leqq c(r,p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1}E_k(f)_p^q\}^{1/q}$$

we get

Corollary 5.2. *Let* $1<p<\infty$, $q=\min\{p, 2\}$, r *be natural,* f *be bounded and measurable. Then we have*

$$K^-(f,n^{-r};L_p,W_p^r)\leqq c(r,p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1}E_k^-(f)_p^q\}^{1/q};$$

$$K^+(f,n^{-r};L_p,W_p^r)\leqq c(r,p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1}E_k^+(f)_p^q\}^{1/q};$$

$$\tilde{K}(f,n^{-r};L_p,W_p^r)\leqq c(r,p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1}\tilde{E}_k(f)_p^q\}^{1/q}.$$

From Theorem 5.1, (3.1) and (5.1) we also deduce the following new properties of the average moduli of smoothness:

Corollary 5.3. *Let* $1\leqq p\leqq\infty$, r *and* m *be natural,* f *be bounded and measurable. Then we have*

$$\tau_r(f,t)_p \leqq c(r,m)\{\tau_{r+m}(f,t)_p+\omega_r(f,t)_p\}.$$

This corollary and Marchaud inequality for ω imply Marchaud type inequalities for the averaged moduli of smoothness.

From Theorem 5.1, Theorem 4.3 and Proposition 3.1 we get the following inequalities of Zamansky type

Corollary 5.4. *Under the assumptions of Theorem* 5.1 *we have*

$$\|P_n^-(f)^{(r)}\|_p \leqq c(r)n^r K^-(f,n^{-r};L_p,W_p^r);$$

$$\|P_n^+(f)^{(r)}\|_p \leqq c(r)n^r K^+(f,n^{-r};L_p,W_p^r);$$

$$\|\tilde{P}_n(f)^{(r)}\|_p \leqq c(r)n^r \tilde{K}(f,n^{-r};L_p,W_p^r),$$

where $P_n^-(f)$, $P_n^+(f)$ *and* $\tilde{P}_n(f)$ *are the corresponding polynomials of best trigonometric onesided approximation to* f.

b) Trigonometric polynomials — multivariate case

Let $1\leqq p\leqq\infty$, d, r, e be natural, e being the smallest integer $\geqq r$ and $>d/p$, $\Omega=[0,2\pi)^d$, $X_0=L_p(\Omega)$, $X_{1,1}=W_p^r(\Omega)$ and $X_{1,2}=W_p^e(\Omega)$. The semi-norm in the Sobolev spaces are defined as the sum of the norms of all partial derivatives of the given order, i. e.

$$\|g\|_{1,1} := \sum_{|\gamma|=r} \|D^\gamma g\|_p, \quad \|g\|_{1,2} := \sum_{|\gamma|=e} \|D^\gamma g\|_p,$$

where γ is a multiindex and D^γ denotes the corresponding differential operator. Spaces $X_{1,1}$ and $X_{1,2}$ coincide iff $r>d/p$. Let G_n denote the set of all trigonometric polynomials of total degree $n-1$. The Jackson type inequality now is (see [3])

$$\text{(5.4)} \qquad \tilde{E}_n(f)_p \leqq c(r)\{n^{-r}\|f\|_{1,1}+n^{-e}\|f\|_{1,2}\} \quad \forall f\in W_p^e(\Omega).$$

In the univariate case we have $r=e$ (except for $r=p=1$ when $e=2$). But for $d\geqq 2$ we have $e>r$ whenever $r\leqq d/p$. In these cases, unlike the non-constrained approximation, an estimate of the type

$$\tilde{E}_n(f)_p \leqq c(r)\, n^{-r}\|f\|_{1,1} \quad \forall f\in W_p^r(\Omega)$$

is not true (see e. g. [11]). Thus we are forced to use the more complicated scheme in which the space X_1 is replaced by the set of spaces $X_{1,1},\dots,X_{1,j}$. The K-functionals are

$$K^-(f,n^{-r},n^{-e};L_p,W_p^r,W_p^e) := \inf\{\|f-g\|_p+n^{-r}\|g\|_{1,1}+n^{-e}\|g\|_{1,2} : g\in W_p^e, g\leqq f\}$$

and analogously for K^+ and $\tilde{K}$. We shall not formulate the statements because of their similarity to these ones from the first part of the section. Let us only mention that the validity of (1.6) in this case ($X_1=W_p^r$) can be established following the univariate case in Section 3.

c) Algebraic polynomials — univariate case

Let G_n denote the set of all algebraic polynomials of degree $n-1$. Keeping the notations of Section 3.2 ($\varphi(x)=\sqrt{1-x^2}$, $X_0=L_p[-1,1]$, $X_{1,1}=W_p^r(\varphi)$ and $X_{1,2}=W_p^r$) denote once more the constrained approximations corresponding to the three cases by $E_n^-(f)_p$, $E_n^+(f)_p$ and $\tilde{E}_n(f)_p$ respectively.

The corresponding K-functionals are

$$K^-(f,n^{-r},n^{-2r};L_p,W_p^r(\varphi),W_p^r)$$
$$:=\inf\{\|f-g\|_p+n^{-r}\|\varphi^r g^{(r)}\|_p+n^{-2r}\|g^{(r)}\|_p : g\in W_p^r, g\leqq f\},$$
$$K^+(f,n^{-r},n^{-2r};L_p,W_p^r(\varphi),W_p^r)$$
$$:=\inf\{\|f-g\|_p+n^{-r}\|\varphi^r g^{(r)}\|_p+n^{-2r}\|g^{(r)}\|_p : g\in W_p^r, g\geqq f\},$$
$$\tilde{K}(f,n^{-r},n^{-2r};L_p,W_p^r(\varphi),W_p^r)$$
$$:=K^-(f,n^{-r},n^{-2r};L_p,W_p^r(\varphi),W_p^r)+K^+(f,n^{-r},n^{-2r};L_p,W_p^r(\varphi),W_p^r).$$

A characterization of the onesided K-functional $\tilde{K}$ is given in [4]. For the K-functionals from below and from above see [5].

The Jackson type inequality

$$\tilde{E}_n(f)_p \leqq c(r)\{n^{-r}\|\varphi^r f^{(r)}\|_p + n^{-2r}\|f^{(r)}\|_p\}, \quad \forall f \in W_p^r \tag{5.5}$$

is proved in [13], [4]. (5.1) implies similar inequalities for the onesided approximations from below and from above and hence Theorems 4.3 and 4.4 together with (3.7), (3.9) and Propositions 3.3 and 3.4 imply

Theorem 5.5. *Let* $1 \leqq p \leqq \infty$, *r be natural, f be bounded (from below, from above) and measurable in* $[-1, 1]$. *Then we have*

$$c(r)E_n^-(f)_p \leqq K^-(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$$
$$\leqq c(r)\{E_n^-(f)_p + K(f, n^{-r}; L_p, W_p^r(\varphi))\};$$
$$c(r)E_n^+(f)_p \leqq K^+(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$$
$$\leqq c(r)\{E_n^+(f)_p + K(f, n^{-r}; L_p, W_p^r(\varphi))\};$$
$$c(r)\tilde{E}_n(f)_p \leqq \tilde{K}(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$$
$$\leqq c(r)\{\tilde{E}_n(f)_p + K(f, n^{-r}; L_p, W_p^r(\varphi))\}.$$

Using Theorem 5.5 and the inverse result of V. Totik [14] ($q = \min\{2, p\}$, $1 < p < \infty$)

$$K(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r) \leqq c(r, p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1} E_k(f)_p^q\}^{1/q}$$

we get

Corollary 5.6. *Let* $1 < p < \infty$, $q = \min\{p, 2\}$, *r be natural, f be bounded and measurable. Then we have*

$$K^-(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r) \leqq c(r, p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1} E_k^-(f)_p^q\}^{1/q};$$

$$K^+(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r) \leqq c(r, p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1} E_k^+(f)_p^q\}^{1/q};$$

$$\tilde{K}(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r) \leqq c(r, p)n^{-r}\{\sum_{k=1}^{n} k^{qr-1} \tilde{E}_k(f)_p^q\}^{1/q}.$$

From Theorem 5.5, Theorem 4.3 and Proposition 3.3 we get the following inequalities of Zamansky type

Corollary 5.7. *Under the assumptions of Theorem 5.5 we have*

$$\|\varphi^r P_n^-(f)^{(r)}\|_p \leqq c(r)n^r K^-(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r);$$

$$\|\varphi^r P_n^+(f)^{(r)}\|_p \leqq c(r)n^r K^+(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r);$$

$$\|\varphi^r \tilde{P}_n(f)^{(r)}\|_p \leqq c(r)n^r \tilde{K}(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r),$$

where $P_n^-(f)$, $P_n^+(f)$ and $\tilde{P}_n(f)$ are the corresponding polynomials of best algebraic onesided approximation to f.

Now we shall show that (5.5) cannot be improved to

$$\tilde{E}_n(f)_p \leqq c(r)n^{-r}\|\varphi^r f^{(r)}\|_p \quad \forall f \in W_p^r(\varphi) \tag{5.6}$$

for $r=1$ and $1 \leqq p < 2$. Assume (5.6) holds. Then Theorem 4.1 gives

$$\tilde{E}_n(f)_p \leqq c\tilde{K}(f, n^{-1}; L_p, W_p^1(\varphi)). \tag{5.7}$$

For the function $f(x)=0$ $(-1 \leqq x < 1)$, $f(1)=1$ we easily see that $K^+(f, n^{-1}; L_p, W_p^1(\varphi))=0$ by taking $g_\varepsilon(x)=0$ for $-1 \leqq x \leqq 1-\varepsilon$ and $g_\varepsilon(x)=1+(x-1)/\varepsilon$ for $1-\varepsilon \leqq x \leqq 1$ and letting $\varepsilon \to 0$. On the other hand $\tilde{E}_n(f)_p = E_n^+(f)_p \geqq cn^{-2/p}$ – a contraction to (5.7). The same arguments show that (5.6) cannot be true even if one replaces $\tilde{E}_n(f)_p$ by $E_n^+(f)_p$ or $E_n^-(f)_p$.

d) Algebraic polynomials — multivariate case

Let us only mention that direct and converse theorems for the best onesided approximations of functions defined on $[-1, 1]^d$ are obtained in [4]. The proper onesided K-functional is also given there. Property (1.6) can be established for example by modifying the ideas from Section 3.1 for trigonometric approximation.

6. Shape-preserving approximation

Here we give some other examples of constrained approximation fitting in the scheme of Section 4. Let X_0 be a set of functions defined on an interval Ω of the real line.

IV. Monotone and comonotone approximation. Let X_0^* consists of all increasing (decreasing) functions from X_0. Then for $f \in X_0^*$ we set $R(f)=X_0^*$. In these cases we have monotone approximation and condition (4.1) is obviously satisfied.

Let $\Omega = U[a_i, a_{i+1}]$, where $a_i < a_{i+1}$. Let X_0^* consists of all functions $f \in X_0$ which are increasing or decreasing in $[a_i, a_{i+1}]$ if i is even or odd and $R(f)=X_0^*$ for every $f \in X_0^*$. In this case we have comonotone approximation.

V. Convex approximation. As in IV X_0^* and $R(f)$ consist of all functions of the same convexity as f.

We shall describe in more details only the case of monotone approximation by algebraic polynomials. Let r be natural, $1 \leqq p \leqq \infty$, $\varphi(x) = \sqrt{1-x^2}$, $X_0 = L_p[-1, 1]$, $X_{1,1} = W_p^r(\varphi)$, $X_{1,2} = W_p^r$ and let G_n be the set of all algebraic polynomials of degree at most $n-1$.

For an increasing function f from L_p we set

$$R(f) := \{g \in L_p[-1, 1] : g \text{ is increasing}\},$$

$$E'_n(f)_p := \inf\{\|f-g\|_p : g \in G_n \cap R(f)\},$$

$$K'(f, n^{-r}, n^{-2r};\ L_p, W_p^r(\varphi), W_p^r)$$

$$:= \inf\{\|f-g\|_p + n^{-r}\|\varphi^r g^{(r)}\|_p + n^{-2r}\|g^{(r)}\|_p : g \in W_p^r \cap R(f)\},$$

$$K'(f, n^{-r};\ L_p,\ W_p^r) := \inf\{\|f-g\|_p + n^{-r}\|g^{(r)}\|_p : g \in W_p^r \cap R(f)\}.$$

A Jackson type inequality

$$(6.1) \qquad E'_n(f)_p \leqq c(r)\{n^{-r}\|\varphi^r f^{(r)}\|_p + n^{-2r}\|f^{(r)}\|_p\} \quad \forall f \in W_p^r \cap R(f)$$

is obtained only for $r=1$ and $r=2$ in [8] and [15]. It is actually shown there that

$$E'_n(f)_p \leqq c(r) K(f, n^{-r};\ L_p,\ W_p^r(\varphi)), \quad r=1,\ 2.$$

It seems true (although not proved) that for these r's the non-constrained and constrained functionals $K(f, n^{-r}; L_p, W_p^r(\varphi))$ and $K'(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$ are equivalent. Thus the weighted moduli used by Z. Ditzian and V. Totik [2] or by the second of the authors [7] will characterize the monotone K-functional.

The situation radically changes when r is bigger than 2. First we do not know whether (6.1) is true. Our conjecture is that (6.1) also holds for $r \geqq 3$. The weaker than (6.1) inequality

$$(6.2) \qquad E'_n(f)_p \leqq c(r) n^{-r}\|f^{(r)}\|_p \quad \forall f \in W_p^r,\ f \text{ increasing},$$

is proved in [1]. (6.2) implies

$$E'_n(f)_p \leqq c(r) K'(f, n^{-r};\ L_p,\ W_p^r)$$

— a direct result with no chances to be inverted. But even in this case A. S. Shvedov [20] shows (using moduli of smoothess) that the inequality

$$E'_n(f)_p \leqq c(r) K(f, n^{-r};\ L_p,\ W_p^r)$$

is not true whenever $r \geqq 3$, that is moduli $\omega_r(f, n^{-1})_p$ are not suitable for characterizing $K'(f, n^{-r};\ L_p,\ W_p^r)$. So the problem for characterizing both weighted and non-weighted monotone functionals — $K'(f, n^{-r}, n^{-2r}; L_p, W_p^r(\varphi), W_p^r)$ and $K'(f, n^{-r}; L_p, W_p^r)$ – via proper moduli is also open if $r \geqq 3$.

7. Remarks and open problems

7.1. All statements in Sections 1, 2 and 4 remain true (after a proper modification) if we require X_0 and X_1 to be quasi-normed spaces. This allows spaces L_p, $0<p<1$, to be also treated as L_p, $1 \leqq p \leqq \infty$, in Sections 3, 5, 6.

7.2. The simplest way for proving (1.6) in the trigonometric case may be via the de la Vallee-Poussin sums $V_n(g)$. One has to use only their linearity, $V_n'(g) = V_n(g')$, $\|g - V_n(g)\|_p \leqq 4E_n(g)_p$ and that $V_n(g)$ is a trigonometric polynomial of degree $2n$. But it seems that the construction of an operator with such properties in other situations will be more elaborate than the proof of (1.6).

7.3. The results in Section 4 separate the investigation of the constrained approximation case into three practically independent parts:

a) obtaining of Jackson type results (4.2) or (4.2′) for the constrained approximation;

b) investigating the non-constrained case — proving of (1.2), (1.3) and (1.6) with the same parameter α as in (4.2);

c) characterization of the constrained K-functional via proper moduli of functions.

The reason we require c) is the easier computation of moduli compared to the evaluation of K-functionals. If this program is fulfilled then we have a characterization of the constrained approximation. Any inverse result for the non-constrained approximation better than (1.5) will automatically provide (as shown in Section 5) a similar result for the constrained approximation.

7.4. As far as we know the onesided K-functional is introduced by V. Popov [10] and the monotone (non-weighted) K-functional is introduced by R. A. DeVore [1].

7.5. The statements in Section 5 look similar for E_n^-, E_n^+ and $\tilde{E}_n$. The reason is that we use the notion of the constrained K-functional. The difference between the onesided approximation from one side and onesided approximations from below and from above from another side appears when one tries to characterize the corresponding K-functionals.

7.6. Open problems. In Section 1 we have asked whether (1.6) is independent from (1.2) and (1.3). We do not know a characterization of K^+ and K^- functionals via moduli of smoothness for $r \geqq 3$ (Section 5). To prove Jackson type inequality (6.1) and to find characterization of the monotone K-functionals for $r \geqq 3$ (Section 6).

One can also consider other type of approximating processes, e. g. splines with fixed or free knots, rational functions, and prove for them results similar to those of Sections 3, 5 and 6.

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