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On the Theorem of H. Whitney in Spaces L^p, $1 \leqq p \leqq \infty$.

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Presented by Bl. Sendov

Whitney's theorem shows that the error of approximation to a function $f(x)$ by algebraic polynomials of degree $n-1$ is estimated by the n-th order modulus of smoothness of f.

In the present paper we prove the following estimates of constants in this theorem.

For uniform norm $W_\infty \leqq 3$.

For integral L^p-norm, $1 \leqq p < \infty$, $W_p \leqq 11$.

I. Introduction

Let $f(x)$ be a measurable function on $I=[0, 1]$,

$$\Delta_h^n f(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+jh), \quad x+jh \in I, \quad n \in N_+$$

is the n-th difference off with step size h.

Define

$$\|f\|_p = \|f\|_{L^p(I)} = \begin{cases} \{\int_0^1 |f(x)|^p \, dx\}^{1/p}, & 1 \leqq p < \infty, \\ \sup_{x \in I} |f(x)|, & p = \infty. \end{cases}$$

We say that $f \in L^p(I)$, if $\|f\|_p < \infty$. We define the n-th order modulus of smoothness of $f \in L^p(I)$, $1 \leqq p \leqq \infty$, as

$$\omega_n(f; \delta) = \sup_{0 \leqq h \leqq \delta} \|\Delta_h^n f\|_{L^p(I_h)},$$

$$I_h = [0, 1-nh], \quad 0 < \delta \leqq 1/n.$$

Let $E_{n-1}(f)_p = \inf_{P_{n-1}} \|f - P_{n-1}\|_p$ denote the error in approximating $f(x)$ by polynomial of degree $n-1$.

In the theory of approximation we consider the well-known

Whitney's theorem. *Let $f \in L^p(I)$, $1 \leqq p \leqq \infty$. Then for each $n \in N$ there is a smallest positive number $W_p(n)$, such that*

$$E_{n-1}(f)_p \leqq W_p(n)\, \omega_n(f; \frac{1}{n})_p.$$

The history of this theorem is as follows. The first results were obtained in L^∞-metric. The case $n=1$ is trivial. In 1952 H. Burkill [3] proved that $E_1(f)_\infty \leqq \omega_2(f, \frac{1}{2})_\infty$ and conjected that $W_\infty(n) < \infty$ for $n \geqq 3$. This conjecture was proved by H. Whitney in 1957 for continuous functions [11] and in 1959 for bounded functions [12]. He also proved that

$$W_\infty(1) = W_\infty(2) = \frac{1}{2}, \quad \frac{8}{15} \leqq W_\infty(3) \leqq \frac{7}{10},$$

$$W_\infty(4) \leqq 3.3, \quad W_\infty(5) \leqq 10.4, \quad W_\infty(n) \geqq \frac{1}{2}$$

and concluded that the problem of finding $W_\infty(n)$ was probably extremely difficult.

Later Ju. A. Brudnyi [2] proved the theorem for $f \in L^p$, $1 \leqq p \leqq \infty$ and gave the estimates

$$W_p(n) \leqq C.n^{2n}, \quad 1 \leqq p \leqq \infty.$$

In 1982 Bl. Sendov [6] proposed a numerical method for the estimation of the Whitney's constants and showed that

$$W_\infty(4) \leqq 1.3, \quad W_\infty(5) \leqq 1.3, \quad W_\infty(6) \leqq 1.7.$$

Bl. Sendov [7] formulated the following conjecture:

$$W_\infty(n) \leqq 1, \quad n \in N.$$

In 1985, in papers [4], [1], [7] the break-through was made in improving estimates of constants $W_\infty(n)$.

Firstly, K. Ivanov and M. Takev [4], developing the ideas of A. Bearling ([11], p. 84), proved that $W_\infty(n) \leqq Cn \ln n$.

The integral operators

$$\int_0^h \Delta_y^n f(x-vy)\,dy$$

$$h=\frac{1}{n+1}, \quad x=vh+t, \quad v=0,\ 1,\dots,n, \quad 0\leqq t\leqq h$$

play an important role in this proof.

Immediately after that P. Binev [1] proved that

$$W_\infty(n)\leqq Cn.$$

Independence constants $W_\infty(n)$ of n were obtained by Bl. Sendov [7]. He proved the remarkable equation

$$f(x)=P_{n-1}(x)+\varphi_n(f;\ x)+\sum_{j=0}^{n}\frac{1}{h}\int_0^t \varphi_n(f;\ jh+y)\,l_n\left(j,\ \frac{x-y}{h}\right)dy,$$

where

$$\varphi_n(f;\ x)=\varphi_n(f;\ vh+t)=\frac{(-1)^{n-v}}{h\binom{n}{v}}\int_0^h \Delta_y^n f(x-vy)\,dy,$$

$$l_n(j,\ x)=\prod_{\substack{i=0\\ i\neq j}}^{n}\frac{x-i}{j-i}$$

and with the help of this equation established [8] that $W_\infty(n)\leqq 6$. In 1986 Bl. Sendov and M. Takev [9] showed that $W_1(n)\leqq 30$.

Now let us consider the results of the present paper.

Let $Q_{n-1}(x)$ be polynomial of degree at most $n-1$ such that

$$\int_{i/n}^{(i+1)/n} (f(t)-Q_{n-1}(t))\,dt=0, \quad i=0,\ 1,\dots,n-1.$$

Polynomials of this type are used by E. A. Storozhenko in the proof of theorem of Whitney in L^p, $1\leqq p<\infty$ [10].

Denote

$$\tilde{W}_p(n)=\sup_{f\in L^p(I)}\frac{\|f-Q_{n-1}\|_p}{\omega_n(f,\ \frac{1}{n})_p}, \quad 1\leqq p\leqq\infty.$$

It is clear that $w_p(n)\leqq W_p(n)$.

In this paper we prove that

$$1\leqq \tilde{W}_\infty(n)\leqq 3 \tag{1.1}$$

$$1\leqq \tilde{W}_p(n)\leqq 11, \quad 1\leqq p<\infty. \tag{1.2}$$

We should also mention that above estimate in (1.1) was independently announced in [13], p. 37, in [14] and in [15]. A sketch of the proof of (1.1) is given in [5].

A word about contents of the paper. Section 2 contains auxiliary results. The main identity (Lemma 1) is analog of identity of Bl. Sendov. Section 3 is devoted to estimates of $\tilde{W}_\infty(n)$. In section 4 we give L^p-estimates.

2. Auxiliary results

Lemma 1. *Let $f\in L^1(I)$, $n\in N$ and $\int_0^{i/n} f(t)\,dt=0$, $i=1,\dots,n$. Then for $x\in(0,\frac{1}{n}]$*

$$f(ix)=\varphi_i(f,x)+\int_{1/n}^{x}\sum_{j=1}^{n}\varphi_j(f,y)\left(\frac{j}{i}\right)[l_n(j,\frac{x}{y}i)]'_x\,dy,$$

where

$$\varphi_i(f,x)=\frac{(-1)^{n-i}}{\binom{n}{i}}\cdot\frac{1}{x}\int_0^x\Delta_y^n f(i(x-y))\,dy,$$

$$l_n(j,t)=\prod_{\substack{k=0\\k\neq j}}^{n}\frac{t-k}{j-k},\quad j=1,\dots,n.$$

Proof. Denote

$$z_i(x)=\frac{1}{i}\int_0^{ix}f(t)\,dt,\quad i=1,\dots,n.$$

It is clear that $f(ix)=z_i'(x)$ and that $z_i(\frac{1}{n})=0$. Using expression $\varphi_i(f,x)$ we get identities

$$f(ix)=\frac{(-1)^{n-i}}{\binom{n}{i}}\cdot\frac{1}{x}\int_0^x\Delta_y^n f(i(x-y))\,dy+\frac{1}{x}\sum_{j=1}^{n}a_{ij}\frac{1}{j}\int_0^{jx}f(t)\,dt.$$

This identities we may regard as initial-value problem of the system of linear differential equations

$$(2.1)\qquad \begin{aligned}&z_i'(x)=\frac{1}{x}\sum_{j=1}^{n}a_{ij}z_j(x)+\varphi_i(f,x),\\&z_i(\frac{1}{n})=0.\end{aligned}$$

It is easy to see that general solution of the homogeneous part of the system is

$$Z^j(x)=[\frac{j}{i}\int_0^{ix} t^{j-1}\,dt]_{j=1}^n=[i^{j-1}x^j]_{i=1}^n.$$

We find the solution of nonhomogeneous system (2.1) by the method of variations of the constants.

$$Z^*_{(x)}=\sum_{j=1}^n C_j(x)Z^j(x).$$

We have

$$\sum_{j=1}^n C_j'(x)Z^j(x)=\varphi(f;\ x)$$

where

$$\varphi(f;\ x)=[\varphi_i(f;\ x)]_{i=1}^n.$$

Solution of the last system give

$$Z^*(x)=\int_{1/n}^{x} F(x,\ y)\,dy$$

where

$$F=[F_i]_{i=1}^n,\quad F_i(x,\ y)=\sum_{j=1}^n \varphi_j(f;\ y)(\frac{j}{i})\,l_n(j,\ \frac{x}{y}i).$$

From condition $Z^*(\frac{1}{n})=0$ there follows that $Z^*(x)$ is a solution of system (2.1). With one differentiation of $Z^*(x)$ we finally obtain the desired identity.

Lemma 1 is proved.

The estimates of integral terms in the main identity will be needed below. For this purpose

Lemma 2. *Let* $\sigma_0=0$, $\sigma_j=1+\ldots+\frac{1}{j}$,

$$A_i(n)=\sum_{j=1}^n \frac{1}{\binom{n}{j}}\left(\frac{j}{i}\right)^{1-1/p}\int_{i\frac{n}{n+1}}^{i} |l_n'(j,\ t)|\,dt.$$

Then

$$A_i(n)\leq\frac{1}{\binom{n}{i}}\left\{\max_{0\leq x\leq\frac{i}{n+1}} e^{x(\sigma_{n-i}-\sigma_i)}+2C(p)\max_{0\leq x\leq\frac{i}{n+1}} xe^{x(\sigma_{n-i}-\sigma_i+1)}\right\}$$

where

$$C(p)=\sum_{\substack{j=1\\ j\neq i}}^n \frac{1}{|i-j|}\left(\frac{j}{i}\right)^{1-1/p}\leq\begin{cases}\sigma_{i-1}+\sigma_{n-i}+\dfrac{n-2i+1}{i}, & p=\infty,\\[2ex] \sigma_{i-1}+\sigma_{n-i}+\dfrac{n-i}{i}, & 1\leq p<\infty.\end{cases}$$

Proof. Let split our sum into two parts

$$\sum_{j=1}^{n}=\sum_{j=i}^{i}+\sum_{j\neq i}.$$

The monotone properties of $l_n(j, t)$ give

$$A_i\leqq\frac{1}{\binom{n}{i}}\max_{i\frac{n}{n+1}\leqq t\leqq i}|l_n(i, t)|$$

$$+2\sum_{j=1}^{n}\frac{1}{\binom{n}{j}}\left(\frac{j}{i}\right)^{1-1/p}\max_{i\frac{n}{n+1}\leqq t\leqq i}|l_n(j, t)|$$

$$\leqq\frac{1}{\binom{n}{i}}\max_{0\leqq x\leqq\frac{i}{n+1}}\left|\frac{(i-x)\dots(1-x)(1+x)\dots(n-i+x)}{i!\ \ (n-i)!}\right|$$

$$+2\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{\binom{n}{j}}\left(\frac{j}{i}\right)^{1-1/p}\max_{0\leqq x\leqq\frac{i}{n+1}}\left|\frac{1-x}{i-j-x}\cdot\frac{(i-x)\dots(2-x)x\dots(n-i+x)}{j!\ \ (n-j)!}\right|.$$

Using simple inequalities

$$\max_{0\leqq x<1}\left|\frac{1-x}{i-j-x}\right|\leqq\frac{1}{|i-j|},\qquad 1+t\leqq e^t,\quad t\in\mathrm{R}$$

it follows that

$$A_i\leqq\frac{1}{\binom{n}{i}}\max_{0\leqq x\leqq\frac{i}{n+1}}e^{x(\sigma_{n-i}-\sigma_i)}+2\frac{1}{\binom{n}{i}}\left\{\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{j}{i}\right)^{1-1/p}\frac{1}{|i-j|}\right\}\max_{0\leqq x\leqq\frac{i}{n+1}}xe^{x(\sigma_{n-i}-\sigma_i+1)}$$

Putting $1-1/p=\alpha$ we obtain

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{j}{i}\right)^{\alpha}\frac{1}{|i-j|}=\frac{1}{i^\alpha}\sum_{j=1}^{i-1}\frac{j^\alpha-i^\alpha+i^\alpha}{i-j}+\frac{1}{i^\alpha}\sum_{j=i+1}^{n}\frac{j^\alpha-i^\alpha+i^\alpha}{j-i}$$

$$\leqq\frac{1}{i^\alpha}\sum_{j=1}^{i-1}\frac{j^\alpha-i^\alpha}{i-j}+\sigma_{i-1}+\frac{1}{i^\alpha}\sum_{j=i+1}^{n}\frac{j^\alpha-i^\alpha}{j-i}+\sigma_{n-1}\leqq\sigma_{i-1}+\sigma_{n-i}+\frac{n-i}{i}.$$

Analogously if $p=\infty$

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \binom{j}{i} \frac{1}{|i-j|} = \sigma_{i-1} + \sigma_{n-i} + \frac{n-2i+1}{i}.$$

Lemma 2 is proved.

3. Constants of H. Whitney in $L^\infty(I)$

In this section we prove the estimates of constants of H. Whitney in uniform metric.

An important role in this estimates is played by the main identity. The main identity leads problem to the computation of the number sums. We use Lemma 2 in the computation of these sums.

Theorem 1. *Let $n \in N$. Then*

$$\tilde{W}_\infty(n) = 2\left(1 + \frac{1}{e}\right) + O\left(\frac{1}{n}\right) \tag{3.1}$$

$$1 \leqq \tilde{W}_\infty(n) \leqq 3. \tag{3.2}$$

Proof. There is no loss in assumptions

$$Q_{n-1}(x) \equiv 0,$$

$$\int_0^{i/n} f(t)\,dt = 0, \quad i = 1, \dots, n \tag{3.3}$$

$$\omega_n\left(f, \frac{1}{n}\right) \leqq 1. \tag{3.4}$$

Thus, it is sufficient to prove that

$$\sup_{y \in I} |f(y)| = 2\left(1 + \frac{1}{e}\right) + O\left(\frac{1}{n}\right) \tag{3.5}$$

for the functions with properties (3.3), (3.4).

From the symmetry of sets

$$J = \bigcup_{i=1}^{n} \left[\frac{i}{n+1}, \frac{i}{n}\right] \text{ and } J^c = I \setminus J$$

there follows that it is sufficient to prove (3.5) for $y \in J$.

We write $y = ix$, $1 \leqq i \leqq n$, $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$ and from Lemma 1 obtain

$$|f(ix)|\leqq|\varphi_i(x)|+\int_x^{1/n}\sum_{j=1}^n\frac{j}{i}|\varphi_j(y)|\cdot|l_n(j, i\frac{x}{y})'_x|\,dy$$

$$\leqq\frac{1}{\binom{n}{i}}+\frac{n+1}{n}\sum_{j=1}^n\frac{1}{\binom{n}{j}}\cdot\frac{j}{i}\cdot\int_{i\frac{n}{n+1}}^{i}|l'_n(j, t)|\,dt.$$

From Lemma 2 it follows that

$$|f(ix)|\leqq\frac{1}{\binom{n}{i}}+\frac{n+1}{n}\cdot\frac{1}{\binom{n}{i}}\Big\{\max_{0\leqq t\leqq\frac{i}{n+1}} e^{t(\sigma_{n-i}-\sigma_i)}$$

$$+2(\sigma_{n-i}+\sigma_{i-1}+\frac{n-2i+1}{i})\max_{0\leqq t\leqq\frac{i}{n+1}} te^{t(\sigma_{n-i}-\sigma_i+1)}\Big\}.$$

It is easy to see that right part of the last inequality is the largest one for $i=n$:

$$|f(ix)|\leqq 1+\frac{n+1}{n}\Big\{1+2(\sigma_n-1)\max_{0\leqq x\leqq\frac{n}{n+1}} xe^{-x(\sigma_n-1)}\Big\}$$

$$\leqq 1+\frac{n+1}{n}(1+\frac{2}{e})=2(1+\frac{1}{e})+\frac{1}{n}(1+\frac{2}{e}).$$

The estimate (3.1) is proved.

Inequality $W_\infty(n)\leqq 3$ for $n\geqq 7$ follows from last estimate because for $n\geqq 7$

$$1+\frac{n+1}{n}(1+\frac{2}{e})<3.$$

For $1\leqq n\leqq 6$ it is sufficient to compute the quantities

$$\frac{n+1}{n}\sum_{j=1}^n\frac{1}{\binom{n}{j}}\frac{j}{i}\int_{i\frac{n}{n+1}}^{i}|l'_n(j, t)|\,dt.$$

The inequality (3.2) is proved.

In conclusion we shall prove that $\tilde{W}_\infty(n)\geqq 1$. For $n\in N$, $\alpha\in(0, \frac{1}{n})$ put

$$f_\alpha(x)=\begin{cases}\frac{1}{\alpha^{n-1}}(\alpha-x)^{n-1}, & 0<x<\alpha\leqq\frac{1}{n}\\ 0, & \alpha\leqq x<1\end{cases}$$

From the identities

$$\Delta_h^n f(x)=\int_0^h du_1\ldots\int_0^h \Delta_h^1 f^{(n-1)}(x+u_1+\ldots+u_{n-1})\,du_{n-1}$$

$$f^{(n-1)}(t)\equiv 0,\quad t\geqq\alpha$$

and the inequalities

$$|\Delta_h^1 f_\alpha^{(n-1)}(x)|\leqq\frac{(n-1)!}{\alpha^{n-1}}$$

$$\Big|\int\limits_{0\leqq u_1+\ldots+u_{n-1}\leqq\alpha}\cdots\int du_1\ldots du_{n-1}\Big|\leqq\frac{\alpha^{n-1}}{(n-1)!}$$

there follows that

$$|\Delta_h^n f_\alpha(x)|\leqq 1.$$

Let $\varepsilon>0$ and $P(x)$ be a polynomial which satisfies

$$\int_{i/n}^{(i+1)/n} P(t)\,dt=0,\quad i=1,\ldots,n-1$$

$$\int_0^{1/n} P(t)\,dt<0,\quad \|P(t)\|_\infty\leqq\varepsilon.$$

Denote

$$g_\alpha(x)=\begin{cases} f_\alpha(x)+P(\alpha), & 0\leqq x\leqq\alpha<\dfrac{1}{n}\\ P(x), & x>\alpha\end{cases}$$

We choose α such that

$$\int_0^{1/n} g_\alpha(t)\,dt=0.$$

It is easy to see that

$$\int_{i/n}^{(i+1)} g_\alpha(t)\,dt=0,\quad i=0,\ 1,\ldots(n-1)$$

From the definition of $g_\alpha(x)$ we have

$$\|g_\alpha(t)\|_\infty=\|f_\alpha(t)+P(\alpha)\|_\infty\geqq\|f_\alpha(t)\|_\infty-\varepsilon\geqq 1-\varepsilon$$

and

$$\|\Delta_h^n g_\alpha(t)\|_\infty\leqq\|\Delta_h^n(g_\alpha(t)-f_\alpha(t))\|_\infty+\|\Delta_h^n f_\alpha(t)\|_\infty\leqq\ 2^n\|P(t)\|_\infty+1\leqq 1+2^n\varepsilon.$$

Since ε may be arbitrary positive then $\tilde{W}_\infty(n)\geq 1$.

Theorem 1 is completely proved.

2. Constants of H. Whitney in $L^p(I)$, $1\leq p<\infty$.

Theorem 2. *For $n\in N$, $p\in[1, \infty.)$*

$$\tilde{W}_p(n)=4(1+\frac{1}{e})+O(\frac{1}{\ln n}), \tag{4.1}$$

$$1\leq\tilde{W}_p(n)\leq 11. \tag{4.2}$$

Let us begin with the asymptotic estimates (4.1). As in the case $p=\infty$ we shall suppose that

$$Q_{n-1}(x)\equiv 0, \quad \int_0^{i/n} f(t)\,dt=0, \quad i=1,\dots,n$$

$$\left\{n\int_0^{1/n} dy \int_0^{1-ny} |\Delta_y^n f(u)|^p\,du\right\}^{1/p}\leq 1.$$

By the symmetry argument, we conclude that sufficient to prove the estimate

$$\|f\|_{L^p(j)}=\sum_{i=1}^n \left\{\int_{i/(n+1)}^{i/n} |f(y)|^p\,dy\right\}^{1/p}=2(1+\frac{1}{e})+O(\frac{1}{\ln n}) \tag{4.3}$$

For proving (4.3) we use the main identity for $x\in[\frac{1}{n+1}, \frac{1}{n}]$, $i=1,\dots,n$:

$$|f(ix)|\leq|\varphi_i(x)|+\sum_{j=1}^n \frac{j}{i}\int_x^{1/n} |\varphi_j(y)|\,|l_n(j, \frac{x}{y}i)'_x|\,dy$$

and estimate $\|f(ix)\|\equiv\|f(ix)\|_{L^p(\frac{1}{n+1}, \frac{1}{n})}$ by the Minkowski's inequalities.

$$\|f(ix)\|\leq\|\varphi_i(x)\|+\sum_{j=1}^n (\frac{j}{i})\left\|\int_{-\infty}^{+\infty} |\varphi_j(\frac{x}{v})|\,\chi_{[xn,1]}(v)\frac{1}{v}|l_n(j, vi)'_v|\,dv\right\|$$

$$\leq\|\varphi_i(x)\|+\sum_{j=1}^n \left(\frac{j}{i}\right)\int_{\frac{n}{n+1}}^{1} |v^{\frac{1}{p}-1}\,l_n(j, vi)'_v|\left(\int_{\frac{1}{n+1}}^{\frac{v}{n}} |\varphi_j(\frac{x}{v})|^p\,\frac{dx}{v}\right)^{1/p} dv$$

$$\leq\|\varphi_i(x)\|+\left(\frac{n+1}{n}\right)^{1-1/p}\sum_{j=1}^n \left(\frac{j}{i}\right)\|\varphi_j(x)\|\,C_{ij}(n)$$

where

$$C_{ij}(n)=\int_{i\frac{n}{n+1}}^{i}|l_n(j,\ t)|\,dt.$$

Thus

$$\|f\|_{L^p(J)}=\sum_{i=1}^{n}\left(\int_{\frac{i}{n+1}}^{\frac{i}{n}}|f(y)|^p\,dy\right)^{1/p}=\sum_{i=1}^{n}i^{1/p}\,\|f(ix)\|$$

$$\leqq\sum_{i=1}^{n}i^{1/p}\,\|\varphi_i(x)\|+\left(\frac{n+1}{n}\right)^{1-1/p}\sum_{j=1}^{n}\|\varphi_j(x)\|\,.j^{1/p}\sum_{i=1}^{n}\left(\frac{j}{i}\right)^{1-1/p}C_{ij}(n).$$

But

$$\|\varphi_i(x)\|=\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}}|\varphi_i(x)|^p dx\right)^{1/p}$$

$$\leqq\left[\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{1}{\binom{n}{i}^p}\cdot\frac{1}{x^p}\left[\int_0^x|\Delta_y^n f((x-y)i)|\,dy\right]^p dx\right]^{1/p}$$

$$\leqq\frac{1}{\binom{n}{i}}\left((n+1)\int_{\frac{1}{n+1}}^{\frac{1}{n}}dx\left[\int_0^x|\Delta_y^n f((x-y)i)|^p dy\right]\right)^{1/p}$$

$$\leqq\frac{1}{\binom{n}{i}}\cdot\left(\frac{n+1}{n}\right)^{1/p}\cdot\frac{1}{i^{1/p}}\left(n\int_0^{1/n}dy\int_0^{1-ny}|\Delta_y^n f(x)|^p\,dx\right)\leqq\frac{1}{\binom{n}{i}}\,\frac{1}{i^{1/p}}\left(\frac{n+1}{n}\right)^{1/p}.$$

Hence

$$\|f\|_{L^p(J)}=\sum_{i=1}^{n}\left\{\int_{i/(n+1)}^{i/n}|f(y)|^p\,dy\right\}^{1/p}\leqq H(p,\ n)$$

$$=\left(\frac{n+1}{n}\right)^{1/p}\sum_{i=1}^{n}\frac{1}{\binom{n}{i}}+\left(\frac{n+1}{n}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\binom{n}{j}}\left(\frac{j}{i}\right)^{1-1/p}C_{ij}(n).$$

To prove that

$$C_1(p,\ n)=\left(\frac{n+1}{n}\right)^{1/p}\sum_{i=1}^{n}\frac{1}{\binom{n}{i}}=1+O\left(\frac{1}{n}\right)\tag{4.5}$$

$$(4.6)\qquad C_2(p,\ n)=\left(\frac{n+1}{n}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\binom{n}{j}}\cdot\left(\frac{j}{i}\right)^{1-1/p}C_{ij}(n)=1+\frac{2}{e}+O\left(\frac{1}{\ln n}\right)\cdot$$

The estimate (4.5) is simple

$$C_1(p,\ n)\leqq C_1(1,\ n)\leqq\frac{n+1}{n}\left(\sum_{i=1}^{n-1}\frac{1}{\binom{n}{i}}+1\right)\leqq 1+\frac{5}{n}\cdot$$

For obtaining (4.6) we shall use Lemma 2:

$$C_2(p,\ n)\leqq\left(\frac{n+1}{n}\right)\sum_{i=1}^{n}\frac{1}{\binom{n}{i}}\max_{0\leqq x\leqq i/(n+1)}e^{x(\sigma_n-i-\sigma_i)}$$

$$+2\left(\frac{n+1}{n}\right)\sum_{i=1}^{n}\frac{1}{\binom{n}{i}}\left(\sigma_{i-1}+\sigma_{n-i}+\frac{n-i}{i}\right)\max_{0\leqq x\leqq\frac{i}{n+1}}xe^{x(\sigma_n-i-\sigma_i+1)}=D_1(n)+D_2(n).$$

$$D_1(n)\leqq\frac{n+1}{n}\left\{\sum_{i=1}^{[\frac{n}{2}]}\frac{1}{\binom{n}{i}}e^{\frac{i}{n+1}(\sigma_n-i-\sigma_i)}+\sum_{i=[\frac{n}{2}]+1}^{n}\frac{1}{\binom{n}{i}}\right\}\leqq$$

$$\frac{n+1}{n}\left\{e\sum_{i=1}^{[\frac{n}{2}]}\frac{1}{\binom{n}{i}}+\sum_{i=[\frac{n}{2}]+1}^{n}\frac{1}{\binom{n}{i}}\right\}\leqq\frac{n+1}{n}\left\{\frac{2e}{n}+1+\frac{2}{n}\right\}\leqq 1+\frac{10}{n}\qquad(n\geqq 10)$$

$$D_2(n)=\frac{2(n+1)}{n}\left\{e\sum_{i=1}^{[\frac{n}{2}]+1}\frac{1}{\binom{n}{i}}\frac{(\sigma_{i-1}+\sigma_{n-i}+\frac{n-i}{i})i}{n+1}\right.$$

$$\left.+\sum_{i=[\frac{n}{2}]+2}^{n-1}\frac{1}{\binom{n}{i}}\frac{\sigma_{i-1}+\sigma_{n-i}+\frac{n-i}{i}}{\sigma_i-\sigma_{n-i}}+\frac{\sigma_{n-1}}{\sigma_n-1}\cdot\frac{1}{e}\right\}$$

$$\leqq\frac{2(n+1)}{n}\left\{\frac{12}{n}+\frac{6}{n}+\frac{1}{e}+\frac{2}{\ln n}\right\}\leqq\frac{2}{e}+\frac{40}{n}+\frac{4}{\ln n}\cdot$$

Thus we get

$$C_2(p,\ n)=1+\frac{2}{e}+\frac{4}{\ln n}+\frac{50}{n}=1+\frac{2}{e}+O\left(\frac{1}{\ln n}\right).$$

The estimate (4.1) is proved.

To prove the inequality $\tilde{W}_p(n)\leqq 11$ we consider three cases: a) $n\geqq 50$, b) $10\leqq n\leqq 49$, c) $n\leqq 9$.

In the "a" case (4.2) follows from (4.1):

$$\tilde{W}_p(n)\leqq 4(1+\frac{1}{e})+\frac{8}{\ln n}+\frac{110}{n}\leqq 10, \qquad n\geqq 50.$$

In the "b" case the calculation on computor constants $C_1(1,\ n), D_1(n)\ D_2(n)$ gives

$$\tilde{W}_p(n)\leqq 2(C_1(1,\ n)+D_1(n)+D_2(n))\leqq 2(C_1(1,\ 10)+D_1(10)+D_2(10))\leqq 11.$$

In the "c" case the direct computation constants

$$C_{ij}(n)=\int\limits_{i\frac{n}{n+1}}^{i}|l'_n(j,\ t)|\,dt, \quad i,\ j=1,\dots,n$$

and sums $H(p,\ n)$ give $\tilde{W}_p(n)\leqq 2H(1,\ 5)\leqq 11$.

The right estimation (4.2) is proved.

The inequality $\tilde{W}_p(n)\geqq 1$, $1\leqq p<\infty$ is proved analogously to the inequality $\tilde{W}_\infty(n)\geqq 1$. Slightly corrected function $f_\alpha(x)$ serves as the opposite example.

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