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Fourier — Stieltjes Series Associated with a Process Belonging to the Domain of Attraction of Stable Law

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The paper defines stochastic integral with respect to stochastic processes with increments belonging to the domain of attraction of the stable law and solves a convergence problem of Fourier — Stieltjes series associated with such a process.

1. Introduction:

Let $X(t, \omega)$, $t \in R$ be a stochastic process with independent increments and continuous in sense of quadratic mean and f be a continuous function in $[a, b]$. Then the stochastic integral $\int_a^b f(t)\, \mathrm{d}X(t,\omega)$ can be defined in the sense of convergence in probability and it is a random variable (cf. E. Lukacs [2], p. 148). Hence the Fourier — Stieltjes coefficient of $X(t, \omega)$,

$$A_n(\omega) = \int_0^1 \mathrm{e}^{-2\pi nit} \mathrm{d}X(t, \omega), \tag{1.1}$$

exists for the orthonormal set $\{\mathrm{e}^{2\pi int}\}_{n=-\infty}^{\infty}$.

It was shown by C. Nayak, S. Pattanayak and M. N. Mishra [3] that for $f \in L^\alpha$, where

$L^\alpha [0, 1] = \{f\colon [0, 1] \to C$; measurable such that $\int_0^1 |f(t)|^\alpha \,\mathrm{d}t < \infty\}$ the stochastic integral $\int_0^1 f(t)\, \mathrm{d}X(t, \omega)$ can be defined for the stable process $X(t, \omega)$ and moreover the random Fourier — Stieltjes series (RFS),

$$\sum_{n=-\infty}^{\infty} a_n A_n(\omega)\, \mathrm{e}^{2\pi iny} \text{ where } a_n = \int_0^1 f(t) \mathrm{e}^{-2\pi int}\, \mathrm{d}t, \tag{1.2}$$

converges in probability to

$$(1.3)\qquad \int_0^1 f(y-t)\, dX(t,\omega) \qquad \text{for } y \in R.$$

They have also shown that the sum function of the RFS series (1.2) is differentiable in probability if a_n satisfies the condition $\sum_{n=-\infty}^{\infty} |na_n|^2 < \infty$.

In this paper we show how to define a stochastic integral with respect to process whose increments belong to the domain of attraction of stable law. Also we describe the mode of convergence of the RFS series for Cauchy processes by imposing a weaker condition of a_n. This series is differentiable under a weaker condition.

2. Definitions:

Definition D_1. The class of functions f satisfying

$$\int_a^b |f(t)|^p\, dt < \infty \text{ is denoted by } L^p[a, b].$$

Definition D_2. If $\Phi(u)$ be a non-negative function defined for $u \geqq 0$, then the set of functions f satisfying $\int_a^b \Phi|f(t)|\, dt < \infty$ will be denoted by $L_\Phi[a, b]$.

Definition D_3. A sequence of random variables $\{X_n, n=0, 1, \dots\}$ converges in $(C, 1)$ probability to a random variable X if

$$\lim_{n\to\infty} \mathbf{P}(|Y_n - X| \geqq \varepsilon) = 0 \text{ for all } \varepsilon > 0,$$

where, $Y_n = \frac{1}{n}(X_0 + X_1 + \dots + X_{n-1})$.

Definition D_4. If $f \in L^p$, $p \geqq 1$, the expression

$$\omega_p(\varepsilon, f) = \sup_{0 \leq h \leq \varepsilon} \left\{\frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p\, dx\right\}^{1/p}$$

is called the integral modulus of continuity (in L^p) of f.

Definition D_5. The random function $f(t,\omega)$ is said to be differentiable in probability at $t = t_0$ if there exists a random function $g(t,\omega)$ such that for all $\varepsilon > 0$,

$$\lim_{h\to 0} \mathbf{P}\left(\left|\frac{1}{h} f(t_0+h, \omega) - f(t_0, \omega) - g(t_0, \omega)\right| > \varepsilon\right) = 0.$$

Definition D_6.

$$\Lambda_\alpha[a, b] = \{f : [a, b] \to \mathbf{C};\ |f(t) - f(s)| = 0(|t-s|^\alpha)\}.$$

3. Main results

Theorem 1. *Let $X(t, \omega)$ be a stochastic process with independent increments which belong to the domain of attraction of the stable law with characteristic function $e^{-\Phi(t)}$ where $\Phi(t)=|t|^\alpha \log^+ t$ and $f\in L_\Phi \cap L^\alpha$. Then the function $\int_0^{2\pi} f\,dX$ can be defined in the sense of convergence in probability. Moreover if*

$$a_n=\int_0^{2\pi} f(t)\, e^{-2\pi int} dt,\ A_n(\omega)=\int_0^1 e^{-2\pi int} dX(t,\ \omega),\ n\in Z$$

then the series

$$\sum_{n=-N}^{N} a_n A_n(\omega)\, e^{2\pi iny} \tag{3.1}$$

converges in probability as $N\to\infty$, to the stochastic integral

$$\int_0^1 f(y-t)\, dX(t,\ \omega). \tag{3.2}$$

To prove the above Theorem we need two preliminary results formulated as Lemmas.

Lemma 1. (Doob Inequality) (Cf. Y. S. Chow and H. Teicher [1], p. 268). *If X is a random variable with characteristic function Φ, then for any positive C and δ,*

$$P\{|X|\geqq \frac{1}{\delta}(1+\frac{2\pi}{C\delta})^2 \int_0^\delta (1-\mathrm{Re}\ \Phi(t))\, dt.$$

In particular,

$$P\{|X|\geqq\varepsilon\}\leqq(1+\frac{2\pi}{\varepsilon})^2 \int_0^1 (1-\mathrm{Re}\,\Phi(t))\ dt.$$

Lemma 2: *Let $X(t, \omega)$ be a process with independent increments belonging to the domain of attraction of the stable law having characteristic function $e^{-|t|^\alpha \log^+ t}$. Then the characteristic function of $\int_a^b g(t)\, dX(t)$ is equal to*

$$e^{-\int_a^b \Phi(u.g(t))\, dt} \quad \textit{where } \Phi(t)=|t|^\alpha \log^+ t.$$

Moreover, for $\varepsilon>0$ we have

$$P\{|\int_a^b g(t)\, dX(t)|\geqq\varepsilon\}\leqq K_1 \int_a^b |g(t)|^\alpha\, dt+K_2 \int_a^b \Phi(g(t))\, dt$$

where $g\in C[a, b]$, K_1 and K_2 are positive constants depending on ε.

Proof of Lemma 2: Applying Lemma 1, we find

$$P(|\int_a^b g(t)\,dX(t)| \geqq \varepsilon) \leqq (1+\frac{2\pi}{\varepsilon})^2 \int_0^1 \int_a^b \Phi(ug(t))\,dt.$$

Since the characteristic function of

$$\int_a^b g(t)\,dX(t) \text{ is}$$

$$\exp[-\int_a^b \Phi(ug(t))\,dt],$$

where $\Phi(t)=|t|^\alpha \log^+ t$. Then,

$$\int_a^b \Phi(ug(t))\,dt = \int_a^b |u|^\alpha \log^+(ug(t))\, |g(t)|^\alpha\,dt$$

$$\leqq \int_a^b |u|^\alpha |g(t)|^\alpha \log^+ u\,dt + \int_a^b |u|^\alpha\, |g(t)|^\alpha \log^+ g(t)\,dt = A\,\Phi(u)+|u|^\alpha B.$$

Here,

$$A=\int_a^b |g(t)|^\alpha\,dt, \quad B=\int_a^b |g(t)|^\alpha \log^+ g(t)\,dt = \int_a^b \Phi(g(t))\,dt.$$

Hence,

$$\mathrm{P}(|\int_a^b g(t)\,dX(t, w)| \geqq \varepsilon) \leqq (1+\frac{2\pi}{\varepsilon})^2 \int_0^1 \int_a^b \Phi(ug(t))\,dt$$

$$\leqq (1+\frac{2\pi}{\varepsilon})^2 \int_0^1 (A\Phi(u)+|u|^\alpha B)\;dt$$

$$=K_1 \int_a^b |g(t)|^\alpha\,dt + K_2 \int_a^b |g(t)|^\alpha \log^+ g(t)\,dt$$

$$=K_1 \int_a^b |g(t)|^\alpha\,dt + K_2 \int_a^b \Phi(g(t))\,dt.$$

Here K_1 and K_2 are positive constants depending on ε.

Proof of the Theorem. It is well known (cf. A. Zygmund [5], p. 146) that if $f \in L_\Phi \cap L^\alpha$, then there exists a sequence of function $f_n \in C\,[0, 2\pi]$ such that

$$\lim_{n\to\infty} \int_0^{2\pi} \Phi(\frac{1}{4}|f_n-f|)\,dt=0 \text{ and } \lim_{n\to\infty} \int_0^{2\pi} |f_n-f|^\alpha\,dt=0.$$

We know that (cf. E. Lukacs [2], p. 148) for $f_n \in C\,[0, 2\pi]$, $\int_0^{2\pi} f_n(t)\,dX(t, \omega)$ is defined in the sense of convergence in probability. So by application of Lemma 2, we get

$$\mathbf{P}\{|\int_0^{2\pi} f_n(t)\,\mathrm{d}X(t,\ \omega)-\int_0^{2\pi} f_m(t)\,\mathrm{d}X(t,\ \omega)|>\varepsilon\}$$

$$\leqq(1+\frac{2\pi}{8\varepsilon})^2\int_0^{2\pi}\Phi(\frac{1}{8}|f_n-f_m|)\,\mathrm{d}t+(1+\frac{2\pi}{\varepsilon})^2\int_0^{2\pi}|f_n-f_m|^\alpha\,\mathrm{d}t.$$

Note firstly that

$$\int_0^{2\pi}|f_n-f_m|^\alpha\,\mathrm{d}t\to 0 \text{ as } m,\ n\to\infty.$$

Further by Jensen inequality

$$\int_0^{2\pi}\Phi(\frac{1}{8}|f_n-f_m|)\,\mathrm{d}t\leqq\frac{1}{2}\int_0^{2\pi}\Phi(\frac{1}{4}|f_n-f|)\,\mathrm{d}t$$

$$+\frac{1}{2}\int_0^{2\pi}\Phi(\frac{1}{4}|f_m-f|)\,\mathrm{d}t.$$

According to A. Zygmund [5], p. 146, if $f\in L_\Phi$, then

$$\int_0^{2\pi}\Phi(\frac{1}{4}|f_n-f|)\,\mathrm{d}t\to 0 \text{ and } \int_0^{2\pi}\Phi(\frac{1}{4}|f_m-f|)\,\mathrm{d}t\to 0, \text{ as } n,\ m\to\infty.$$

Thus the sequence $\int_0^{2\pi} f_n(t)\,\mathrm{d}X(t,\ \omega)$ converges in probability as $n\to\infty$. We can easily show that the limit does not depend on the choice of the sequence f_n. Therefore the limit of $\int_0^{2\pi} f_n(t)\,\mathrm{d}X(t,\ \omega)$, denoted by $\int_0^{2\pi} f(t)\,\mathrm{d}X(t,\ \omega)$, is called a stochastic integral of f with respect to the process $X(t,\ \omega)$. Let

$$\Sigma_n=\frac{1}{n}(S_0+S_1+\ldots+S_{n-1})$$

and

$$\sigma_n=\frac{1}{n}(f_0+f_1+\ldots+f_{n-1})$$

where

$$S_n(y,\ \omega)=\sum_{k=-n}^{n} a_k A_k(\omega)\,\mathrm{e}^{2\pi iky}$$

and

$$f_n(t)=\sum_{k=-n}^{n} a_k\,\mathrm{e}^{2\pi ikt}.$$

Hence we get

$$S_n(y,\ \omega)=\int_0^1 f_n(y-t)\,\mathrm{d}X(t,\ \omega).$$

Also we get

$$\Sigma_n(y,\ \omega)=\int_0^1 \sigma_n(y-t)\,\mathrm{d}X(t,\ \omega).$$

Now

$$\mathbf{P}\{|\Sigma_n(y,\ \omega)-\int_0^1 f(y-t)\,\mathrm{d}X(t,\ \omega)|\geqq\varepsilon\}$$

$$=\mathbf{P}(|\int_0^1 \sigma_n(y-t)\,\mathrm{d}X(t,\ \omega)-\int_0^1 f(y-t)\,\mathrm{d}X(t,\ \omega)|\geqq\varepsilon)$$

$$\leqq K_1\int_0^1 |\sigma_n-f|^\alpha \mathrm{d}t+K_2\int_0^1 \Phi(\frac{1}{4}|\sigma_n-f|)\,\mathrm{d}t \text{ (by lemma 2).}$$

However recalling that (cf. A. Zygmund [5], p. 146) if $S[f]$ is a Fourier series of f in the form of $\sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{inx}$, where

$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-int}\mathrm{d}t$ and if $\Sigma A_n(x)$ is an $S[f]$ with $f\in L_\Phi$, then as $n\to\infty$

$$\int_0^1 \Phi(\frac{1}{4}|\sigma_n-f|)\mathrm{d}t\to 0 \quad\text{and}\quad \int_0^1 |\sigma_n-f|^\alpha \mathrm{d}t\to 0.$$

Thus for any $\varepsilon>0$,

$$\lim_{n\to\infty} P\{|\Sigma_n(y,\omega)-\int_0^1 f(y-t)\mathrm{d}X(t,\omega)|\geqq\varepsilon\}=0.$$

Hence the series (3.1) converges in probability to the stochastic integral (3.2).

Theorem 2. *Suppose*

$$\int_0^{2\pi} f(t)\mathrm{d}t=0,\quad \int_0^t f(u)\mathrm{d}u=F(t)$$

and for the series

$$a_n=\frac{1}{2\pi}\int_0^{2\pi} F(t)\mathrm{e}^{-int}\,\mathrm{d}t$$

(3.3) $$\sum_{n=-N}^{N} a_n A_n(\omega)e^{iny}$$

converges in probability to $\int_0^{2\pi} F(y-t)\ \mathrm{d}X(t,\omega)$ *as* $N\to\infty$. *Then the series*

(3.4)
$$\sum_{n=-N}^{N} ina_n A_n(\omega) e^{iny}$$

converges in probability as $N \to \infty$ *to*

(3.5)
$$\int_0^{2\pi} f(y-t) dX(t, \omega).$$

(*Thus we see that the series* (3.3) *is termwise differentiable*).

Proof. By assumption we have

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} F(t) dt = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} (\int_0^t f(u) du) dt.$$

Now,

$$\text{in } a_n = \frac{i}{2\pi} \int_0^{2\pi} n e^{-int} (\int_0^t f(u) du) dt$$

$$= \frac{i}{2\pi} \{[-\frac{e^{-int}}{i} \int_0^t f(u) du]_0^{2\pi} + \frac{1}{i} \int_0^{2\pi} e^{-int} f(t) dt\}$$

$$= \frac{i}{2\pi i} \int_0^{2\pi} e^{-int} f(t) dt = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t) dt = b_n.$$

Let

$$S(y, \omega) = \sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{iny}.$$

Then

$$\frac{1}{h}\{S(y+h, \omega) - S(y, \omega)\} = \sum_{n=-\infty}^{\infty} a_n A_n(\omega) \frac{1}{h} (e^{in(y+h)} - e^{iny})$$

$$= \sum_{-\infty}^{\infty} a_n A_n(\omega) e^{iny} (\frac{e^{inh}-1}{h}) = \sum_{-\infty}^{\infty} in\ a_n A_n(\omega) e^{iny} (\frac{e^{inh}-1}{inh}) = \sum_{-\infty}^{\infty} d_n A_n(\omega) e^{iny}$$

where

$$d_n = ina_n (\frac{e^{inh}-1}{inh}) = b_n (\frac{e^{inh}-1}{inh})$$

which is a RFS series with weights d_n. Again

$$d_n = b_n (\frac{e^{inh}-1}{inh}) = \frac{b_n}{h} \int_{-h}^{0} e^{-int} dt = \int_0^{2\pi} \frac{1}{h} \int_{-h}^{0} f(y-t) dy\, e^{-iny} dt.$$

Thus d_n is the Fourier coefficient of an integral which is absolutely continuous and hence belongs to L^p, $p > 0$. Further, we khow (cf. C. Nayak, S. Pattanayak and M. N. Mishra [3]) that the series

$$\sum_{n=-N}^{N} d_n A_n(\omega) e^{iny}$$

converges in probability as $N\to\infty$, to

$$\int_0^{2\pi}\frac{1}{h}\int_{-h}^{0} f(y-t-u)\,du\,dX(t,\omega).$$

Thus

$$P\{|\frac{1}{h}(S(y+h,\omega)-S(y,\omega))-\int_0^{2\pi} f(y-t-u)dX(t,\omega)|\geqq\varepsilon\}$$

$$\leqq K_\alpha\int_0^{2\pi}\int_0 |f(x+hu)-f(x)|^\alpha du\,dx$$

by the Theorem 3 of C. Nayak, S. Pattanayak and M. N. Mishra [3]. But for $f\in L^p$, $p\geqq 1$, we know (cf. A. Zygmund [5], p. 37) that

$$\lim_{x\to y}(\int_0^1 |f(x-t)-f(y-t)|^p dt)=0$$

$$\Rightarrow\lim_{h\to 0}\int_0^1 |f(x+hu)-f(x)|^\alpha dx=0.$$

Thus we get

$$\lim_{h\to 0} P\{|\frac{1}{h}(S(y+h,\omega)-S(y,\omega))-\int_0^{2\pi} f(y-t)\,dX(t,\omega)|\geqq\varepsilon\}=0$$

which means that the RFS series (3.4) is differentiable in probability.

S. Pattanayak and S. K. Sharma [4] have shown the following result.

Theorem A. *If $a_0, a_1\ldots$ is a convex sequence such that $a_n=o(1/\log n)$, then the series*

$$\frac{1}{2}a_0 A_0+\sum_{n=1}^{N} A_n(\omega)\,a_n\cos 2\pi nt$$

converges in probability, as $N\to\infty$, to

$$\int_0^1\frac{1}{2}(f(y-t)+f(y+t))dX(t,\omega)$$

where $f(x)$ is almost everywhere the sum of the series

$$\frac{1}{2}a_0+\Sigma a_n\cos 2\pi nt$$

and

$$A_n(\omega)=\int_0^1\cos 2\pi nt\,dX(t,\omega).$$

Let us mention that now we can show that the series (1.2) is convergent in probability under a weaker condition.

Theorem 3. *If $X(t,\omega)$ is a symmetric stable process of index $\alpha=1$ and if $f\in L^1$,*

$$a_n = \int_0^{2\pi} f(t)\, e^{-2\pi i n t}\, dt$$

and

$$\omega_1(\varepsilon, f) = o\left(\frac{1}{\log 1/\varepsilon}\right),$$

then the series

$$\sum_{n=-N}^{N} a_n A_n(\omega)\, e^{2\pi i n y}$$

converges in probability, as $N \to \infty$, *to the stochastic integral* (3.2) *where* a_n *and* $A_n(\omega)$ *have the same meaning as in Theorem* 1.

To prove Theorem 3 we need the following Lemma due to A. Zygmund [5], p. 180.

Lemma 3. *If the integral modulus of continuity satisfies the condition*

$$\omega_1(\varepsilon, f) = o\left(\frac{1}{\log 1/\varepsilon}\right),$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} |f - S_n|\, dx \to 0 \quad \text{as } n \to \infty.$$

Proof of Theorem 3. Let

$$S_n(y, \omega) = \sum_{-n}^{n} a_k A_k(\omega)\, e^{2\pi i k y}$$

and

$$f_n(t) = \sum_{-n}^{n} a_k\, e^{2\pi i k t}.$$

Thus

$$S_n(y, \omega) = \sum_{-n}^{n} a_k\, e^{2\pi i k (y-t)} dX(t, \omega) = \int_0^1 f_n(y-t)\, dX(t, \omega).$$

Now

$$\mathbf{P}\{|S_n(y, \omega) - \int_0^1 f(y-t)\, dX(t, \omega)| > \varepsilon\} = \mathbf{P}\{|\int_0^1 (f_n(y-t) - f(y-t))\, dX(t, \omega)| > \varepsilon\}$$

$$\leqq K \int_0^1 |f_n(y-t) - f(y-t)|\, dt \quad \text{(by lemma 2)}.$$

But since

$$\omega_1(\varepsilon; f) = o\left(\frac{1}{\log 1/\varepsilon}\right),$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} |S_n - f|\, dx \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty}\int_0^1 |f_n(y-t)-f(y-t)|\,dt=0$$

$$\lim_{n\to\infty} \mathbf{P}\{|S_n(y,\omega)-\int_0^1 f(y-t)\,dX(t,\omega)|>\varepsilon\}=0$$

for any $\varepsilon>0$.
Thus the series (3.1) converges in probability to the stochastic integral (3.2).

Theorem 4. *Let $X(t,\omega)$ be a symmetric stable process of index α, $1<\alpha<2$, and let*

$$A_n(\omega)=\int_0^1 e^{-2\pi int}\,dX(t,\omega),\quad n\in Z. \tag{3.6}$$

Then the RFS series

$$\sum_{n=-N}^{N} a_n A_n(\omega)\,e^{2\pi iny}=S_N \tag{3.7}$$

converges in probability, as $N\to\infty$, to the stochastic integral

$$\int_0^1 f(y-t)\,dX(t,\omega),\ \text{for}\ \ f\in\Lambda_\alpha, \tag{3.8}$$

where a_n is the Fourier Coefficient of f and

$$\mathbf{P}(|S_n(y,\omega)-\int_0^1 f(y-t)\,dX(t,\omega)|\geqq\varepsilon)=O(n^{-\alpha}). \tag{3.9}$$

Proof: Let

$$S_n(y,\omega)=\sum_{k=-n}^{n} a_k A_k(\omega)\,e^{2\pi iky}$$

and

$$S_n'(t)=\sum_{k=-n}^{n} a_k e^{2\pi ikt}.$$

Thus

$$S_n(y,\omega)=\sum_{k=-n}^{n}\int_0^1 a_k e^{2\pi ik(y-t)}\,dX(t,\omega)$$
$$=\int_0^1 S_n'(y-t)\,dX(t,\omega).$$

Now

$$\mathbf{P}\{|S_n(y,\omega)-\int_0^1 f(y-t)\,dX(t,\omega)|>\varepsilon\}$$
$$=\mathbf{P}\{|\int_0^1 (S_n'(y-t)-f(y-t))\,dX(t,\omega)|>\varepsilon\}$$
$$\leqq \mathrm{K}\int_0^1 |S_n(y-t)-f(y-t)|^\alpha\,dt \ \text{(by lemma 2).}$$

It is well known (cf. A. Zygmund [5], p. 91) that for $f \in \Lambda_\alpha$, $0<\alpha<1$,

$$S'_n(x)-f(x)=O(n^{-\alpha}).$$

$$P\{|S_n(y,\omega)-\int_0^1 f(y-t)\,dX(t,\omega)|>\varepsilon\}=O(n^{-\alpha}).$$

Hence for all $\varepsilon>0$

$$\lim_{n\to\infty} \mathbf{P}\{|S_n(y,\omega)-\int_0^1 f(y-t)\,dX(t,\omega)|>\varepsilon\}=0. \tag{3.10}$$

Therefore the series (3.7) is convergent in probability and the rate of convergence is given by (3.9).

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