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# Local Structure of a Riemannian Manifold that Admits a Type of Semi-Symmetric Metric Connection

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M. C. Chaki and A. Konar [1] have given an expression for the curvature tensor of a Riemannian manifold $M^n$ that admits a semi-symmetric metric connection $D$ with zero curvature and recurrent torsion tensor. Later A. Konar [2] has shown it is locally a hypersurface of $R^{n+1}$. In this paper we have obtained the local structure of such a manifold, using a result due to J. A. Schouten [3], S. Nishikawa and J. Maeda [4].)

## Introduction

We consider an $n$-dimensional Riemannian manifold $M^n$ with Levi—Civita connection $D$. A linear connection $D$ on $M^n$ is said to be a semi-symmetric metric connection if the torsion tensor $T$ of the connection $D$ and the metric tensor $g$ of the manifold satisfy the following conditions:

(1.1) $T(X, Y)=\omega(Y)X-\omega(X)Y$ for any two vector fields $X$, $Y$, where $\omega$ is a 1-form associated with the torsion tensor of the connection $D$ and

(1.2) $(D_z g)(X, Y)=0$ and further, if

(1.3) $(D_z T)(X, Y)=B(Z)T(X, Y)$, then the torsion tensor $T$ is said to be recurrent with $B$ as its 1-form of recurrence.

Then, by [5], we have for any vector fields $X$, $Y$, $Z$

(1.4) $D_X Y=\tilde{D}_X Y+\omega(Y)X-g(X, Y)V$, where

(1.5) $g(X, V)=\omega(X)$ for every vector field $X$ and

(1.6) $(D_X\omega)(Y)=(\tilde{D}_X\omega)(Y)-\omega(X)\omega(Y)+\omega(V)g(X, Y)$.

Also we have by [1]

(1.7) $$R(X, Y)Z=K(X, Y)Z+B(X)[\omega(Z)Y-g(Y, Z)V]$$

$$-B(Y)[\omega(Z)X-g(X, Z)V]+\omega(V)[g(X, Z)X-g(X, Z)Y]$$

where $R$ and $K$ are the respective curvature tensors for the connections $D$ and $\tilde{D}$.

If in particulars $R=0$, then by [5] the manifold is conformally flat and then by [2] we have

(1.8) $$K(X,Y)Z=a\omega(Y)[\omega(Z)X-g(X,Z)V+a\omega(X)[g(Y,Z)V-\omega(Z)Y] + \omega(V)[g(X,Z)Y-g(Y,Z)X],$$

(1.9) where $B(V)=a\omega(V)$, $\omega(V)\neq 0$ and $\omega$ is closed i.e. $d\omega(X,Y)=0$ for all $X$ and $Y$.

We shall use these results in the sequel. Further, we state the following theorems due to J. A. Schonten [3], S. Nishikawa and J. Maeda [4] which will be used in the following section:

**Theorem 1.1** (J. A. Schouten). *Let $M^n$ be a subspace of a Euclidean space $R^{n+1}(n>3)$. Then $M^n$ is conformally flat if and only if at each point of $M^n$, the second fundamental operator $N$ of $M^n$ is one of the following types:*

(1.10) (A) *$N=\lambda I$, $I=$identity transformation*

(B) *$N$ has two distinct eigenvalues of multiplicity ($n$-1) and 1 respectively.*

S. Nishikawa and J. Maeda classified conformally flat hypersurfaces in a Euclidean space $H^{n+}$ for different cases as follows:

Case I. $N=\lambda I$, $I$ being the identity transformation.

Case II. $N$ has two distinct eigenvalues $\lambda$ and $\mu$ which are of multiplicity ($n$-1) and 1 respectively at every point of $M^n$.

They treated the Case II for different subclasses II(A), II(B) and II(C) as follows:

Case II(A): $V.\lambda\neq 0$ and $X_i.\mu=0$, for all $i=1, 2,\ldots,(n-1)$

II(B): $V. =0$ and $X_i.\mu\neq 0$ for some $i$.

II(C): $V.\lambda=0$ and $X_i.\mu=0$ for all $i$.

Where $V$ is an eigen vector corresponding to the eigen value $\mu$ and $X_i$, $i=1, 2,\ldots,(x-1)$ are other eigen vectors corresponding to the same eigen value $\lambda$.

Now, we state the theorem due to S. Nishikawa and J. Maeda:

**Theorem 1.2** (S. Nishikawa and J. Maeda) [4]. *Let $M^n$ $(n>3)$ be a conformally flat hypersurface of a Euclidean space $R^{n+1}$. Then $M^n$ is locally one of the following:*

*Case I: A totally umbilical hypersurface (hence of constant curvature).*

*Case II: A surface of revolution–[let $(x^1, x^2,\ldots,x^{n+1})$ be cannonical co-ordinate system of $R^{n+1}$ and v, a curve in $(x^1-x^2)$ plane defined by $x^1=v(x^2)$, $x^2>0$. Rotating v about $x^2$ axis, we get a surface of revolution G.v, where G is rotation group $G=SO(n)=SO(x^1, x^2,\ldots,x^{n+1})$].*

*Case IIB: A tube.*

*Case IIC: A product manifold $S^{n-1}X\ R$ or a cylinder $R^{n-1}Xv$ built over a plane curve v.*

**Section 2.**

This section is concerned with the determination of the local structure of the manifold $M^n$ (upto isometry) when it is a hypersurface of $R^{n+1}$ as obtained by the theorem due to A. Konar [2] as follows:

**Theorem 2.1.** *Let p be a point on an n (>3) dimensional Riemannian manifold $M^n$ which admits a semi-symmetric metric connection D whose curvature tensor $R=0$ and torsion tensor T is recurrent. Then there exists an isometric imbedding of a neighbourhood ∪ of p in $R^{n+1}$ and N is the corresponding symmetric linear operator of the second fundamental form L given by*

(2.1) $L(X,Y)=\frac{1}{n-2}\left(\frac{\mathrm{Ric}(X,Y)}{b}-c\,g(X,Y)\right)$ *where* $b(\neq 0)$ *and C are given by the following relations*

(2.2) $b^2=\omega(V)$ *and* $C=(a-1)b$ i.e. $bc=(a-1)\omega(V)$ *and also*

(2.3) $L(X,Y)=g(N(X),Y)=g(X,N(Y))$ *for all X, Y, N being a linear map*: $M_p\to M_p$ *for every point p in* $M^n$.

(2.4) *So* $N(x)=\frac{1}{(n-2)b}P(X)-\frac{C}{(n-2)}X$

(2.5) *where* $\mathrm{Ric}(X,Y)=g(P(X),Y)$.

From (1.8), we get

(2.6) $\mathrm{Ric}(X,Y)=(a-n+1)\omega(V)g(X,Y)+(n-2)a\,\omega(X)\omega(Y)$

and also we have by [2]

(2.7) $Z(b)=\frac{a}{b}\omega(V)\omega(Z),\ Z(c)=bZ(a)+\frac{a(a-1)}{b}\omega(V)\omega(Z)$

$\omega(V)X(a)=V(a)\omega(X)$.

In virtue of $R=0$, the conformal curvature tensor $C$ of the manifold $M^n$ is zero i.e. $C(X,Y)Z=0$ by [5].
Again by Theorem 2.1 the manifold is a hypersurface of $R^{n+1}$ and then the fundamental operator $N$ will be of the type indicated by (1.10). Now, from (2.4) we have

(2.8) $$N(X)=\left(\frac{(a-n+1)\omega(V)}{(n-2)b}-\frac{c}{(n-2)}\right)X+\frac{a}{b}\omega(X)V$$

for every vector field $X$,

where $b$ and $c$ are given by (2.2).
Thus, the fundamental operator $N$ will be of the type $N(X)=\lambda X$ if and only if $a=0$ i.e. when it is a group manifold [6] with respect to the connection $D$. So, by case I of theorem 1.2 [4] we get the following:

**Theorem 2.2.** *If a Riemannian manifold $M^n$ admits a semi-symmetric metric connection for which it is a group manifold, then $M^n$ is a totally umbilical hypersurface of the Euclidean space $R^{n+1}$ and hence of constant curvature.*

Next, we assume that $a \neq 0$ and then by Th. 1.1 [3] the second fundamental operator $N$ of the manifold $M^n$ has two distinct eigen values of multiplicity ($n$-1) and 1 respectively.

In (2.8), we put $X = V$ and we find (on using (2.2)) that

$$NV = CV. \tag{2.9}$$

This shows that $C = (a-1)b$ where $b^2 = \omega(V)$ is one of the eigenvalues of the operator $N$ with $V$ as the corresponding eigenvector.

Again, as the manifold is of dimension $n(>3)$, it is always possible to get ($n$-1) mutually orthogonally vectors $X_1, X_2, \ldots, X_{n-1}$ such that

$$g(X_i, V) = 0, \ i = 1, 2, \ldots, (n-1) \text{ i.e. } \omega(X_i) = 0. \tag{2.10}$$

We shall now show that each such $X_i$, $i = 1, 2, \ldots (n-1)$ is an eigenvector of the fundamental operator $N$ with the same eigenvalue $(-b)$.

From (2.8), it follows that

$$N(X_i) = -bX_i, \ i = 1, 2, \ldots (n-1), \text{ [using 2.2]}. \tag{2.11}$$

Thus, the fundamental operator $N$ has two distinct eigenvalues $(-b)$ and $C$ of multiplicity $(n-1)$ and 1 respectively. Now, using the result due to S. Nishikawa and J. Maeda [4] we find that here $\lambda = -b$ and $\mu = C$. We assume that $C \neq 0$, hence from (2.7) it follows that

$$V(-b) = -\frac{a}{b}(\omega(V))^2 \neq 0 \text{ for } a, b, \omega(V) \neq 0 \tag{2.12}$$

$$\text{and } X_i(c) = 0 \text{ for all } i = 1, 2, \ldots, (n-1). \tag{2.13}$$

Therefore, by Case IIA Th. 1.2 [4] we get the following:

**Theorem 2.2.** *Let $M^n$ $(n > 3)$ be a Riemannian manifold of dimension n which admits a semi-symmetric metric connection D whose curvature tensor $R = 0$ and torsion tensor is recurrent. Then the manifold is a surface of revolution.*

Next, we shall prove the following:

**Theorem 2.3.** *If a Riemannian manifold $M^n$ of dimension n $(>3)$ admits a semi-symmetric metric connection D whose curvature tensor $R = 0$ and torsion tensor is recurrent, then for each point p on $M^n$ has an open neighbourhood U such that $U = U' \times U''$, where $U'$* (resp. *$U''$*) *is an open neighbourhood of p in $M'$* (resp. *$M''$*) *and the Riemannian metric in U is the direct product of the Riemennian metrics in $U'$ and $U''$, $M'$ and $M''$ being the respective maximal integral manifolds of the distributions $T'$ and $T''$ which respectively correspond to the vector subspaces $T'_p$(generated by $V_p$) and $T''_p$ (generated by* ($n$-1) *vectors $(X_i)_p$, $i-1, 2, \ldots,$* ($n$-1), *where $\omega(X_i) = 0$*), *$\omega$ being the i-form associated with the torsion tensor.*

Proof. Let $p$ be a point on $M^n (n>3)$. Thus $T_p(M^n)$ is the tangent space of the manifold $M^n$ at $p$.

Since $V$ and $X_i$ are the vector fields on $M^n$, the vector $V_p$ and each vector $(X_i)_p$, $i=1, 2, \ldots, (n-1)$ are the elements of $T_p(M^n)$.

Now, let us denote the vector sub-space of $T_p(M^n)$ generated by $V_p$ by $T'_p$ and the vector subspace generated by $(X_i)_p$, $i=1, 2, \ldots (n-1)$ by $T''_p$. Thus we obtain two distributions $T'$ and $T''$ obtained from $T'_p$ and $T''_p$ and the distributions $T'$ and $T''$ are complementary and orthogonal to each other at every point $p$ of $M^n$ as $\omega(X_i)=g(X_i, V)=0$ for $i=1, 2, \ldots, (n-1)$. For this, we are only to prove that the distribution $T''$ is involutire i.e. if $(X_i)_p$, $(X_j)_p$, $i\neq j$, $i, j=1, 2, \ldots (n-1)$ belong to $T''_p$ then $[X_i, X_j]_p$ also belongs to $T''_p$.

Since $\omega$ is closed then $d\omega(X, Y)=0$ for all vector fields $X$, $Y$.
So $d\omega(X_i, X_j)=0$, $i\neq j$, $i, j=1, 2, \ldots n-1$

i.e. $-\frac{1}{2}(X_i\omega(X_j)-X_j\omega(X_i)-\omega([X_i, X_j]))=0.$

Since $\omega(X_i)=0$ for $i=1, 2, \ldots (n-1)$, we find that

$$\omega([X_i, X_j])=0, \quad i\neq j, \quad i, j=1, 2, \ldots n-1, \tag{2.14}$$

which shows that $[X_i, X_j]_p$ is an element of $T''_p$. Hence the distribution $T''$ is involutive.

This completes the proof.

Finally, we consider the case when the eigenvalue $C$ of the fundamental operator $N$ of the manifold associated with eigenvalue $V$ is zero i.e. when $C=(a-1)b=0$.

Since $b\neq 0$, $C=0\Rightarrow a=1$,
i.e. 1-form $B$ associated with the recurrent torsion tensor $T$ is equal to the associated 1-form $\omega$.

Also when $a=1$, from (1.8) and (2.6) we have

$$K(X, Y)Z=\omega(Y)[\omega(Z)X-g(X, Y)V]-\omega(X)[\omega(Z)Y-g(Y, Z)V] -\omega(V)[g(Y, Z)X-g(X, Z)Y] \tag{2.15}$$

$$\mathrm{Ric}(X, Y)=-(n-2)\omega(V)g(X, Y)+(n-2)\omega(X)\omega(Y). \tag{2.16}$$

Putting $Y=V$ in (2.15) we find that

$$K(X, V)Z=0. \tag{2.17}$$

Also

$$K(X, Y;\ Z, U)\overset{\mathrm{def}}{=}g(K(X, Y)Z, U)$$

$$=\omega(Y)[\omega(Z)g(X, U)-g(X, Z)\omega(U)]-\omega(X)[\omega(Z)g(Y, U) -\omega(U)g(Y, Z)]-\omega(V)[g(Y, Z)g(X, U)-g(X, Z)g(Y, U)]. \tag{2.18}$$

Also from (2.16), we find that the scalar curvature $r$ of the manifold is given by $r=-(n-1)(n-2)\omega(V)$.
Thus (2.18) can be expressed as follows:

$$K(X, Y; Z, U)=\frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, U)-g(X, Z)g(Y, U)]$$

$$\text{(2.19)} \qquad +g(X, U)\omega(Y)\omega(Z)+g(Y, Z)\omega(X)\omega(U)$$

$$-g(X, Z)\omega(Y)\omega(U)-g(Y, U)\omega(X)\omega(Z).$$

Now from (2.19) we can easily find after covariant differentiation that

$$(\tilde{D}_W K)(X, Y; Z, U)=4\omega(W)K(X, Y; Z, U)+2\omega(X)K(W, Y, Z, U)$$

$$\text{(2.20)} \qquad +2\omega(Y)K(X, W; Z, U)+2\omega(Z)K(X, Y; W, U)$$

$$+2\omega(U)K(X, Y; Z, W).$$

The relation (2.20) shows that the curvature tensor of the manifold satisfies the condition of Pseudo Symmetric manifold introduced by M. C. Chaki [6] and is named as Chaki's Space $C(PS)_n$.

Thus we have the following:

**Theorem 2.4.** *If a Riemannian manifold $M^n$ $(n>3)$ admits a semi-symmetric metric connection D whose curvature tensor $R=0$ with recurrent torsion tensor and 1-form of recurrence coincides with the associated 1-form of the connection D then the manifold is a Chaki's space $C(PS)_n$ and the curvature tensor K of the manifold satisfies the following properties.*

(i) $K(X, V)Z=0$ *where* $g(X, V)=\omega(X)$ *for all* $X$.

(ii) $K(X, Y; Z, U)=\frac{r}{(n-1)(n-2)}[g(Y, Z)\, g(X, U)-(X, Z)g(Y, U)]$

$+g(X, U)\omega(X)\omega(Z)+g(Y, Z)\omega(X)\omega(U)-g(X, Z)\omega(Y)\omega(U)-g(Y, U)\omega(X)\omega(U),$

*where r the scalar curvature of the manifold.*

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