Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 4

Some Remarks on a Paper of A. Verma and C. M. Joshi

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Presented by P. Kenderov

1. Introduction

In a recent paper A. Verma and C. M. Joshi [6] obtained the following transformation

(1.1)
$$ {}_3\varphi_2\left(\begin{matrix} a, b, q^{-n}; q^{1+N} \\ e, abq^{1+N-n}/e \end{matrix}\right) = \frac{(eq^{-N}/a)_n (eq^{-N}/b)_n}{(e)_n (eq^{-N}/ab)_n} q^{Nn} \cdot \sum_{j=0}^{N} \frac{(q^{-N})_j (q^{-n})_j (q^{-N}/ab)_j}{(q)_j (eq^{-N}/a)_j (eq^{-N}/b)_j} q^j, $$

where n, N are non-negative integers with $N<n$ and

$$ {}_r\varphi_s\left(\begin{matrix} a_1, a_2, \dots, a_r; t \\ b_1, b_2, \dots, b_s \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n t^n}{(q)_n (b_1)_n (b_2)_n \dots (b_s)_n}, $$

$$ (a; q)_n \equiv (a)_n = (1-a)(1-aq)\dots(1-aq^{n-1}); \ (a)_0 = 1, $$

$$ (a; q)_\infty \equiv (a)_\infty = \prod_{r=0}^{\infty} (1-aq^r), \ |q|<1. $$

As an application of (1.1), the transformation formula

(1.2)
$$ {}_3\varphi_2\left(\begin{matrix} a, b, q^{-n}; q \\ e, abq^{1+N-n}/e \end{matrix}\right) = \frac{(eq^{-N}/a)_n (eq^{-N}/b)_n q^{nN}}{(e)_n (eq^{-N}/ab)_n} \cdot \sum_{p=0}^{N} \sum_{j=0}^{N-p} \frac{(a)_p (b)_p (eq^{-N}/ab)_j (q^{-n})_{p+j} (q^{-N})_{p+j}}{(q)_p (q)_j (eq^{-N}/a)_{j+p} (eq^{-N}/b)_{p+j}} (eq^{1-N}/ab)^p q^j $$

was deduced by them.

They gave a rather lengthy proof of (1.1), by using the technique of W. N. Bailey [5].

In the present note, we have shown, in what follows, that a generalization of (1.1) can be obtained directly as an application of a result due to D. B. Sears [4]. We also give a generalization of (1.2).

2. A generalization of (1.1)

We prove that

$$(2.1)\qquad {}_3\varphi_2\begin{pmatrix} a, b, q^{-N}; t \\ e, abtq^{-N}/e \end{pmatrix} = \frac{(eq/bt)_N (eq/at)_N t^N q^{-N}}{(e)_N (eq/abt)_N}.$$

$$\cdot\, {}_3\varphi_2\begin{pmatrix} q/t, eq/abt, q^{-N}; q \\ eq/at, eq/bt \end{pmatrix}.$$

Proof: A well-known result of D. B. Sears [4] is that

$$(2.2)\qquad {}_3\varphi_2\begin{pmatrix} a\, b, c; ef/abc \\ e, f \end{pmatrix} = \frac{(e/c)_\infty (ef/ab)_\infty}{(e)_\infty (ef/abc)_\infty}.$$

$$\cdot\, {}_3\varphi_2\begin{pmatrix} c, f/a, f/b; e/c \\ f, ef/ab \end{pmatrix}.$$

Setting $e=q^{-N}$, $f=abtq^{-N}/e$ in (2.2) and then writing the terminating series ${}_3\varphi_2$ on the right hand side in the reverse order, we get (2.1) after some simplification.

3. A generalization of (1.2)

We, next, prove that

$$(3.1)\qquad {}_3\varphi_2\begin{pmatrix} a, b, q^{-N}; t \\ e, abtq^{-N+n}/e \end{pmatrix} = \frac{(eq^{1-n}/bt)_N (eq^{1-n}/at)_N t^N q^{-N} q^{nN}}{(e)_N (eq^{1-n}/abt)_N}.$$

$$\cdot \sum_{p=0}^{n} \sum_{j=0}^{n-p} \frac{(a)_p (b)_p (eq^{1-n}/abt)_j (q^{-N})_{p+j} (q^{-n})_p (q^{1-n+p}/t)_j}{(q)_p (q)_j (eq^{1-n}/at)_{p+j} (eq^{1-n}/bt)_{p+j}} (eq^{2-n}/abt)^p.$$

Proof: For proving (3.1), we observe that

$$(3.2)\qquad {}_3\varphi_2\begin{pmatrix} a, b, q^{-N}; t \\ e, abtq^{n-N}/e \end{pmatrix} = \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix} \frac{(a)_p (b)_p (q^{-N})_p}{(e)_p (abtq^{n-N}/e)_p} t^p.$$

$$\cdot\, {}_3\varphi_2\begin{pmatrix} aq^p, bq^p, q^{-N+p}; tq^{n-p} \\ eq^p, abtq^{n-N+p}/e \end{pmatrix}.$$

The truth of (3.2) can be easily verified by substituting the series definition of ${}_3\varphi_2$ on the right hand side, interchanging the order of summations and summing the inner terminating ${}_2\varphi_0$ by the known sum

$$(3.3)\qquad {}_2\varphi_0(q^{-n}, a; —; q) = a^n.$$

Now summing the inner ${}_3\varphi_2$-series on the right hand side in (3.2) by using (2.1), we obtain the required result (3.1), on simplification.

4.

Setting $e=abt$ in (2.1), we get the partial sum of a ${}_2\varphi_1$ in the form

$$\sum_{n=0}^{N}\frac{(a)_n(b)_n}{(q)_n(abt)_n}t^n=\frac{(a)_{N+1}(b)_{N+1}}{(q)_N(abt)_N}t^Nq^{-N}\cdot\sum_{n=0}^{N}\frac{(q/t)_n(q^{-N})_n}{(a)_{n+1}(b)_{n+1}}q^n. \tag{4.1}$$

For $t=q$, (4.1) reduces to a result, due to R. P. Agarwal [1, p. 443(iii)], namely

$$\sum_{n=0}^{N}\frac{(a)_n(b)_nq^n}{(q)_n(abq)_n}=\frac{(aq)_N(bq)_N}{(q)_N(abq)_N}. \tag{4.2}$$

For $t=q^2$, (4.1) yields the sum of a partial ${}_2\varphi_1$-function, namely

$$\sum_{n=0}^{N}\frac{(a)_n(b)_nq^{2n}}{(q)_n(abq^2)_n}=\frac{(aq)_N(bq)_Nq^N}{(q)_N(abq^2)_N}\cdot\left[1+\frac{(1-q)(1-q^N)q^{-N}}{(1-aq)(1-bq)}\right]. \tag{4.3}$$

Similarly, by taking $t=q^3$, q^4, etc., we get different sums for different partial ${}_2\varphi_1$-functions.

Replacing a by aq, b by q/a in (4.1), we get a partial theta function [for definition, see G. E. Andrews (2)] for a partial sum of a ${}_2\varphi_1$-function, namely

$$\sum_{n=0}^{N}\frac{(aq)_n(q/a)_nt^n}{(tq^2)_n(q)_n}=\frac{(qa)_{N+1}(q/a)_{N+1}}{(q)_N(tq^2)_N}t^Nq^{-N}\cdot\sum_{n=0}^{N}\frac{(q/t)_n(q^{-N})_nq^n}{(aq)_{n+1}(q/a)_{n+1}}. \tag{4.4}$$

Next, setting $e=aq/b$, $t=q^2$ in (2.1), we get another known sum of a nearly-poised ${}_3\varphi_2$, due to W. N. Bailey [3], namely

$${}_3\varphi_2\left(\begin{matrix}a, b, q^{-N}; q^2\\ aq/b, b^2q^{1-N}\end{matrix}\right)=\frac{(a/b^2)_N(1/b)_N}{(aq/b)_N(1/b^2)_N}q^N\cdot\left[1-\frac{(1-1/b^2)(1-q^{-N})}{(1-a/b^2)(1-1/b)}\right]. \tag{4.5}$$

Similarly taking $t=q^3$, q^4, etc. in (2.1), we get a number of other sums of different nearly-poised ${}_3\varphi_2$-series.

Acknowledgement: This work has been done as a J. R. F. under the supervision of Prof. R. P. Agarwal, Emeritus Scientist, in his C. S. I. R. Scheme, entitled, 'Basic hypergeometric series, continued fractions, Pade' Approximants and their applications'.

References

1. R. P. Agarwal. On the partial sums of series of hypergeometric type. *Proc. Camb. Phil. Soc.* (3), **49**, 1953, 441-445.
2. G. E. Andrews. Ramanujan 'Lost' Note book I: Patial theta functions. *Adv. in Maths*, **41**, 1981, 137-172.
3. W. N. Bailey. A transformation of nearly-poised basic hypergeometric series. *J. London Math. Soc.*, **22**, 1947, 237-240.
4. D. B. Sears. On the transformation theory of basic hypergeometric functions. *Proc. London Math. Soc.*, **53**, 1951, 158-180.
5. L. J. Slater. Generalized Hypergeometric Functions. Cambridge University Press (1966).
6. A. Verma, C. M. Joshi. Some remarks on summation of basic hypergeometric series. *Houston J. Math.*, **5**, 1979, 277-294.

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Received 27.09.1989