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Cosine Functions of Unbounded Operators and Second Order Differential Equations

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In the paper the correlation between cosine functions of unbounded operators in a Banach space and the theory of differential equations is investigated. The main theorem characterizes the connection between the second order Cauchy problem and the generator of unbounded cosine function.

In the previous paper of the author [N] the cosine functions of unbounded operators have been defined and their fundamental properties investigated. Cosine families of unbounded families have appeared first in the papers of G. Maltese [M] and H. Grabmuller [G], but a general theory has not been yet fully developed. The difficulties that always appear when one tries to deal with algebraic properties of a system of unbounded operators are not easy to overcome, especially in the case of cosine families. The reader can consult the paper of D. Lutz [L] for the bounded operators' case and judge how much can and have been saved from the standard theory.

The definition of a cosine family of unbounded operators (given in [N]) is as follows. First of all for a given family $\mathscr{C}=\{\mathscr{C}(t):t\in \mathsf{R}\}$ of unbounded operators in a Banach space X we define its cosine domain $\mathscr{D}(\mathscr{C})$ as the set

$$\mathscr{D}(\mathscr{C})=\left\{x\in \bigcap_{r,q\in \mathsf{R}} D[\mathscr{C}(r)\mathscr{C}(q)]: \begin{array}{l}\text{(i) } 2\mathscr{C}(t)\mathscr{C}(s)x=\mathscr{C}(t+s)x+\mathscr{C}(t-s)x \\ \qquad \text{for any } s,t\in \mathsf{R} \\ \text{(ii) } \mathsf{R}\ni t\to\mathscr{C}(t)x \text{ is continuous}\end{array}\right\},$$

and then we say that $\mathscr{C}$ is a cosine function if the set $\mathscr{D}(\mathscr{C})$ is dense in X and $\mathscr{C}(0)x=x$ for any $x\in\mathscr{D}(\mathscr{C})$. The (*i*) condition is called the d'Alembert equation. For such a cosine function there exists a natural topology on the domain $\mathscr{D}(\mathscr{C})$, called the *$\mathscr{C}$-topology*, and given as the topology of the family of seminorms $n_K(x)=\sup\{\|\mathscr{C}(t)x\|:|t|\leqq K\}$, where K varies through all nonnegative real numbers. A cosine function is called closed, iff $\mathscr{D}(\mathscr{C})$ is complete in this topology. In [N] it has been proved that if all the operators $\mathscr{C}(t)$, $t\in\mathsf{R}$, are closed then $\mathscr{C}$ is closed. One can thus see that this property is quite natural and for this reason in the current paper, as in [N], only closed cosine families are investigated.

For a closed cosine family $\mathscr{C}=\{\mathscr{C}(t):t\in\mathsf{R}\}$ its generator $\mathfrak{C}$ is defined as an operator with values given by

$$\mathfrak{C}x=\mathscr{C}-\lim_{h\to 0}\frac{2}{h^2}[\mathscr{C}(h)x-x]$$

and the domain equal to the set of all those $x\in\mathscr{D}(\mathscr{C})$, for which the above limit, taken in the $\mathscr{C}$-topology, exists.

One should notice here that the definitions above coincide with the standard ones when all the operators $\mathscr{C}(t)$, $t \in \mathrm{R}$ are bounded ($\mathscr{C}$-topology is then equal to the norm topology).

We will need one more property, common for cosine families of bounded and unbounded operators, namely that given in Preposition 3 of [N]. It states that for any $x \in D(\mathfrak{C})$ and $t \in \mathrm{R}$ the vector $\int_0^t (t-s)\mathscr{C}(s)x\mathrm{d}s$ as well as $\mathscr{C}(t)x$ belong to the domain of the generator, that $\mathfrak{C}$ and $\mathscr{C}(t)$ commute on x and that

$$\mathscr{C}(t)x - x = \mathfrak{C}\int_0^t (t-s)\mathscr{C}(s)x\mathrm{d}s = \int_0^t (t-s)\mathscr{C}(s)\mathfrak{C}x\mathrm{d}s,$$

where the integral is taken as the Riemann integral in the (complete) $\mathscr{C}$-topology.

A scalar function $b\,\mathrm{coh}\,(t\sqrt{a})$ is the solution of the differential equation $f'' = af$, with the initial conditions $f(0) = b, f'(0) = 0$. The above remark is also true for a cosine function $\mathscr{C}$ of bounded operators in a Banach space X. The function $f(t) = \mathscr{C}(t)x$ is then a solution of the equation $f'' = \mathfrak{C}f$, $f(0) = x$, $f'(0) = 0$, where $\mathfrak{C}$ denotes the generator of the cosine function $\mathscr{C}$. In the case when the operators $\mathscr{C}(t)$ are bounded it implies the continuous dependence of the solutions on the initial conditions. The inverse procedure is possible as well: if the solutions of the eqution $f'' = \mathfrak{B}f$ continuously depend on the initial conditions, then one can construct a cosine function with the generator equal to $\mathfrak{B}$. The reader can find this process fully described in the H. O. Fattorini papers [F1], [F2].

A part of these results is valid also for the cosine functions of unbounded operators, although it is obvious that one cannot expect, say, the continuous dependence of the solutions on initial conditions.

Corollary 1. *Let $\mathscr{C} = \{\mathscr{C}(t) : t \in \mathrm{R}\}$ be a closed cosine family of operators in a Banach space X, and $\mathfrak{C}$ its generator. Then the Cauchy problem*

$$\text{(PC)} \qquad f''(t) = \mathfrak{C}f(t), \quad f(0) = x, \quad f'(0) = y$$

for any $x, y \in D(\mathfrak{C})$ has a unique solution in the class of the functions $f\colon \mathrm{R} \to D(\mathfrak{C})$ twice continuously differentiable in the $\mathscr{C}$-topology. This solution is given by

$$f(t) = \mathscr{C}(t)x + \mathscr{S}(t)y, \quad \textit{where} \quad \mathscr{S}(t)y = \int_0^t \mathscr{C}(s)y\,\mathrm{d}s.$$

Proof. We use the proposition from [N] that we have mentioned before stating that for a vector $x \in \mathscr{D}(\mathfrak{C})$ the value of $\mathscr{C}(t)$ on x is equal to $\mathscr{C}(t)x = x + t\int_0^t \mathscr{C}(s)\mathfrak{C}x\mathrm{d}s - \int_0^t s\mathscr{C}(s)\mathfrak{C}x\mathrm{d}s$ and that the operators $\mathfrak{C}$ and $\mathscr{C}(t)$ commute on $\mathscr{D}(\mathfrak{C})$. Then

$$f'(t) = \int_0^t \mathscr{C}(s)\mathfrak{C}x\mathrm{d}s + \mathscr{C}(t)y, \quad f''(t) = \mathscr{C}(t)\mathfrak{C}x + \int_0^t \mathscr{C}(s)\mathfrak{C}y\,\mathrm{d}s = \mathfrak{C}f(t)$$

and $f(0) = \mathscr{C}(0)x = x$, $f'(0) = \mathscr{C}(0)y = y$. The function f is twice continuously $\mathscr{C}$-differentiable, by the same proposition, or one can deduce this from the $\mathscr{C}$-continuity of the operators $\mathscr{C}(t)$ and the formula above. The uniquness of the solution in the class of twice continuously $\mathscr{C}$-differentiable functions can be proved in a standard manner. One can use the method presented in the paper of D. Lutz [L] pp. 86-87, if only one notices that whenever the function f is $\mathscr{C}$-differentiable then the same is true of the function $t \to \mathscr{C}(t)f(t)$. ■

Let consider the way the closedness condition for the cosine function $\mathscr{C}$ influences the solutions of the Cauchy problem (PC). It is easy to see that the following closability condition (CC) is fulfilled.

(CC): *Whenever* $\{f_n\}$ *is a sequence of the solutions of* (*PC*) *from the class of twice continuously* $\mathscr{C}$*-differentiable functions, such that* $f_n(0)\to 0$, $f_n'(0)=0$, *then for any function* $g:\mathrm{R}\to X$ *fulfilling*

$$\forall K\geqq 0 \lim_{n\to\infty} \sup_{|t|\leqq K} \|f_n(t)-g(t)\|=0$$

there has to be $g(t)\equiv 0$.

Following H. Grabmüller [G], we can call the function $g(t)$ the generalized solution of the Cauchy problem (PC).

One can notice here that the above condition (CC) is fulfilled under the conditions of the Corollary 1. Indeed, we already know that the functions f_n are of the form $f_n(t)=\mathscr{C}(t)x_n$, where $x_n=f_n(0)$. Therefore, for any $K\geqq 0$ we have

$$n_K(x_n-x_m)=\sup_{|t|\leqq K}\|\mathscr{C}(t)x_n-\mathscr{C}(t)x_m\|=\sup_{|t|\leqq K}\|f_n(t)-f_m(t)\|$$

$$\leqq \sup_{|t|\leqq K}\|f_n(t)-g(t)\|+\sup_{|t|\leqq K}\|f_m(t)-g(t)\|\to 0, \quad n,m\to\infty,$$

hence the sequence $\{x_n\}$ is a Cauchy sequence in the $\mathscr{C}$-topology. The $\mathscr{D}(\mathscr{C})$ being $\mathscr{C}$-complete provides then a vector x that is the limit of the sequence $\{x_n\}$ in the $\mathscr{C}$-topology. Thus

$$\mathscr{C}(t)x=\lim\mathscr{C}(t)x_n=\lim f_n(t)=g(t) \quad \text{for any} \quad t\in\mathrm{R}.$$

But if $t=0$ then $x=\mathscr{C}(0)x=\lim f_n(0)=0$, and it follows that $g(t)=\lim f_n(t)=\mathscr{C}(t)x=0$, and the condition (CC) is fulfilled.

We will use the condition (CC) to construct a closed cosine function from the set of the solutions of the Cauchy problem.

From now on $\mathfrak{B}$ will denote a fixed linear operator in a Banach space X, with its domain $D(\mathfrak{B})=\mathfrak{D}$ being dense in X.

A function $f:\mathrm{R}\to X$ we will call the solution of the Cauchy problem

(PC) $$f''=\mathfrak{B}f,\quad f(0)=x,\quad f'(0)=y,$$

if f satisfies the initial conditions, is twice continuously differentiable, has values in $\mathfrak{D}$ and $f''(t)=\mathfrak{B}f(t)$, $t\in\mathrm{R}$.

Definition 2. The Cauchy problem $f''=\mathfrak{B}$ will be called closable posed, if firstly: for any x, $y\in\mathfrak{D}$ there exists a solution of $f''=\mathfrak{B}f$ that satisfies the initial condition $f(0)=x$, $f'(0)=y$, and secondly: the closability condition is fulfilled, that is:

For any sequence $\{f_n\}$ of the solutions of $f''=\mathfrak{B}f$ such that $f_n(0)\to 0$, $f_n'(0)=0$, if there exists a function $g(t)$ with the property $\forall K\geqq 0 \lim_{n\to\infty}\sup_{|t|\leqq K}\|f_n(t)-g(t)\|=0$, then it has to vanish, i.e. $g\equiv 0$.

The closability condition implies the uniqueness of the solutions. Indeed, it is enough for two solutions f and $\tilde{f}$ to take the functions f_n, g as $f_n=f-\tilde{f}$, $g=f-\tilde{f}$.

Denote then, for x, $y\in\mathfrak{D}$, by $f_{x,y}$ the solution of (PC) with the initial condition $f_{x,y}(0)=x$, $f'_{x,y}(0)=y$.

Definition 3. Let $f''=\mathfrak{B}f$ be a closable posed Cauchy problem. The cosine function generated by this problem is a cosine function $\mathscr{C}=\{\mathscr{C}(t): t\in \mathsf{R}\}$ defined as follows:

The common domain of the operators $\mathscr{C}(t)$, $t\in\mathsf{R}$, is the set $\mathscr{D}$ of all such $x\in X$, for which there exist a sequence $\{x_n\}\subset\mathfrak{D}$ and a function $g_x:\mathsf{R}\to X$ with the property:

$$\lim x_n = x \quad \text{and} \quad \forall K\geqq 0 \lim_{n\to\infty} \sup_{|t|\leqq K} \|f_{x_n,0}(t)-g_x(t)\|=0.$$

(From Definition 2 it follows that the function g_x is uniquely determined.) Using the function g_x we define the values of the operators $\mathscr{C}(t)$ by the formula $\mathscr{C}(t)x=g_x(t)$.

The term cosine function used in the above definition is fully justified, as the following theorem shows.

Theorem 4. *Let $\mathscr{C}=\{\mathscr{C}(t): t\in\mathsf{R}\}$ be the family of operators generated by a closable posed Cauchy problem $f''=\mathfrak{B}f$, as described in Definition 3. Then $\mathscr{C}$ is a closed cosine function with the domain $\mathscr{D}(\mathscr{C})$ equal to $\mathscr{D}$.*

If now we denote by $\mathfrak{C}$ its second generator, then the inclusion $\mathfrak{B}\subset\mathfrak{C}$ is true. This inclusion becomes an equality if, in addition, we assume that the operator $\mathfrak{B}$ is closed in the original Banach space X.

Proof. We start from showing $\mathscr{D}$ to be complete in the $\mathscr{C}$-topology. Take then a sequence $\{x_n\}\subset\mathscr{D}$ that is a Cauchy sequence in the $\mathscr{C}$-topology,

$$n_K(x_n-x_m)=\sup_{|t|\leqq K}\|\mathscr{C}(t)x_n-\mathscr{C}(t)x_m\|\to 0.$$

For any $t\in\mathsf{R}$ $\{\mathscr{C}(t)x_n\}$ is then a Cauchy sequence in the norm topology. Put $g(t)=\lim\mathscr{C}(t)x_n$.

Using the definition of $\mathscr{D}$, we can find a sequence $\{y_n\}\subset\mathfrak{D}$, such that

$$\|x_n-y_n\|\leqq\frac{1}{n} \text{ and } \sup_{|t|\leqq n}\|f_{y_n,0}(t)-\mathscr{C}(t)x_n\|\leqq\frac{1}{n}.$$

Define $x=g(0)=\lim x_n$ to obtain $x=\lim y_n$. For any $K\geqq 0$ and $n\geqq K$ it holds

$$\sup_{|t|\leqq K}\|f_{y_n,0}(t)-g(t)\|\leqq\sup_{|t|\leqq n}\|f_{y_n,0}(t)-\mathscr{C}(t)x_n\|+\sup_{|t|\leqq K}\|\mathscr{C}(t)x_n-g(t)\|.$$

As both terms on the right hand side tend to zero (the first being majorized by $\frac{1}{n}$ and the second by $\limsup_{m\to\infty} n_K(x_n-x_m)$), the left side tends to zero with $n\to\infty$ as well. By the definition of $\mathscr{D}$ we obtain $x\in\mathscr{D}$ and $C(t)x=g(t)=\lim\mathscr{C}(t)x_n$. Letting m tend to infinity in the Cauchy condition we see that $\lim n_K(x_n-x)=0$, and therefore the sequence $\{x_n\}$ has in $\mathscr{D}$ a $\mathscr{C}$-topology limit, still equal to x.

If now $x\in\mathscr{D}$ and $\mathfrak{D}\ni x_n\to x$ is the sequence as in the definition of $\mathscr{D}$, then the functions $f_{x_n,0}(t)$ are converging almost uniformly, with respect to t, to the function $\mathsf{R}\ni t\to\mathscr{C}(t)x$. Therefore this function is continuous, as an almost uniform limit of the continuous functions. It follows now that any vector from $\mathscr{D}$ satisfies the continuity condition that appears in the definition of the domain of a cosine function.

As $\mathscr{C}(0)x = \lim f_{x_n,0}(0) = \lim x_x = x$, then in order to show that $\mathscr{C}$ is a cosine function with the domain equal to $\mathscr{D}$ we must check, for any $x \in \mathscr{D}$, the d'Alembert equation. We start with checking it for the vectors from $\mathfrak{D}$.

For any fixed $x \in \mathfrak{D}$ and $s \in \mathsf{R}$ let us consider the function

$$f(t) = f_{x,0}(s+t) + f_{x,0}(s-t).$$

We get

$$f''(t) = f''_{x,0}(s+t) + f''_{x,0}(s-t) = \mathfrak{B} f_{x,0}(s+t) + \mathfrak{B} f_{x,0}(s-t) = \mathfrak{B} f(t),$$

hence f is a solution of (PC) with the initial values

$$f(0) = 2f_{x,0}(s), \quad f'(0) = 0.$$

For $x \in \mathfrak{D}$ we have $\mathscr{C}(t)x = f_{x,0}(t)$, and therefore

$$2\mathscr{C}(t)\mathscr{C}(s)x = \mathscr{C}(t)[2f_{x,0}(s)] = f(t) = f_{x,0}(s+t) + f_{x,0}(s-t) = \mathscr{C}(t+s)x + \mathscr{C}(t-s)x.$$

The d'Alembert equation is, for $x \in \mathfrak{D}$, fulfilled.

Now let $x \in \mathscr{D}$ and $\{x_n\} \subset \mathfrak{D}$ be the sequence associated with x as in Definition 3, that is $x_n \to x$ and $f_{x_n,0}(t) \to \mathscr{C}(t)x$ almost uniformly. Fix $s \in \mathsf{R}$ and take $y_n = 2\mathscr{C}(s)x_n = 2f_{x_n,0}(s) \in \mathfrak{D}$. Obviously $y_n = 2\mathscr{C}(s)x_n \to 2\mathscr{C}(s)x$. Let $g(t) = \mathscr{C}(t+s)x + \mathscr{C}(t-s)x$ and take any $K \geqq 0$. Then one has

$$\begin{aligned}\sup_{|t| \leqq K} \|f_{y_n,0}(t) - g(t)\| &= \sup_{|t| \leqq K} \|\mathscr{C}(t)[2\mathscr{C}(s)x_n] - \mathscr{C}(t+s)x - \mathscr{C}(t-s)x\| \\ &= \sup_{|t| \leqq K} \|\mathscr{C}(t+s)x_n + \mathscr{C}(t-s)x_n - \mathscr{C}(t+s)x - \mathscr{C}(t-s)x\| \\ &\leqq 2 \sup_{|t| \leqq K+|s|} \|\mathscr{C}(t)x_n - \mathscr{C}(t)x\| = 2 \sup_{|t| \leqq K+|s|} \|f_{x_n,0}(t) - \mathscr{C}(t)x\| \to 0,\end{aligned}$$

as $n \to \infty$. It follows that the conditions from the definition of $\mathscr{D}$ are satisfied and in this way we obtain $2\mathscr{C}(s)x \in \mathscr{D}$ and $\mathscr{C}(t)[2\mathscr{C}(s)x] = \mathscr{C}(t+s)x + \mathscr{C}(t-s)x$. That means that the d'Alembert equation is valid for x. In this way we have ended the proof of the first part of the theorem, $\mathscr{C}$ is shown to be a closed cosine function.

It remains to prove the part about the generator $\mathfrak{C}$. Take then $x \in \mathfrak{D}$, fix $K \geqq 0$, $t \in \mathsf{R}$, $|t| \leqq K$, and consider

$$\| \frac{1}{h^2} [f_{x,0}(t+h) + f_{x,0}(t-h) - 2f_{x,0}(t)] - f''_{x,0}(t)\|.$$

The function $f''_{x,0}(t)$ is continuous, hence uniformly continuous on compact subintervals of the real line. It follows that the above term tends to zero with $h \to 0$, uniformly respectively t, $|t| \leqq K$. In other words for any $K \geqq 0$, if $h \to 0$, then

$$\sup_{|t| \leqq K} \| \frac{1}{h^2} [f_{x,0}(t+h) + f_{x,0}(t-h) - 2f_{x,0}(t)] - f''_{x,0}(t)\| \to 0.$$

But $f_{x,0}(t+h) + f_{x,0}(t-x) = \mathscr{C}(t+h)x + \mathscr{C}(t-h)x = 2\mathscr{C}(t)\mathscr{C}(h)x$ hence the above can be reformulated as

$$\sin_{|t| \leqq K} \|\mathscr{C}(t) \left[\frac{2}{h^2} [\mathscr{C}(h)x - x] \right] - f''_{x,0}(t)\| \to 0 \quad \text{when} \quad h \to 0.$$

We have proved that $\mathcal{D}$ is complete in the $\mathcal{C}$-topology, so it follows that there exists a $\mathcal{C}$-topology limit $\mathcal{C}\text{-}\lim_{h\to 0} \frac{2}{h^2}[\mathcal{C}(h)x-x]$. In this way we have proved that $x\in\mathcal{D}(\mathfrak{C})$, and thus the inclusion $\mathfrak{D}=D(\mathfrak{B})\subset D(\mathfrak{C})$ holds. It is enough to put $t=0$ in the previous limit to obtain $f''_{x,0}(0)=\mathcal{C}-\lim_{h\to 0}\frac{2}{h^2}[\mathcal{C}(h)x-x]=\mathfrak{C}x$. But $f''_{x,0}(0)=\mathfrak{B}f_{x,0}(0)=\mathfrak{B}x$ and the required inclusion $\mathfrak{B}\subset\mathfrak{C}$ now follows.

We need the inverse inclusion $\mathfrak{C}\subset\mathfrak{B}$, in the case when we additionally know about $\mathfrak{B}$ that it is closed. Of course it is sufficient to show the inclusion of domains, $\mathfrak{D}(\mathfrak{C})\subset\mathfrak{D}$.

The definition of $\mathcal{D}$ assures that the set $\mathfrak{D}$ is dense in $\mathcal{D}$ in the $\mathcal{C}$-topology. Fix then $x\in D(\mathfrak{C})$ and let $\{x_n\}$ be a sequence of points from $\mathfrak{D}$ converging to x in the $\mathcal{C}$-topology.

Before we use the sequence $\{x_n\}$ defined above, we must prove certain equality for the operator $\mathfrak{B}$. Denote by $\mathcal{I}(t)y=\int_0^t\mathcal{C}(S)y\,ds$, $y\in\mathfrak{D}$, the corresponding sine function. As $f''_{x,0}(s)=\mathfrak{B}\mathcal{C}(s)y$ is, being continuous, Riemann integrable on the interval $[0,t]$, and $\mathfrak{B}$ is a closed operator, then

$$\mathcal{I}(t)y\in D(\mathfrak{B})=\mathfrak{D} \text{ and } \mathfrak{B}\mathcal{I}(t)y=\int_0^t\mathfrak{B}\mathcal{C}(s)y\,ds=\int_0^t\mathcal{C}(s)\mathfrak{B}y\,ds=I(t)\mathfrak{B}y.$$

(Here we use the inclusion $\mathfrak{B}\subset\mathfrak{C}$, and the commutativity of $\mathfrak{C}$ with the operators $\mathcal{C}(t)$, $t\in\mathrm{R}$.) Consider now, for a fixed $s\in\mathrm{R}$ and $y\in\mathfrak{D}$, a function f defined by $f(t)=\mathcal{C}(s+t)y=f_{y,0}(s+t)$. We have $f''(t)=\mathfrak{B}f(t)$, as $f_{y,0}$ is a solution of (PC). What is more, f is also $\mathcal{C}$-differentiable and fulfills the equation $f''=\mathfrak{C}f$ (see Corollary 1), with the initial conditions

$$f(0)=\mathcal{C}(s)y,\quad f'(0)=\mathcal{C}'(s)y=\mathcal{I}(s)\mathfrak{C}y=\mathfrak{I}(s)\mathfrak{B}y=\mathfrak{B}\mathcal{I}(s)y.$$

It is true that we cannot be quite certain if $f'(0)\in\mathfrak{D}(\mathfrak{C})$, but it suffices to reconsider the proof of the uniqueness in the Corollary 1 and conclude that

$$f(t)=\mathcal{C}(t)f(0)+\mathcal{I}(t)f'(0).$$

In this way we obtain

$$\mathcal{C}(s+t)y=\mathcal{C}(t)\mathcal{C}(s)y+\mathcal{I}(t)\mathfrak{B}\mathcal{I}(s)y,$$

in other words the equality

$$\mathfrak{B}\mathfrak{I}(t)\mathcal{I}(s)y=\mathcal{C}(s+t)y-\mathcal{C}(t)\mathcal{C}(s)y,\quad y\in\mathfrak{D},\ s,t\in\mathrm{R}.$$

Returning now to our initial sequence $\{x_n\}$ and using the equality we have shown with $t=s$ and $y=x_n$, we get

$$\mathfrak{B}\mathcal{I}(t)^2x_n=\mathcal{C}(2t)x_n-\mathcal{C}(t)^2x_n\to\mathcal{C}(2t)x-\mathcal{C}(t)^2x$$

(the operators $\mathcal{C}(t)$ are continuous in the $\mathcal{C}$-topology). In the same way we can get $\mathcal{I}(t)^2x_n\to\mathcal{I}(t)^2x$. As the operator $\mathfrak{B}$ is closed it follows that $\mathcal{I}(t)^2x\in D(\mathfrak{B})$ and

$$\mathfrak{B}\mathcal{I}(t)^2x=\mathcal{C}(2t)x-\mathcal{C}(t)^2x.$$

Therefore for $h^{-2}\mathcal{I}(h)^2x$ we have $h^{-2}\mathcal{I}(h)^2x\in\mathfrak{D}$ and

$$\mathfrak{B}[h^{-2}\mathcal{I}(h)^2x]=2(2h)^{-2}[2\mathcal{C}(2h)x-2\mathcal{C}(h)^2x]=2(2h)^{-2}[\mathcal{C}(2h)x-x]\to\mathfrak{C}x,$$

(we must use the equality $2\mathscr{C}(h)^2x=\mathscr{C}(2h)x+\mathscr{C}(0)x$ and $x\in D(\mathfrak{C})$). The next step is to show that $h^{-2}\mathscr{I}(h)^2x\to x$, as $h\to 0$. But indeed,

$$2\mathscr{I}(h)^2x-2h^2x=2\int_0^h\mathscr{C}(t)\int_0^h\mathscr{C}(s)x\,ds\,dt-2\int_0^h\int_0^h x\,ds\,dt$$

$$=\int_0^h\int_0^h[2\mathscr{C}(t)\mathscr{C}(s)x-2x]\,ds\,dt=\int_0^h\int_0^h[\mathscr{C}(t+s)x+\mathscr{C}(t-s)x-2x]\,ds\,dt$$

hence

$$\|h^{-2}\mathscr{I}(h)^2x-x\|\leqq\sup_{|u|\leqq 2h}\|\mathscr{C}(u)x-x\|\to 0,\ \text{as}\ h\to 0.$$

Now we know that $h^{-2}\mathscr{I}(h)^2x\to x$, $h\to 0$ and that the values of $\mathfrak{B}$ on these vectors converge, as $h\to 0$, to $\mathfrak{C}x$. Once more we can refer to the fact that $\mathfrak{B}$ is closed and obtain $x\in D(\mathfrak{B})=\mathfrak{D}$, and $\mathfrak{B}x=\mathfrak{C}x$. The equality $\mathfrak{B}=\mathfrak{C}$ is now wholly proved.

Example. To understand better the meaning of the theorem consider the following simple example. As the Banach space X we take $C_0(\mathsf{R}_+)$, the set of all continuous functions from $[0,\infty)$ to C that vanish at infinity, with the supremum norm. For the dense subspace $\mathfrak{D}$ on which we define the operator $\mathfrak{B}$ take the set of all functions from X that have compact support, that is that vanish outside certain finite subinterval. And finally the operator $\mathfrak{B}$ we define by $(\mathfrak{B}x)(u)=u\cdot x(u)$, $u\in\mathsf{R}_+$, $x\in\mathfrak{D}$.

Let consider the equation $f''=\mathfrak{B}f$. Whenever f is a solution of this equation with the initial condition $f0)=x$, $f'(0)=y$, then, as $\delta_u: X\ni x\to x(u)$ is a continuous functional on X, we obtain $[\delta_u f(t)]''=\delta_u[f''(t)]=\delta_u[\mathfrak{B}f(t)]=u\cdot f(t)(u)=u\cdot\delta_u f(t)$. Therefore the function $g(t)=\delta_u f(t)=f(t)(u)$ is a solution of the scalar equation $g''=u\cdot g$ with the initial values $g(0)=f(0)(u)=x(u)$, $g'(0)=f'(0)(u)=y(u)$. That means that g is of the form

$$g(t)=f(t)(u)=x(u)\operatorname{coh}(t\sqrt{u})+y(u)\operatorname{sh}(t\sqrt{u}).$$

For any x, y from $\mathfrak{D}$ that is x, y vanishing outside a finite interval, let say $[0,K]$, the t-derivative will exist uniformly respectively $u\in[0,K]$. Hence for $x,y\in\mathfrak{D}$ and a function $f_{x,y}(t)$ given by

$$f_{x,y}(t)(u)=x(u)\operatorname{coh}(t\sqrt{u})+y(u)\operatorname{sh}(t\sqrt{u})$$

there exists its derivative in the topology of X and, what is more, fulfills the equation $f''_{x,y}=\mathfrak{B}_{x,y}$.

In order to have checked all the conditions from Definition 2 we have to prove the closability condition. Take then a sequence $\{f_n\}$ of solutions of (PC) and a function g such that

$$f_n(0)\to 0,\ f'(0)=0,\ \text{and}\ \forall K\geqq 0\ \sup_{|t|\leqq K}\|f_n(t)-g(t)\|\to 0.$$

We want to prove that $g\equiv 0$. But we already know that

$$f_n(t)(u)=f_n(0)(u)\cdot\operatorname{coh}(t\sqrt{u})\to 0,\ n\to\infty.$$

On the other hand $f_n(t)(u)-g(t)(u)\to 0$, for the reason that evaluating in a given point is a continuous functional in X, and $\|f_n(t)-g(t)\|$ tends to zero. In that way we obtain that $g(t)(u)=0$, u, $t\in\mathsf{R}$, that is indeed $g\equiv 0$.

We have shown that with X, $\mathfrak{D}$, $\mathfrak{B}$ defined as above the Cauchy problem $f''=\mathfrak{B}f$ is closable posed. We are in the position to use Theorem 3 to generate a closed cosine function $\mathscr{C}$. From what we have said it follows that for any $x\in\mathfrak{D}$ we have $[\mathscr{C}(t)x](u)=x(u)\operatorname{coh}(t\sqrt{u})$. The domain $\mathscr{D}(\mathscr{C})$ of the cosine function $\mathscr{C}$ is equal to the set of all functions vanishing at the infinity faster than any function $u\to\operatorname{coh}(t\sqrt{u})$, $t\in\mathsf{R}$.

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