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Quasidiagonal Operators in Tridiagonal Algebras

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1. Introduction

A given subset $\mathcal{R}$ of the algebra $\mathcal{L}(\mathcal{H})$ of all (bounded linear) operators acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$, the question —

What kinds of operators do we have in $\mathcal{R}$? —

admits several possible answers. The most obvious one is the description of the unitary orbit of $\mathcal{R}$,

$$\mathcal{U}(\mathcal{R})=\{UTU^*: T\in\mathcal{R},\ U \text{ is unitary}\}.$$

But this can be too restrictive. A slightly weaker version is a description of the approximate unitary orbit, $\mathcal{U}(\mathcal{R})^-$ (= the norm-closure of $\mathcal{U}(\mathcal{R})$ in $\mathcal{L}(\mathcal{H})$).

Another possible answer is the description of $\mathcal{U}(\mathcal{R})$ modulo compact operators, that is

$$\hat{\mathcal{R}}=\mathcal{U}(\mathcal{R})+\mathcal{K}(\mathcal{H}),$$

where $\mathcal{K}(\mathcal{H}))$ denotes the ideal of all compact operators. Finally, in a combination of the last two, we can also consider the set

$$\hat{\mathcal{R}}_0=\{A\in\mathcal{L}(\mathcal{H}): \text{given } \varepsilon>0 \text{ there exists } K_\varepsilon\in\mathcal{K}(\mathcal{H}),$$
$$\text{with } \|K_\varepsilon\|<\varepsilon, \text{ such that } A-K_\varepsilon\in\mathcal{U}(\mathcal{R})\}$$

(Clearly, $\hat{\mathcal{R}}_0\subset\hat{\mathcal{R}}\cap\mathcal{U}(\mathcal{R})^-$.)

In [13] and [14], D. A. Herrero analyzed these sets for the case when $\mathcal{R}$ is the nest algebra alg $\mathcal{N}$ associated with a nest of subspaces $\mathcal{N}$ of $\mathcal{H}$. (The reader is referred to [3] for definition and properties of nests and nest algebras). In this particular case, $\hat{\mathcal{N}}$ and $\hat{\mathcal{N}}_0$ are always closed subsets of $\mathcal{L}(\mathcal{H})$, and therefore $\hat{\mathcal{N}}_0=\mathcal{U}(\text{alg}\,\mathcal{N})^-$; moreover, in "most cases" $\hat{\mathcal{N}}=\hat{\mathcal{N}}_0=\mathcal{L}(\mathcal{H})$. In [18], D. A. Herrero and D. R. Larson obtained spectral characterizations of $\mathcal{R}^\infty(\mathcal{N})^\wedge$ and

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$\mathscr{R}^{\infty}(\mathscr{N})\hat{}_0$, where $\mathscr{R}^{\infty}(\mathscr{N})$ is the Larson ideal of a nest algebra (see [20]). Once again, $\mathscr{R}^{\infty}(\mathscr{N})\hat{}$ and $\mathscr{R}^{\infty}(\mathscr{N})\hat{}_0$ are closed subsets of $\mathscr{L}(\mathscr{H})$ and so $\mathscr{R}^{\infty}(\mathscr{N})\hat{}_0=\mathscr{U}(\mathscr{R}^{\infty}(\mathscr{N}))^-$.

Z. Y. Wang extended Herrero's results to the case when $\operatorname{alg}\mathscr{N}$ is replaced by $\operatorname{alg}(\otimes_{i=1}^{k}\mathscr{N}_i)$ (the algebra associated with the tensor product of finitely many nests $\mathscr{N}_1,\mathscr{N}_2,\dots,\mathscr{N}_k$ [25]). In this case $(\otimes_{i=1}^{k}\mathscr{N}_i)\hat{}$ is always closed (and coincides with $\mathscr{L}(\mathscr{H})$ in "most cases"). Indeed, except for a few other peculiar cases, it is also known that $(\otimes_{i=1}^{k}\mathscr{N}_i)\hat{}_0=\mathscr{U}(\operatorname{alg}\otimes_{i=1}^{k}\mathscr{N}_i)^-$. However, the results of J. Froelich [8], [9] seem to indicate that, in general, $\hat{\mathscr{R}}_0$ is a proper subset of $\mathscr{U}(\mathscr{R})^-$ (in particular, for the case when $\mathscr{R}$ is the algebra associated with a denumerable tensor product of continuous nests).

Finally, observe that D. Voiculescu's theorem [22] can be used to deduce, for instance, that if $\mathscr{H}=\Sigma\otimes_{k=1}^{\infty}\mathscr{H}_k(\mathscr{H}_k\simeq\mathscr{H}$ for all $k=1,2,\dots)$ and $\mathscr{R}$ is the algebra of all operators that leave invariant $\Sigma\otimes_{k\in\Gamma}\mathscr{H}_k$ for each subset Γ of natural numbers, then

$$\hat{\mathscr{R}}=\hat{\mathscr{R}}_0=\mathscr{U}(\mathscr{R})^-=\mathscr{L}(\mathscr{H}).$$

In this article we provide some partial answers to the same problems for the case when $\mathscr{R}=\mathscr{T}(\{m_k\})$ is one of the tridiagonal algebras studied by F. Gilfeather in [10]. Here $\mathscr{H}=\Sigma\otimes_{k=1}^{\infty}\mathscr{M}_k$, where $1\leqq M_k=\dim\mathscr{M}_k\leqq\infty$, and $\mathscr{T}(\{m_k\})$ is the algebra of all "staircase operators".

$$T=\left|\begin{array}{cccccccc}
A_1 & B_1 & & & & & & \\
 & C_1 & & & & & & \\
 & D_1 & A_2 & B_2 & & & 0 & \\
 & & & C_2 & & & & \\
 & & & D_2 & A_3 & B_3 & & \\
 & & & & & \ddots & & \\
 & & & & D_{k-1} & A_k & B_k & \\
 & 0 & & & & & C_k & \\
 & & & & & & D_k & A_{k+1} \\
 & & & & & & & \ddots
\end{array}\right|\begin{array}{c}
\mathscr{M}_1\\ \mathscr{M}_2\\ \mathscr{M}_3\\ \mathscr{M}_4\\ \mathscr{M}_5\\ \vdots\\ \mathscr{M}_{2k-1}\\ \mathscr{M}_{2k}\\ \mathscr{M}_{2k+1}\\ \vdots
\end{array}$$

(Equivalently, T leaves invariant each of the subspaces $\mathscr{M}_1$, $\mathscr{M}_1\oplus\mathscr{M}_2\oplus\mathscr{M}_3,\dots,$ $\mathscr{M}_1\oplus\mathscr{M}_2\oplus\dots\oplus\mathscr{M}_{2k-1}$, and $(\mathscr{M}_1\oplus\mathscr{M}_2)^{\perp}$, $(\mathscr{M}_1\oplus\mathscr{M}_2\oplus\mathscr{M}_3\oplus\mathscr{M}_4)^{\perp},\dots,$ $(\mathscr{M}_1\oplus\mathscr{M}_2\oplus\dots\oplus\mathscr{M}_{2k})^{\perp},\dots;$ $k=1,2,\dots$. Observe that $\mathscr{T}(\{m_k\})$ is the intersection of two nest algebras.)

These algebras have a long tradition. C. Foiaş, C. M. Pearcy and D. Voiculescu [6], [7] proved that if $A\in\mathscr{L}(\mathscr{H})$ is biquasitriangular and $\varepsilon>0$, then there exists $K_\varepsilon\in\mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\|<\varepsilon$, such that $T=A-K_\varepsilon\in\mathscr{T}(\{m_k\})$ for some sequence $\{m_k\}$ of natural numbers. This staircase structure was later exploited by D. A. Herrero [16] and by K. R. Davidson and D. A. Herrero [4] in rather different contexts (see also the survey articles [15], [17]). After the above mentioned references, one might suspect that if $\{m_k\}$ is a sequence of natural numbers tending to infinity fast enough, then $\mathcal{T}(\{m_k\})_0^\wedge$ and $\mathcal{U}(\mathcal{T}(\{m_k\}))^-$ should contain every biquasitriangular operator. But this is definitely FALSE.

Let

$$(\mathrm{BD})_{\mathrm{nor}} = \{B = \Sigma \oplus_{n=1}^{\infty} B_n : B_n \text{ acts on a space of finite dimension } d_n, \text{ and } \sup_n d_n < \infty\}$$

(= the set of all "n-normal" block-diagonal operators; this set plays a very interesting role in quasidiagonal approximation [5]).

It will be shown that if $(\mathrm{BD})_{\mathrm{nor}} \subset \mathcal{T}(\{m_k\})$, then

$$(*) \qquad m_k + m_{k+1} + m_{k+2} = \infty \text{ for all } k \geqq 0.$$

(Here $m_0 = 0$).

On the other hand, if $\{m_k\}$ satisfies (*), then

$$\mathcal{U}(\mathcal{T}(\{m_k\}))^- = \mathcal{T}(\{m_k\})_0^\wedge = \mathcal{L}(\mathcal{H}).$$

For $\mathcal{T}(\{m_k\})^\wedge$ we only have weaker results: If $(\mathrm{BD})_{\mathrm{nor}} \subset \mathcal{T}(\{m_k\})^\wedge$, then

$$\lim_{k\to\infty} (m_k + m_{k+1} + m_{k+2}) = \infty.$$

If $\lim_{k\to\infty} m_k = \infty$, then $(\mathrm{BD})_{\mathrm{nor}} \subset \mathcal{T}(\{m_k\})^\wedge$.

However, it is known that $(\mathrm{BD})_{\mathrm{nor}}$ is not dense in the class (QD) of all quasidiagonal operators [21], [23], [24]. Indeed, $(\mathrm{QD})_{\mathrm{nor}} := [(\mathrm{BD})_{\mathrm{nor}}]^-$ is a nowhere dense subset of (QD) [5] (and (QD) is well-known to be a nowhere dense subset of the class (BQT) of all biquasitriangular operators [11]).

Open problem. Does $(\mathrm{QD}) \subset \mathcal{T}(\{m_k\})^\wedge$ imply that $m_{k-1} + m_k + m_{k+1} = \infty$ for all k large enough?

2. When is $(\mathrm{BD})_{\mathrm{nor}} \subset \mathcal{U}(\mathcal{T}(\{m_k\}))^-$?

Theorem 2.1. *The following are equivalent*

(i) $$\mathcal{U}(\mathcal{T}(\{m_k\}))^- \supset (\mathrm{BD})_{\mathrm{nor}};$$

(ii) $$m_k + m_{k+1} + m_{k+2} = \infty \text{ for all } k \geqq 0 \ (m_0 = 0);$$

(iii) $$\mathcal{T}(\{m_k\})_0^\wedge = \mathcal{L}(\mathcal{H}).$$

Let

$$A_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ & 2 & 1 & \dots & 1 \\ & & 3 & \dots & 1 \\ & & & \ddots & \vdots \\ & 0 & & & \vdots \\ & & & & n \end{vmatrix}$$

with respect to the orthogonal direct sum decomposition $\mathscr{H}=\Sigma\oplus_{k=1}^{n}\mathscr{H}_k(\mathscr{H}_k\simeq\mathscr{H}$, $k=1,2,\dots,n)$. It is obvious that $A_n\in(\mathrm{BD})_{\mathrm{nor}}$, A_n is unitarily equivalent to the direct sum of infinitely many copies of the $n\times n$ complex matrix (with the same entries with respect to the decomposition $\mathbf{C}^n=\mathbf{C}\oplus\mathbf{C}\oplus\dots\oplus\mathbf{C}$), and the spectrum of A_n is equal to $\sigma(A_n)=\{1,2,\dots,n\}$.

Let $\mathscr{G}_k=\ker(A_n-k)$ and $\mathscr{G}_k^*=\ker(A_n-k)^*$ $(k=1,2,\dots,n)$. Clearly, $\mathscr{G}_k=\mathscr{H}(A_n;k)$, where $\mathscr{H}(T;\sigma)$ denotes the Riesz spectral invariant subspace of the operator T associated with the clopen subset σ of $\sigma(T)$ (if $\sigma=\{\lambda\}$ is a singleton, we simply write $\mathscr{H}(T;\lambda)$).

By Gauss-Jordan elimination [1], we obtain

$$\mathscr{G}_k=\{x\oplus x\oplus\dots\oplus x\,(k\text{-copies})\oplus 0\oplus 0\oplus\dots\oplus 0:x\in\mathscr{H}\}$$

$(k=1,2,\dots,n)$,

$$\mathscr{G}_k^*=\{0\oplus\dots\oplus 0\,(k-1\text{ copies})\oplus x\oplus(-x)\oplus 0\oplus\dots\oplus 0:x\in\mathscr{H}\}$$
$$(k=1,2,\dots,n-1),\text{ and } \mathscr{G}_n=\{0\oplus 0\oplus\dots\oplus 0\oplus x:x\in\mathscr{H}\}.$$

We have $\mathscr{H}=\Sigma\dot{+}_{k+1}^{n}\mathscr{G}_k=\Sigma\dot{+}_{k=1}^{n}\mathscr{G}_k^*$, where $\dot{+}$ denotes algebraic (but not necessarily orthogonal) direct sum.

Lemma 2.2. *Let $\mathscr{X}$ be the direct sum of r $(0<r<n)$ subspaces of the family $\{\mathscr{G}_1,\mathscr{G}_2,\dots,\mathscr{G}_n\}$, and let $\mathscr{X}^*$ be the direct sum of $n-r$ subspaces of the family $\{\mathscr{G}_1^*,\mathscr{G}_2^*,\dots,\mathscr{G}_n^*\}$. Then*

$$\mathscr{H}=\mathscr{X}\dot{+}\mathscr{X}^*,$$

that is, $\mathscr{X}\cap\mathscr{X}^*=\{0\}$, $\mathscr{H}=\mathscr{X}+\mathscr{X}^*$ and

$$\inf\{\|x-x^*\|:x\in\mathscr{X},\ x^*\in\mathscr{X}^*,\ \|x\|=\|x^*\|=y\}\geqq\delta$$

for some positive constant δ depending only n.

Proof. As mentioned above, $A_n=B_n^{(\infty)}$, where B_n is the operator defined by the obvious $n\times n$ matrix with respect to the canonical orthogonal basis of $\mathbf{C}^n$. Given $\mathscr{X}$ and $\mathscr{X}^*$ as above, let $\mathscr{Y}$ and $\mathscr{Y}^*$ be the intersection of $\mathscr{X}$ and, respectively, $\mathscr{X}^*$ with the "first copy" of $\mathbf{C}^n$.

It is straightforward to check that $\mathscr{Y}\cap\mathscr{Y}^*=\{0\}$. Since $\dim\mathscr{Y}+\dim\mathscr{Y}^*=r+(n-r)=n=\dim\mathbf{C}^n$, we infer that $\mathbf{C}^n=\mathscr{Y}\dot{+}\mathscr{Y}^*$, and therefore

$$\min\{\|y-y^*\|:y\in\mathscr{Y},\ y^*\in\mathscr{Y}^*,\ \|y\|=\|y^*\|=1\}=\delta(\mathscr{Y},\mathscr{Y}^*)$$

for some $\delta(\mathscr{Y},\mathscr{Y}^*)>0$.

Since there are only finitely many possible choices for $\mathscr{Y}$ and $\mathscr{Y}^*$, we see that

$$\delta:=\min_{\mathscr{Y},\mathscr{Y}^*}\delta(\mathscr{Y},\mathscr{Y}^*)>0.$$

Now the result follows by observing that $\mathscr{H}=\Sigma\oplus_{k=1}^{n}\mathscr{H}_k\simeq\mathscr{H}\otimes\mathbf{C}^n\simeq\mathbf{C}^n\oplus\mathbf{C}^n\oplus\dots$ ■

The following is the key result of this article.

Theorem 2.3. *Let $\mathscr{R}$ be a subset of $\mathscr{L}(\mathscr{H})$ such that $\mathscr{U}(\mathscr{R})^-\supset(\mathrm{BD})_{\mathrm{nor}}$. Let $\mathscr{M}$ and $\mathscr{N}$ be two subspaces of $\mathscr{H}$ such that $\mathscr{M}$ is invariant under $\mathscr{R}$ and $\mathscr{N}$ is invariant under $\mathscr{R}^*(=\{T^*:T\in\mathscr{R}\})$.*

(*i*) *If* $\mathcal{N} \supset \mathcal{M} \neq \{0\}$, *then* $\mathcal{N}$ *is infinite dimensional;*
(*ii*) *If* $\mathcal{M} \supset \mathcal{N} \neq \{0\}$, *then* $\mathcal{M}$ *is infinite dimensional.*

Proof. Let A_n be as above. For each ε, $0<\varepsilon\ll 1$, there exist R_n in $\mathcal{R}$ and U unitary such that $\|A_n - UR_nU^*\| < \varepsilon$.

By replacing, if necessary, $\mathcal{M}$ and $\mathcal{N}$ by $U\mathcal{M}$ and, respectively, $U\mathcal{M}$, we can directly assume that $U=1$, that is, $\|A_n - R_n\| < \varepsilon$. If ε is small enough, then $\sigma(R_n)$ is the disjoint union of n compact subsets $\sigma_1, \sigma_2, \ldots, \sigma_n$, with σ_k included in the interior of the circle γ_k of radius 1/4 centered at $k (k=1,2,\ldots,n)$.

Let $\mathcal{J}_k$ be the range of the Riesz idempotent

$$\frac{1}{2\pi i}\int_{\gamma_k}(\lambda - R_n)^{-1}\mathrm{d}\lambda,$$

and let $\mathcal{J}_k^*$ be the analogously defined subspace corresponding to R_n^*. If $P_{\mathcal{L}}$ denotes the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{L}$, we have

$$\|P_{\mathcal{L}_k} - P_{\mathcal{J}_k}\| = O(\varepsilon) \text{ and } \|P_{\mathcal{G}_k^*} - P_{\mathcal{J}_k^*}\| = O(\varepsilon)$$

$(k=1,2,\ldots,n)$.

Since $\mathcal{M} \in \operatorname{Lat} R_n$, $\mathcal{M}$ "splits" as $\mathcal{M} = \Sigma \dotplus_{k=1}^{n} \mathcal{M}_k$, where $\mathcal{M}_k = \mathcal{M} \cap \mathcal{J}_k$ $(k=1,2,\ldots,n$; see [2]). Let $\mathcal{Q}_k = [P_{\mathcal{G}_k}\mathcal{M}_k]^-$, and let $\mathcal{S}(\mathcal{L})$ denote the unit sphere of the subspace $\mathcal{L}$; then

$$\begin{aligned}\delta(\mathcal{M}_k, \mathcal{Q}_k) := \sup_{x\in S(\mathcal{M}_k)} \operatorname{dist}[x, \mathcal{Q}_k] &= \sup_{x\in S(\mathcal{M}_k)} \operatorname{dist}[x, \mathcal{G}_k] \\ &\leqq \sup_{x\in S(\mathcal{J}_k)} \operatorname{dist}[x, \mathcal{G}_k] = \delta(\mathcal{J}_k, \mathcal{G}_k) \\ &\leqq \|P_{\mathcal{J}_k} - P_{\mathcal{G}_k}\| = O(\varepsilon).\end{aligned}$$

Thus, $\dim \mathcal{M}_k \leqq \dim \mathcal{Q}_k$ provided ε is small [19]. On the other hand, it is obvious from the definition of $\mathcal{Q}_k$ that the dimension of this subspace cannot exceed that of $\mathcal{M}_k$; therefore

$$\dim \mathcal{M}_k = \dim \mathcal{Q}_k \quad (k=1,2,\ldots,n).$$

Define $\mathcal{Q} = \Sigma \dotplus_{k=1}^{n} \mathcal{Q}_k$. Clearly, $\dim \mathcal{Q} = \dim \mathcal{M}$.

Similarly, $\mathcal{N} = \Sigma \dotplus_{k=1}^{n} \mathcal{N}_k (\mathcal{N}_k = \mathcal{N} \cap \mathcal{J}_k^*)$ and, if we define $\mathcal{Q}_k^* = [P_{\mathcal{G}_k^*}\mathcal{N}_k]^-$ $(k=1,2,\ldots,n)$ and $\mathcal{Q}^* = \Sigma \dotplus_{k=1}^{n} \mathcal{Q}_k$, then $\dim \mathcal{Q}^* = \dim \mathcal{N}$.

Moreover, we actually have $\mathcal{Q}_k = P_{\mathcal{G}_k}\mathcal{M}_k$, $\mathcal{Q}_k^* = P_{\mathcal{G}_k^*}\mathcal{N}_k$, $\|P_{\mathcal{M}_k} - P_{\mathcal{Q}_k}\| = O(\varepsilon)$ and $\|P_{\mathcal{N}_k} - P_{\mathcal{Q}_k^*}\| = O(\varepsilon)$ $(k=1,2,\ldots,n$; see [19]).

(i) Assume that $\mathcal{N} \supset \mathcal{M} \neq \{0\}$; then $\mathcal{Q}_h \neq \{0\}$ for some h, $1 \leqq h \leqq n$. Pick $x \in \mathcal{S}(\mathcal{Q}_h)\,(\subset \mathcal{S}(\mathcal{G}_h))$ and $y \in \mathcal{M}_h$ such that $x = P_{\mathcal{G}_h}y$, and let $x' = \Sigma_{k=1}^{n} P_{\mathcal{G}_h^*}P_{\mathcal{J}_k^*}y\ (\in \mathcal{Q}^*)$.

Since $y = P_{\mathcal{J}_h}y = \Sigma_{k=1}^{n} P_{\mathcal{J}_k^*}y$, we have

$$\begin{aligned}\|x - x'\| &= \|P_{\mathcal{G}_h}y - \sum_{k=1}^{n} P_{\mathcal{G}_k^*}P_{\mathcal{J}_k^*}y\| \\ &\leqq \|(P_{\mathcal{G}_h} - P_{\mathcal{J}_h})y\| + \left\|\left(\sum_{k=1}^{n}[P_{\mathcal{J}_k^*} - P_{\mathcal{G}_k^*}]\right)P_{\mathcal{J}_k^*}y\right\| = O(\varepsilon)\|y\| = O(\varepsilon).\end{aligned}$$

Observe that $x \in \mathcal{G}_h$ and x' belongs to the subspace spanned by some subset of $\{\mathcal{G}_1^*, \mathcal{G}_2^*, \ldots, \mathcal{G}_n^*\}$. If ε is sufficiently small, it follows from Lemma 2.2 that x' does not belong to the direct sum of any proper subset of the $\mathcal{G}_k^*$'s, and therefore $P_{\mathcal{G}_h^*} x' \neq 0$ for all $k=1,2,\ldots,n$. Since $0 \neq P_{\mathcal{G}_h^*} x' \in \mathcal{Q}_k^*$, we conclude that

$$\dim \mathcal{N} = \dim \mathcal{Q}^* \geqq n.$$

Since n can be chosen arbitrarily large, $\mathcal{N}$ must be an infinite dimensional subspace.

(ii) follows by the same argument. ∎

Now we are in a position to prove the main result.

Proof of Theorem 2.1. (i)⇒(ii). This follows immediately from Theorem 2.4. Observe that if $R \in \mathcal{T}(\{m_k\})$ then for each $k \geqq 0$

$$\mathcal{M}_{2k+1} \in \operatorname{Lat} R, \ \mathcal{M}_{2k+1} \subset \mathcal{M}_{2k} \oplus \mathcal{M}_{2k+1} \oplus \mathcal{M}_{2k+2} \in \operatorname{Lat} R^*$$

and

$$\mathcal{M}_{2k} \in \operatorname{Lat} R^*, \ \mathcal{M}_{2k} \subset \mathcal{M}_{2k-1} \oplus \mathcal{M}_{2k} \oplus \mathcal{M}_{2k+1} \in \operatorname{Lat} R$$

($\mathcal{M}_0 = \{0\}$). By theorem 2.4,

$$m_{2k} + m_{2k+1} + m_{2k+2} = \infty \text{ and } m_{2k-1} + m_{2k} + m_{2k+1} = \infty.$$

Thus,

$$m_k + m_{k+1} + m_{k+2} = \infty \text{ for all } k = 0, 1, 2, \ldots$$

(ii)⇒(iii). Let $T \in \mathcal{L}(\mathcal{H})$ and let $\varepsilon > 0$ be given. By Voiculescu's theorem [22] there exists $K_0 \in \mathcal{K}(\mathcal{H})$, with $\|K_0\| < \varepsilon/2$, such that $T - K_0 \simeq T \oplus A^{(\infty)}$ for a suitable A in $\mathcal{L}(\mathcal{H})$.

Assume $\mathcal{M}_2, \mathcal{M}_5, \mathcal{M}_8, \mathcal{M}_{11}, \ldots$ are infinite dimensional; then we can write $\mathcal{M}_{3k-1} = \mathcal{R}_k \oplus \mathcal{S}_k$, where $\mathcal{R}_k$ and $\mathcal{S}_k$ are infinite dimensional ($k = 1, 2, \ldots$). We can find compact perturbations F_0 and F_1 such that $\max\{\|F_0\|, \|F_1\|] < \varepsilon/2$, and $T - F_0$ is unitarily equivalent to an operator of the form

$$\begin{pmatrix} A_1 & B_1' \\ 0 & C_1' \end{pmatrix}$$

(with respect to the decomposition $\mathcal{M}_1 \oplus \mathcal{R}_1$, where A_1 is a normal diagonal operator), and $A - F_1$ is unitarily equivalent to an operator of the form

$$\begin{pmatrix} C_1'' & 0 \\ D_1'' & A_2 \end{pmatrix}$$

(with respect to the decomposition $\mathcal{S}_1 \oplus \mathcal{M}_3$), where C_1'' is a normal diagonal operator (see [12, Chapter 3] for details). Thus $T \oplus A - F_0 \oplus F_1$ is unitarily equivalent to

$$\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & C_1 & 0 \\ 0 & D_1 & A_2 \end{pmatrix}$$

(with respect to the decomposition $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$), where $B_1 = (B_1' \, 0)$, $C_1 = C_1' \oplus C_1''$ and $D_1 = (0 \, D_1'')$. Define $B_2 = 0$.

Similarly, we can find compact perturbations F_2 and F_3 such that $\max[\|F_2\|, \|F_3\|]<\varepsilon/4$, $A-F_2$ is unitarily equivalent to an operator of the form

$$\begin{pmatrix} C_2 & 0 \\ D_2' & A_3' \end{pmatrix}$$

(with respect to the decomposition $\mathcal{M}_4 \oplus \mathcal{R}_2$, where C_2 is a normal diagonal operator), and $A-F_3$ is unitarily equivalent to an operator of the form

$$\begin{pmatrix} A_3'' & B_3'' \\ 0 & C_3 \end{pmatrix}$$

(with respect to the decomposition $\mathcal{S}_2 \oplus \mathcal{M}_6$, where C_3 is a normal diagonal operator). Thus, $A \oplus A-F_2 \oplus F_3$ is unitarily equivalent to

$$\begin{pmatrix} C_2 & 0 & 0 \\ D_2 & A_3 & B_3 \\ 0 & 0 & C_3 \end{pmatrix}$$

(with respect to the decomposition $\mathcal{M}_4 \oplus \mathcal{M}_5 \oplus \mathcal{M}_6$), where

$$D_2=\begin{pmatrix} D_2' \\ 0 \end{pmatrix}, \quad A_3=A_3' \oplus A_3'' \text{ and } B_3=\begin{pmatrix} 0 \\ B_3'' \end{pmatrix}.$$

Define $D_3=0$.

By an obvious inductive argument, we can find compact perturbations $F_4, F_5, \ldots$, so that

$$F=\Sigma \oplus_{k=0}^{\infty} F_k \in \mathcal{K}(\mathcal{H}), \quad \|F\|<\varepsilon/2,$$

and $T-(K_0+F)$ is unitarily equivalent to a staircase operator in $\mathcal{T}(\{m_k\})$.

Since K_0+F is compact and $\|K_0+F\|<\varepsilon$, this proves the result for the particular case when $\mathcal{M}_{3k-1}$ is infinite dimensonal for all $k \geqq 1$. The general case follows by exactly the same argument, just by interpolating $\{0\}$-subspaces in the direct sum decomposition $\mathcal{H}=\Sigma \oplus_{k=1}^{\infty} \mathcal{M}_k$. The details are left to the reader.

Since (iii)⇒(i) is a trivial implicaion, the proof of Theorem 2.1 is now complete. ■

Corollary 2.5. *If*

$$m_k+m_{k+1}+m_{k+2}=\infty$$

for all k sufficiently large, then

$$(\mathcal{T}(\{m_k\}))^{\wedge}=\mathcal{L}(\mathcal{H}).$$

3. The analysis of $\mathcal{T}(\{m_k\})^{\wedge}$

Proposition 3.1. *If* $(\mathrm{BD})_{\mathrm{nor}} \subset \mathcal{T}(\{m_k\})^{\wedge}$, *then*

$$m_k+m_{k+1}+m_{k+2} \to \infty \ (k \to \infty).$$

Proof. Let A_n be defined as usual and let K_n be a compact operator such that $R_n=A_n-K_n$ (or some unitarilly equivalent operator) belongs to $\mathcal{T}(\{m_k\})$.

Let P_h denote the orthogonal projection of $\mathscr{H}$ onto $\Sigma\oplus_{k=1}^{h}\mathscr{M}_k$. Given $\varepsilon>0$, we can find $h(\varepsilon)$ such that the norm of the compact operator

$$K_{n,\varepsilon}=K_n-P_{h(\varepsilon)}K_nP_{h(\varepsilon)}$$

does not exceed ε. Clearly, we can assume that $h(\varepsilon)$ is odd.

It is easily seen that

$$A_n-K_{n,\varepsilon}=\begin{bmatrix} A_0 & B_{h(\varepsilon)} & & & \\ & C_{h(\varepsilon)} & & & \\ & D_{h(\varepsilon)} & A_{h(\varepsilon)+1} & B_{h(\varepsilon)+1} & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \cdot \end{bmatrix}$$

with respect to the decomposition $\mathscr{H}=(\Sigma\oplus_{k=1}^{h(\varepsilon)}\mathscr{M}_k)\oplus\mathscr{M}_{h(\varepsilon)+1}\oplus\mathscr{M}_{h(\varepsilon)+2}\oplus\cdots$.

By using the same arguments as in the proofs of Theorems 2.3 and 2.1 (i)$\Rightarrow$(ii), we deduce that

$$m_k+m_{k+1}+m_{k+2}\geqq n$$

for all $k>h(\varepsilon)$. (The details are left to the reader.)

Therefore,

$$m_k+m_{k+1}+m_{k+2}\to\infty\quad(k\to\infty).\ \blacksquare$$

Proposition 3.2. *If* $m_k\to\infty$ $(k\to\infty)$, *then* $(\mathrm{BD})_{\mathrm{nor}}\subset\mathscr{T}\{(m_k)\}^{\wedge}$.

Proof. Assume that all the m_k's are finite and let $M=\Sigma\oplus_{j=1}^{\infty}M_j$, where M_j acts on a space of dimension d_j, and $d_j\leqq d$ for some d.

Define $j_0\geqq 1$ and $k_0=2h_0+1\geqq 1$ so that $m_k>d$ for all $k>k_0$ and

$$\sum_{k=1}^{k_0}m_k\leqq\sum_{j=1}^{j_0}d_j<\sum_{k=1}^{k_0+1}m_k.$$

Now we define $j_1, j_2,\dots,$ so that

$$\sum_{k=1}^{k_0+p}m_k\leqq\sum_{j=1}^{j_p}d_j<\sum_{k=1}^{k_0+p+1}m_k.$$

After a finite rank perturbation, we can directly assume that $M_j=0$ for $1\leqq j\leqq j_0$.

If $p=2m-1$ $(p=2m)$ and M_{j_p} is written as a lower (upper, resp.) triangular matrix with respect to a suitable orthogonal basis of the corresponding d_{j_p}-dimensional space, and the space is decomposed as $\mathscr{R}_p\oplus\mathscr{S}_p$, where $\mathscr{R}_p$ is the span of the first

$$\sum_{j=1}^{j_p}d_j-\sum_{k=1}^{k_0+p}m_k$$

vectors, then

$$M_{j_p}=\begin{pmatrix} C'_{h_0+m-1} & 0 \\ D'_{h_0+m-1} & A'_{h_0+m} \end{pmatrix}$$

$$\left(M_{j_p}=\begin{pmatrix} A''_{h_0+m} & B''_{h_0+m} \\ 0 & C''_{h_0+m} \end{pmatrix}, \text{ resp.}\right).$$

Define $A_1, B_1, C_1, D_1, A_2, \ldots, A_{h_0}, B_{h_0}$ equal to 0,

$$C_{h_0}=0\oplus M_{j_0+1}\oplus M_{j_0+2}\oplus\cdots\oplus M_{j_1-1}\oplus C'_{h_0}$$

(where 0 acts on a space of dimension $\Sigma_{j=1}^{j_0} d_j-\Sigma_{k=1}^{k_0} m_k$),

$$C_{h_0+m}=C''_{h_0+m}\oplus M_{j_p+1}\oplus M_{j_p+2}\oplus\cdots\oplus M_{j_{p+1}-1}\oplus C'_{h_0+m}\ (p=2m),$$

$$A_{h_0+m}=A'_{h_0+m}\oplus M_{j_p+1}\oplus M_{j_p+2}\oplus\cdots\oplus M_{j_{p+1}-1}\oplus A''_{h_0+m}\ (p=2m-1),$$

$B_{h_0+m}=(B''_{h_0+m}00\ldots0)$ and D_{h_0+m} is the

column of operators $(D'_{h_0+m-1}00\cdots0)$ $(m=1,2,\ldots)$.

Now it is a straightforward exercise to check that M (or M minus some finite rank perturbation) has a staircase representation associated with the sequence $A_1, B_1, C_1, D_1, \ldots, A_r, B_r, C_r, D_r, \ldots$, and this operator belongs to $\mathscr{U}(\mathscr{T}(\{m_k\}))$. Hence, $M\in\mathscr{T}(\{m_k\})^{\wedge}$.

If $m_k=\infty$ for some k, the result follows by similar arguments. The details are left to the reader. ■

Let $\mathscr{T}(\{m_k\})$ be a given tridiagonal algebra. For each infinite m_k, drop m_{k-1}, m_k and m_{k+1} from the sequence, and let $\{n_r\}$ be the (finite or infinite) sequence formed by the remaining terms. If $\{n_r\}$ is finite, then we still have $\mathscr{T}\{(m_k\})^{\wedge}\supset(\mathrm{BD})_{\mathrm{nor}}$. (Indeed, $\mathscr{T}(\{m_k\})^{\wedge}=\mathscr{L}(\mathscr{H})$.)

The authors conjecture that if m_k is finite for all k, then $(\mathrm{BD})_{\mathrm{nor}}\subset\mathscr{T}(\{m_k\})^{\wedge}$ if and only if $m_k\to\infty$ $(k\to\infty)$.

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