Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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in memory of our friend
and collegue Prof. Vasil Popov

Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 4

Extinction of Controlled Branching Processes in Random Environments

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Presented by V. Popov

In the present paper sufficient conditions for extinction and nonextinction of branching processes in random environments with an general set of random control functions are given.

1. Introduction

The controlled branching process is a model of a branching process having an interaction among the individuals which is expressed by control functions. The process considered in the present paper may be described mathematically as follows:

Let us have on the probability space $(\Omega, \mathfrak{F}, P)$ two independent sets of integer-values random variables (r.v.) $\{X_i(j,t)\}$ and $\{\varphi_{it}(n)\}$ for $j=1,2,3,\dots$; $n,t=0,1,2,\dots$ and $i\in I$. Let us assume that $\{X_i(j,t)\}$ are independent and for every i and t they are identically distributed with p.g.f.

$$F_{it}(s)=Es^{X_i<j,t>}=\sum_{k=0}^{\infty} p_{ik}s^k,\quad |s|\leqq 1.$$

Next we shall consider the controlled branching process $\{Z(t)\}$ defined as follows:

$$(1)\qquad Z(t+1)=\sum_{i\in I}\sum_{j=1}^{\varphi_{it}(Z(t))} X_i(j,t),\quad t=0,1,2;\quad Z_0=m,\ m\geqq 1,$$

where I is an index set and $\{\varphi_{i_t}(n)\}$ is the set of control functions. Definition (1) describes a very large class of random processes (for example all Markov chains). If $I=\{1\}$, $F_{1t}(s)\equiv F(s)$ and $\varphi_{1t}(n)\equiv n$ a.s., then $\{Z(t)\}$ is a classical Galton-Watson process. If $I=\{1,2\}$ and a.s. $\varphi_{1t}(n)\equiv n$, $\varphi_{2t}(n)\equiv 1$, then $\{Z(t)\}$ is a branching process with immigration. If $I=\{1,2\}$, $F_{1t}(s)\equiv f(s)$, $F_{2t}(s)\equiv g(s)$ and a.s. $\varphi_{1t}(n)\equiv n$, $n\geqq 0$, $\varphi_{2t}(n)=0$, $n\geqq 1$, $\varphi_{2t}(0)=1$, then we obtain the model of J. H. Foster [3] and A. G. Pakes [9]. The Foster-Pakes processes for $F_{2t}(s)\equiv g_t(s)$ are investigated by K. V. Mitov and N. M. Yanev [6], [7]. V. A. Vatutin [13] have considered a case $I=\{1\}$ and $\varphi_{1t}\equiv \max(n-1, 0)$ a.s., i.e. a branching process with constant emigration of one particle. Note that N. M. Yanev, K. V. Mitov [16-17] and S. V. Nagaev, L. V. Han [8], [5] have proved some asymptotic results for some particular cases of definition (1).

One of the main characteristics of each branching process is the probability of extinction. A. G. Pakes [10] and D. R. Grey [4] have found some conditions for almost sure extinction for a process with a special case of emigration. B. A. Sevastyanov and A. M. Zubkov [11] have studied the probability of extinction or non-extinction of the process (1) in case $I=\{1\}$ and $\varphi_{1t}(n)\equiv\varphi(n)$ a.s., where the control function $\varphi(n)$ is nonrandom. A. M. Zubkov [19] has considered the case when $\varphi_{it}(n)\equiv\varphi_i(n)$ a.s., where also $\varphi_i(n)$ are nonrandom functions. N. M. Yanev [14] has obtained some conditions for extinction or non-extinction for $I=\{1\}$, and $\varphi_t=\{\varphi_{1t}(0), \varphi_{1t}(1), \varphi_{1t}(2)\ldots\}$, $t=0,1,2,\ldots$ are independent identically distributed random processes. E. T. Bruss [2] has investigated also some sufficient conditions for extinction. These results are extended for controlled processes in a random environment (cf. N. M. Yanev [15]). The authors in [18] have obtained some conditions for extinction when I being an arbitrary index set.

The present paper deals with a complete criterion for extinction or non-extinction of the general process (1) in i.i.d. random environments, under some assumptions for the control functions. In this case $\{Z(t)\}$ remains a temporally homogeneous Markov chain.

2. Model and main results

Let $(\Omega, \mathfrak{F}, P)$ be a given probability space. Let

$$U=\{\bar{\mathbf{p}}=\{p_{ik}\}_{k=0}^{\infty},\ i\in I: p_{ik}\geqq 0,\ \sum_{k=0}^{\infty} p_{ik}=1,\ \sum_{k=0}^{\infty} kp_{ik}<\infty\}.$$

Clearly U is a subset of the Banach space $(l_\infty, \mathfrak{B}_\infty)$ of all bounded sequences of real numbers with the Borel σ-algebra $\mathfrak{B}_\infty$, generated by the product topology. Then the "random environment" process $\eta_{it}(\omega)$, $t=0,1,2,\ldots$ $i\in I$ is a random mapping from $(\Omega, \mathfrak{F}, P)$ into $(l_\infty, \mathfrak{B}_\infty)$ such that $P\{\eta_{it}\in U,\ i\in I,\ t\geqq 0\}=1$. For any $\eta_{it}\in U$ associate the p.g.f.

$$F_{it}(\omega, s)=\sum_{k=0}^{\infty} p_{ik}(\eta_{it}(\omega))s^k,\quad |s|\leqq 1. \tag{2}$$

Now a branching process with random environment is characterized by the environmental process $\{\eta_{it}\}$ whose realization determines a sequence of generational offspring p.g.f.'s $F_{it}(\omega, s)$, by (2).

Smith-Wilkinson [12] and Athreya-Karlin [1] were the first to invesigate the Galton-Watson process in random environment.

We will consider the process (1) in i.i.d. random invironment, under some assumptions for $\{\varphi_{it}(n)\}$. That is,

$$Z(t+1)=\sum_{i\in I}\ \sum_{j=1}^{\varphi_{it}(Z(t))} X_i(j,t),\quad t=0,1,2;\ Z_0=m,\ m\geqq 1,$$

where $X_i(j,t)$ have p.g.f. $\{F_{it}(\omega, s)\}$ which for fixed s, $|s|\leqq 1$ are:

i) independent;

ii) identically distributed r.v. for fixed $i\in I$.

Note that $X_i(j, t)$ can be non-identically distributed with respect to i.

Furthermore let us assume that the control functions $\{\varphi_{it}(n)\}$ are:

i) independent with respect to t;

ii) identically distributed r. v. for fixed n and $i \in I$. Note that $\varphi_{it}(n)$ can be dependent with respect to i:

Let to the end the random environment $\{\eta_{it}(\omega)\}$ be independent on the control functions $\{\varphi_{it}(n)\}$.

Under these assumptions $\{Z(t)\}$ is a temporally homogeneous Markov chain (cf. Smith-Wilkinson [12]).

We will investigate the probabilities for extinction

$$q_m = \lim_{t \to \infty} P\{Z(t) = 0 | Z(0) = m\}, \quad m \geqq 1.$$

If $q_m < 1$ for some integer $m \geqq 1$, then $\{Z_t\}$ is called non-extinct and contrary-extinct.

Let the set of control functions be connected with a set of r. v. $\{\alpha_{it}\}$ which are:

i) independent;

ii) identically distributed r. v. for fixed $i \in I$. Namely, there exists $N < \infty$, such that for every $i \in I$

$$\inf_{n \geqq N} \frac{\varphi_{it}(n)}{n} \geqq \alpha_{it} \quad \text{a. s.,} \quad t = 0, 1, 2, \dots$$

or

$$\sup_{n \geqq N} \frac{\varphi_{it}(n)}{n} \leqq \alpha_{it} \quad \text{a. s.,} \quad t = 0, 1, 2, \dots$$

Clearly $\{\alpha_{it}\}$ are independent of $\{X_i(j, t)\}$ and $\{\eta_{it}\}$.

Further we will need the following assumptions.

Assumption A. *For every integer* $k \geqq 1$, $H_n(F_t^{(k)}(\omega, 0)) > 0$, $t \geqq 0$, *where*

$$H_n(s^{(k)}) = E s_{i_1}^{\varphi_{i_1 t}(n)} \dots s_{i_k}^{\varphi_{i_k t}(n)}, \quad i_j \in I, \quad 1 \leqq j \leqq k.$$

Denote $f_t(\omega, s) = \sup_{i \in I} F_{it}^{1/F'_{it}}(\omega, s)$, $|s| \leqq 1$, $t \geqq 0$, where $F'_{it} = \frac{d}{ds} F_{it}(\omega, 1)$.

Assumption B. *The following expectations are finite*

$$E\{-\log(1 - f_t(\omega, 0))\}, \quad t \geqq 0;$$

$$E\{-\log \zeta\}, \text{ where } \zeta = \inf_{i \in I} \min \left\{ \frac{1 - F_{i0}(\omega, 0)}{F'_{i0}}, \ 1 - F_{i0}(\omega, 0) \right\}$$

and

$$E\left\{-\log \sum_{i \in I} \frac{F'_{i0} \alpha_{i0}}{F_{i0}(\omega, 0)}\right\}, \text{ where } P\{F_{i0}(\omega, 0) > 0\} = 1, \ i \in I.$$

Now if we define the critical parameter $\rho = E \log \sum_{i \in I} \alpha_{i0} F'_{i0}$, then the following theorem will be obtained which will be the main result of the paper.

Theorem 1. *Let* $N < \infty$ *be such that for every* $i \in I$

i) $\sup_{n \geqq N} \frac{\varphi_{it}(n)}{n} \leqq \alpha_{it}$ a. s., $t = 0, 1, 2, \dots$

and let Ass. A be valid.

Hence if $\rho \leqq 0$, then $q_m = 1$, $m \geqq 1$, i. e. $\{Z(t)\}$ extincts.

ii) $\inf\limits_{n \geqq N} \frac{\varphi_{it}(n)}{n} \geqq \alpha_{it}$ a.s., $t=0,1,2,\ldots$

and Ass. B holds.

Hence if $\rho > 0$, then $\{Z(t)\}$ does not extinct.

Theorem 1 follows from Theorems 2-5 which might be of some interest themselves.

3. Extinction

Lemma 1. *The states $m=1,2,3,\ldots$ of a Markov process $\{Z(t)\}$ are transient, under the Ass. A*

Proof. For every integer $m \geqq 1$

$$(3) \quad P\{Z(t+n)=m, \text{ for some } n \geqq 1 | Z(t)=m\} \leqq 1 - P\{Z(t+1)=0 | Z(t)=m\}$$

$$= 1 - P\{\sum_{j=1}^{\varphi_{1t}(m)} X_1(j,t)=0, \ldots, \sum_{j=1}^{\varphi_{it}(m)} X_i(j,t)=0, \ldots; \ i \in I\}$$

$$= 1 - \Sigma^* P\{\varphi_{1t}(m)=m_1, \ldots, \varphi_{it}(m)=m_i, \ldots\} \prod_{i \in I} P\{\sum_{j=1}^{m_i} X_i(j,t)=0\}$$

$$= 1 - \Sigma^* P\{\varphi_{1t}(m)=m_1, \ldots, \varphi_{it}(m)=m_i, \ldots\} \prod_{i \in I} EF_{it}^{m_i}(\omega, 0)$$

$$= 1 - E\Sigma^* P\{\varphi_{1t}(m)=m_1, \ldots, \varphi_{it}(m)=m_i, \ldots\} \prod_{i \in I} F_{it}^{m_i}(\omega, 0),$$

where

$$\Sigma^* . = \sum_{m_1, \ldots, m_i, \ldots = 0, i \in I}^{\infty} .$$

Hence, for some $m \geqq 1$, under Ass.A,

$$(4) \qquad P\{Z(t+n)=m, \quad \text{for some } n \geqq 1 \ | Z(t)=m\} < 1,$$

which proves the lemma.

Now using the well known results of the theory of Markov chains we obtain that for every $k, m=1,2,3,\ldots$

$$(5) \qquad \lim_{t \to \infty} P\{Z(t)=k | Z(0)=m\} = 0$$

$$(6) \qquad \lim_{t \to \infty} P\{Z(t)=0 | Z(0)=m\} + \lim_{t \to \infty} P\{Z(t)=\infty | Z(0)=m\} = 1.$$

Note that if the Ass.A does not hold, then for every integer $m \geqq 1$

$$\lim_{t \to \infty} P\{Z(t)=0 | Z(0)=m\} = 0.$$

Theorem 2. *Under the Ass.A and if there exists $N < \infty$ such that for all $n \geqq N$, $\sum_{i \in I} E\{F'_{i0}\varphi_{i0}(n)\} \leqq n$, then the process $\{Z(t)\}$ extincts, i.e.. $q_m = 1$, $m \geqq 1$.*

Proof. From (6) it follows that it is sufficient to show that $\lim_{t\to\infty} P\{Z(t)=\infty|Z(0)=m\}=0$, for every $m=1,2,3,\dots$

For this probability we have (cf. A. M. Zubkov [19])

$$\lim P\{Z(t)=\infty|Z(0)=m\} \tag{7}$$

$$\leqq \sum_{t=0}^{\infty} \sum_{k=N_1}^{\infty} P\{Z(t)=k|Z(0)=m\}P\{\min_{t>0} Z(t)\geqq N_1|Z(0)=k\}, \quad \text{for every } N_1<\infty.$$

Hence it is sufficient to prove that for every $m\geqq N_1$

$$r_m=P\{\min_{t>0} Z(t)\geqq N_1|Z(0)=m\}=0.$$

Choose $N_1\geqq N$ and set for any $i\in I$

$$\varphi^*_{it}(n)=\begin{cases}0 & , \quad n<N_1\\ \varphi_{it}(n) & , \quad n\geqq N_1.\end{cases}$$

Let us construct the auxiliary process $\{Z^*(t)\}$ in the following way

$$Z^*(0)=m\geqq N_1, \quad Z^*(t+1)=\sum_{i\in I}\sum_{j=1}^{\varphi^*_{it}(Z^*(t))} X_i(j,t), \quad t=0,1,2,\dots$$

If $k, m\geqq N_1$, then

$$P\{Z(t+1)=k|Z(t)=m\}=P\{Z^*(t+1)=k|Z^*(t)=m\},$$

and hence

$$P\{\min_{t>0} Z(t)\geqq N_1|Z(0)=k\}=P\{\min_{t>0} Z^*(t)\geqq N_1|Z^*(0)=k\}$$

$$=1-\lim_{t\to\infty} P\{Z^*(t)=0|Z^*(0)=m\}=1-q^*_m.$$

To complete the prove it is sufficient to show that $q^*_m=1$ for every $m\geqq N_1$. We have

$$P\{Z^*(t)=0|Z^*(0)=m\}=1-P\{Z^*(t)\geqq M|Z^*(0)=m\}-P\{1\leqq Z^*(t)<M|Z^*(0)=m\}$$

for every $M<\infty$.

Since $Z^*(t)$ satisfies (5), hence for every $\varepsilon>0$ and large enough t

$$P\{1\leqq Z^*(t)<M|Z^*(0)=m\}<\frac{\varepsilon}{2}, \quad \text{for every } M<\infty.$$

The Chebyshev's inequality implies that

$$P\{Z^*(t)\geqq M|Z^*(0)=m\}\leqq\frac{E\{Z^*(t)|Z^*(0)=m\}}{M}. \tag{8}$$

Let $A_{it}=\sigma\{\eta_{ij}, 0\leqq j\leqq t\}$, $i\in I$. Therefore

$$E\{Z^*(t)|Z^*(t-1)=k\}=\sum_{i\in I} EE\{X_i(1,t)+\dots X_i(\varphi^*_{it-1}(k),t)|A_{it-1}\}$$

$$=\sum_{i\in I} E\{F'_{it-1}(\omega,1)E\varphi^*_{it-1}(k)\}\leqq k.$$

Hence

(9) $$E\{Z^*(t)|Z^*(0)=m\} \leqq \sum_{i\in I} P\{Z^*(t-1)=k|Z^*(0)=m\}k$$
$$=E\{Z^*(t-1)=k|Z^*(0)=m\} \leqq \ldots \leqq m.$$

It follows from (8) and (9) that for every $\varepsilon>0$ there exists a large enough M such that $P\{Z^*(t)\geqq M|Z^*(0)=m\}\leqq\frac{\varepsilon}{2}$ for all $t\geqq 1$. Hence for every $\varepsilon>0$ and a large enough t, $P\{Z^*(t)=0|Z^*(0)=m\}\geqq 1-\varepsilon$ which proves the theorem.

Theorem 3. *Under Ass.A, suppose that for every $i\in I$ and $t\geqq 0$*

$$\sup_{n\geqq N} \frac{\varphi_{it}(n)}{n} \leqq \alpha_{it} \quad \text{a.s.,} \quad \textit{for some } N<\infty.$$

Then $\{Z(t)\}$ extincts if $\rho\leqq 0$.

Proof. a) Let $\sum_{i\in I} EF'_{i0}\alpha_{i0}\leqq 1$. From this assumption and the Jensen's inequality we obtain $\rho=E\log\sum_{i\in I}\alpha_{i0}F'_{i0}\leqq\log\sum_{i\in I}EF'_{i0}\alpha_{i0}\leqq 0$. On the other hand for $n\geqq N$, $\sum_{i\in I}E\{F'_{i0}\varphi_{i0}(n)\}\leqq\sum_{i\in I}EF'_{i0}\alpha_{i0}n\leqq n$. Now from Theorem 2 it follows that $\{Z(t)\}$ extincts.

b) Let $\sum_{i\in I}EF'_{i0}\alpha_{i0}>1$ and $\rho\leqq 0$. Let $\{Z^*(t)\}$ be the same auxiliary process as in Theorem 2. Hence for $\{Z^*(t)\}$ we have for every $i\in I$ and $t\geqq 0$

$$\sup_{n\geqq N}\frac{\varphi_{it}(n)}{n}\leqq\alpha_{it} \text{ a.s. for some } N<\infty.$$

Defining $X_k=\log\sum_{i\in I}\alpha_{ik}F'_{ik}$, $k\geqq 1$, which are i.i.d.r.v. we will construct the random walk $S_0=0$, $S_n=\sum_{k=0}^{n-1}X_k$, $n\geqq 1$. Since $\sum_{i\in I}EF'_{i0}\alpha_{i0}>1$, then $P\{X_k=0\}<1$. Thus since $\rho=EX_k\leqq 0$, then it follows that for every $T>0$ there exists non-negative integer-valued r.v. (stopping time)

$$\tau_T=\inf\{t>0\colon S_t\leqq -T\},\ P\{\tau_T<\infty\}=1, \quad \text{i.e.}$$

(10) $$S_{\tau_T}\leqq -T \quad \text{a.s.}$$

Let $A_t=\sigma\{\alpha_{i0},\eta_{i0},\alpha_{i1},\eta_{i1},\ldots\alpha_{it-1},\eta_{it-1};\ i\in I\}$. We will prove that for all $t\geqq 0$

(11) $$E\{Z^*(t)|A_t;\ Z^*(0)=m\}\leqq me^{S_t} \text{ a.s.}$$

Indeed a.s.

$$E\{Z^*(1)|A_1;\ Z^*(0)=m\}=E\{\sum_{i\in I}X_i(1,0)+\ldots+X_i(\varphi^*_{i0}(m),0)|A_1\}$$
$$=\sum_{i\in I}F'_{i0}E\{\varphi^*_{i0}(m)|A_1\}\leqq\sum_{i\in I}F'_{i0}mE\{\alpha_{i0}|A_1\}=m\sum_{i\in I}F'_{i0}\alpha_{i0}=me^{S_1}.$$

Let us assume (11) for some $t\geqq 1$. Since $Z^*(t)$, α_{it} and η_{it} are independent for $i\in I$, then a.s.

$$E\{Z^*(t+1)|A_{t+1};\ Z^*(0)=m\}=E\{\sum_{i\in I}\sum_{j=1}^{\varphi^*_{it}(z^*(t))} X_i(j,t)|A_{t+1};\ Z^*(0)=m\}$$
$$=\sum_{i\in I} F'_{it}E\{\varphi^*_{it}(Z^*(t)|A_{t+1};\ Z^*(0)=m\}\leqq \sum_{i\in I} F'_{it}E\{\alpha_{it}Z^*(t)|A_{t+1};\ Z^*(0)=m\}$$
$$=\sum_{i\in I} F'_{it}\alpha_{it}E\{Z^*(t)|A_{t+1}Z^*(0)=m\}=\sum_{i\in I} F'_{it}\alpha_{it}E\{\{Z^*(t)|A_t;\ Z^*(0)=m\}$$
$$\leqq me^{S_t}\sum_{i\in I} F'_{it}\alpha_{it}=me^{S_{t+1}}.$$

Now from (10) and (11) we will show that

$$E\{Z^*(t)|Z^*(0)=m\}\leqq me^{-T} \text{ a.s.} \tag{12}$$

Indeed since $\{\tau_T=t\}\in A_t$, then

$$E\{Z^*(\tau_T)|Z^*(0)=m\}=EE\{Z^*(\tau_T)|\tau_T;\ Z^*(0)=m\}$$
$$=\sum_{t=1}^{\infty} P\{\tau_T=t\}E\{E\{Z^*(t)|A_t;\ Z^*(0)=m\}|\tau_T=t\}$$
$$\leqq \sum_{t=1} P\{\tau_T=t\}E\{me^{S_t}|\tau_T=t\}=mE\exp\{S_{\tau_T}\}\leqq me^{-T}.$$

From (12) and the Chebyshev's inequality it follows that

$$P\{Z^*(\tau_T)\geqq 1|Z^*(0)=m\}\leqq me^{-T}.$$

If T is large enough this probability will be arbitrarily small. From here and Theorem 3 it follows that $\{Z(t)\}$ ectincts.

4. Non-extinction

Theorem 4. *Under the Ass. B and if $\rho>0$ and for every $i\in I$*

$$\inf_{n\geqq 1}\frac{\varphi_{it}(n)}{n}\geqq\alpha_{it} \ a.s.,\ t=0,1,2,\dots,$$

then $q_m<1$ for every integer $m\geqq 1$.

Proof. We will use the standard symbols $g^+(x)=\max(0,g(x))$ and $g^-(x)=\max(0,-g(x))$.

From $\rho=E\log\sum_{i\in I}\alpha_{i0}F'_{i0}=E\log^+\sum_{i\in I}\alpha_{i0}F'_{i0}+E\log^-\sum_{i\in I}\alpha_{i0}F'_{i0}>0$ it follows that $E\log^-\sum_{i\in I}\alpha_{i0}F'_{i0}<E\log^+\sum_{i\in I}\alpha_{i0}F'_{i0}\leqq\infty$.

Now if we assume that $P\{\alpha_{10}=0,\ \alpha_{20}=0,\dots,\alpha_{i0}=0,\dots;\ i\in I\}>0$, then

$$E\log^-\sum_{i\in I}\alpha_{i0}F'_{i0}=\int_A-\log\sum_{i\in I}\alpha_{i0}F'_{i0}dP=\infty,\quad\text{where } A=\{\sum_{i\in I}\alpha_{i0}F'_{i0}\leqq 1\}.$$

This contradiction implies

$$P\{\alpha_{10}=0,\ \alpha_{20}=0,\dots,\alpha_{i0}=0,\dots;\ i\in I\}=0$$

and hence

(13) $P\{\varphi_{1t}(n)=0,\ \varphi_{2t}(n)=0,\dots,\ \varphi_{it}(n)=0,\dots;\ i\in I\}=0,$ for all $t, n\geqq 0.$

If we assume that for every $i\in I$, $P\{F_{i0}(\omega,0)>0\}=0$, then from (3) and (13) we obtain $q_m=0$, $m\geqq 1$ and thus the theorem holds.

Further we suppose that there exists $i\in I$ such that $P\{F_{i0}(\omega,0)>0\}>0$ and moreover $P\{F_{i0}(\omega,0)>0\}=1$ holds for every $i\in I$.

We will use induction in order to prove that for arbitrary $t\geqq 0$ and $0\leqq s\leqq 1$

$$\Phi_m(t,s)\leqq E\{\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,\prod_{i\in I} F_{i1}^{\alpha_{i1}}(\dots(\omega,\prod_{i\in I} F_{it-1}^{\alpha_{it-1}}(\omega,s))\dots))\}^m, \tag{14}$$

where $\Phi_m(t,s)=E\{s^{Z(t)}|Z(0)=m\}$.

Indeed

$$E\{s^{Z(t)}|Z(0)=m\}=E\prod_{i\in I}\prod_{j=1}^{\varphi_{i0}(m)} s^{X_i(j,0)}=E\prod_{i\in I} F_{i0}^{\varphi_{i0}(m)}(\omega,s)\leqq E\Big(\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,s\Big)^m.$$

If (14) holds for some $t\geqq 1$, then

$$\Phi_m(t+1,s)=E\Big\{\prod_{i\in I}\prod_{j=1}^{\varphi_{i0}(m)} s^{X_i(j,t)}|Z(0)=m\Big\}=E\Big\{\prod_{i\in I} F_{i0}^{\varphi_{it}(Z(t))}(\omega,s)|Z(0)=m\Big\} \tag{15}$$

$$\leqq E\Big\{\Big(\prod_{i\in I} F_{i0}^{\alpha_{it}}(\omega,s)\Big)^{Z(t)}|Z(0)=m\Big\}$$

$$\leqq E\Big\{\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,\prod_{i\in I} F_{i1}^{\alpha_{i1}}(\dots\ (\omega,\prod_{i\in I} F_{it}^{\alpha_{it}}(\omega,s))\dots))\Big\}^m.$$

This completes (14).

Let us set $\alpha_i=\{\alpha_{i0},\alpha_{i1},\dots\}$, $\eta_i=\{\eta_{i0},\eta_{i1},\dots\}$, $i\in I$ and let T be the shift transformation $T\alpha_i=\{\alpha_{i1},\alpha_{i2},\dots\}$, $T\eta_i=\{\eta_{i1},\eta_{i2},\dots\}$. From this definition it is not difficult to obtain

$$\begin{aligned} W_t(\alpha_i,\eta_i)&=\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,\prod_{i\in I} F_{i1}^{\alpha_{i1}}(\dots(\omega,\prod_{i\in I} F_{it-1}^{\alpha_{it-1}}(\omega,s))\dots))\\ &=\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,W_{t-1}(T\alpha_i,T\eta_i)). \end{aligned} \tag{16}$$

Now we will prove that the identity

$$\begin{aligned} &-\log(1-W_t(\alpha_i,\eta_i))\\ &=-\log\frac{1-\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,W_{t-1}(T\alpha_i,T\eta_i))}{1-W_{t-1}(T\alpha_i,T\eta_i)}-\log(1-W_{t-1}(T\alpha_i,T\eta_i)) \end{aligned} \tag{17}$$

can be integrable. Setting

$$b_t=E-\log\frac{1-\prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega,W_{t-1}(T\alpha_i,T\eta_i))}{1-W_{t-1}(T\alpha_i,T\eta_i)}\quad\text{and}$$

$c_t=E-\log(1-W_t(\alpha_i,\eta_i))$ we obtain the recurrent relation

(18) $$c_t = b_t + c_{t-1} = \sum_{k=1}^{t} b_k + c_0.$$

Next (15) will be proved. Defining $A = \left\{ \sum_{i\in I} \alpha_{i0} F'_{i0} \leqq 1 \right\}$ we get

$$c_0 = \int_{\bar{A}} -\log\left(1 - \prod_{i\in I} F_{it}^{\alpha_{it}}(\omega, 0)\right) dP + \int_{A} -\log\left(1 - \prod_{i\in I} F_{it}^{\alpha_{it}}(\omega, 0\right) dP.$$

Since $\prod_{i\in I} F_{it}^{\alpha_{it}}(\omega, s) = F_{1t}^{F'_{1t}\alpha_{1t}/F'_{1t}}(\omega, s)\, F_{2t}^{F'_{2t}\alpha_{2t}/F'_{2t}}(\omega, s) \ldots \leqq f_t^{\sum_{i\in I} F'_{it}\alpha_{it}}(\omega, s)$, it follows that

$$1 - \prod_{i\in I} F_{it}^{\alpha_{it}}(\omega, 0) \geqq 1 - f_t^{\sum_{i\in I} F'_{it}\alpha_{it}}(\omega, 0) \geqq 1 - f_t(\omega, 0) \text{ on } \bar{A} \text{ and}$$

$$1 - \prod_{i\in I} F_{it}^{\alpha_{it}}(\omega, 0) \geqq 1 - f_t^{\sum_{i\in I} F'_{it}\alpha_{it}}(\omega, 0) \geqq \left(\sum_{i\in I} \alpha_{it} F'_{it}\right)(1 - f_t(\omega, 0)) \text{ on A.}$$

Therefore, by using the Ass. B, we obtain

$$c_0 \leqq \int_{\bar{A}} -\log(1 - f_t(\omega, 0))\, dP + \int_{A} -\log\left(\sum_{i\in I} \alpha_{it} F'_{it}\right)(1 - f_t(\omega, 0)\, dP$$
$$= E\{-\log(1 - f_t(\omega, 0))\} + E \log^- \sum_{i\in I} \alpha_{i0} F'_{i0} < \infty.$$

Further, since $F_{i0}(\omega, s) \leqq 1 - (1-s)(1 - F_{i0}(\omega, 0))$ a.s. it follows that a.s.

$$F_{i0}^{1/F'_{i0}}(\omega, s) \leqq [1 - (1-s)(1 - F_{i0}(\omega, 0))]^{1/F'_{i0}} \leqq \begin{cases} 1 - (1-s)\dfrac{1 - F_{i0}(\omega, 0)}{F'_{i0}}, & \text{when } F'_{i0} \geqq 1 \\ 1 - (1-s)(1 - F_{i0}(\omega, 0)), & \text{when } F'_{i0} < 1. \end{cases}$$

Therefore

$$f_0(\omega, s) = \sup_{i\in I} F_{i0}^{1/F'_{i0}}(\omega, s) \leqq 1 - (1-s)\zeta(\omega),$$

where

$$\zeta(\omega) = \inf_{i\in I} \min\left\{\frac{1 - F_{i0}(\omega, 0)}{F'_{i0}},\ 1 - F_{i0}\omega, 0)\right\}.$$

Now on $\bar{A}$ we obtain that a.s.

$$\frac{1 - \prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega, W_{t-1}(T\alpha_i, T\eta_i))}{1 - W_{t-1}(T\alpha_i, T\eta_i)} \geqq \frac{1 - f_0(\omega, W_{t-1}(T\alpha_i, \eta_i))}{1 - W_{t-1}(T\alpha_i, T\eta_i)} \geqq \zeta$$

and on A a.s.

$$\frac{1 - \prod_{i\in I} F_{i0}^{\alpha_{i0}}(\omega, W_{t-1}(T\alpha_i, T\eta_i))}{1 - W_{t-1}(T\alpha_i, T\eta_i)} \geqq \left(\sum_{i\in I} \alpha_{i0} F'_{i0}\right) \frac{1 - f_0(\omega, W_{t-1}(T\alpha_i, \eta_i))}{1 - W_{t-1}(T\alpha_i, T\eta_i)} \geqq \sum_{i\in I} \alpha_{i0} F'_{i0}.$$

Using again Ass. B, we see that for every $t = 0, 1, 2, \ldots$

$$b_t=\int_{\bar{A}}-\log\zeta\,\mathrm{d}P+\int_{A}-\log\left(\zeta\sum_{i\in I}\alpha_{i0}F'_{i0}\right)\mathrm{d}P=E-\log\zeta+E\log^{-}\sum_{i\in I}\alpha_{i0}F'_{i0}<\infty.$$

Now from $0\leqq c_t=b_t+c_{t-1}$ by induction it follows that $c_t<\infty$ for every $t=1,2,3,\dots$ Thus (18) holds.

Assume contrary to the assertion of the theorem that $q_m=P\{\lim_{t\to\infty}Z(t)=0|Z(0)=m\}=1$, for some integer $m\geqq 1$.

Then from (15) (when $s=0$) we get

$$P\{Z(t)=0|Z(0)=m\}\leqq E\,W_t^m(\alpha_i,\eta_i),\ \text{for all } t\geqq 0 \text{ and } m\geqq 1$$

and since

$$0\leqq W_t(\alpha_i,\eta_i)\leqq W_{t+1}(\alpha_i,\eta_i)\leqq 1 \quad \text{a. s.,}$$

we obtain $E\,W_t^m(\alpha_i,\eta_i)\uparrow 1$ and furthermore $E\,W_t(\alpha_i,\eta_i)\uparrow 1$.

Since for every $\varepsilon>0$,

$$P\{1-W_t(\alpha_i,\eta_i)\geqq\varepsilon\}\leqq\frac{1}{\varepsilon}E\{1-W_t(\alpha_i,\eta_i)\},$$

then $W_t(\alpha_i,\eta_i)\uparrow 1$ in probability and hence almost sure. This is true for $W_t(T\alpha_i,T\eta_i)$ as well.

On the other hand for some $0\leqq u\leqq 1$ a. s.

$$\frac{1-\prod_{i\in I}F_{i0}^{\alpha_{i0}}(\omega,s)}{1-s}=\frac{\mathrm{d}}{\mathrm{d}s}\left[\prod_{i\in I}F_{i0}^{\alpha_{i0}}(\omega,s\right]_{s=u}\geqq\begin{cases}\zeta, & \text{on } \bar{A}\\ \zeta\sum_{i\in I}\alpha_{i0}F'_{i0}, & \text{on } A\end{cases}.$$

But

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\prod_{i\in I}F_{i0}^{\alpha_{i0}}(\omega,s)\right]_{s=u}=\sum_{i\in I}\frac{\alpha_{i0}F'_{i0}(\omega,u)}{F_{i0}(\omega,u)}\prod_{j\in I}F_{j0}^{\alpha_{j0}}(\omega,u)\leqq\sum_{i\in I}\frac{\alpha_{i0}F'_{i0}}{F_{i0}(\omega,0)},\quad\text{a. s.}$$

Therefore a. s.

$$-\log\sum_{i\in I}\frac{\alpha_{i0}F'_{i0}}{F_{i0}(\omega,0)}\leqq-\log\frac{1-\prod_{i\in I}F_{i0}^{\alpha_{i0}}(\omega,s)}{1-s}\leqq-\log\zeta+\log^{-}\sum_{i\in I}\alpha_{i0}F'_{i0}.$$

From these inequalities, we obtain

$$\lim_{s\uparrow 1}E\left\{-\log\frac{1-\prod_{i\in I}F_{i0}^{\alpha_{i0}}(\omega,s)}{1-s}\right\}=E\left\{-\log^{-}\sum_{i\in I}\alpha_{i0}\mathrm{F}'_{i0}\right\}=-\rho<0$$

using the Ass. B and Fatou-Lebesgue's theorem and hence $\lim_{t\to\infty}b_t=-\rho$. This is a contradiction to (18), since $c_t\to+\infty$ by definition. This completes the proof.

Theorem 5. *Under the Ass. B, if $\rho>0$ and for every $i\in I$ and $t\geqq 0$, $\inf_{n\geqq N}\frac{\varphi_{it}(n)}{n}\geqq\alpha_{it}$* a. s., for some $N<\infty$, then $\{Z(t)\}$ does not extinct.

Proof. Let us construct the auxiliary process $\{Z^*(t)\}$ in the following way

$$Z^*(t+1)=\begin{cases}\sum_{i\in I}\sum_{j=1}^{\varphi^*_{it}(Z^*(t))} X_i(j,t) & \text{if } \varphi^*_{it}(Z^*(t))>0,\\ 0 & \text{if } \varphi^*_{it}(Z^*(t))=0,\end{cases}\qquad t=0,1,2,\ldots$$

where for every $i\in I$,

$$\varphi^*_{it}(n)=\begin{cases}\max\{\varphi_{it}(n),\ \langle\alpha_{it}n\rangle\}, & n<N\\ \varphi_{it}(n), & n\geqq N.\end{cases}$$

Here $\langle u\rangle$ is the smallest integer greater than or equal to u.

From $\inf\limits_{n\geqq 1}\dfrac{\varphi^*_{it}(n)}{n}\geqq\alpha_{it}$ a. s. $i\in I$ and Theorem 4 it follows that for all $m\geqq 1$,

$$q^*_m=\lim_{t\to\infty}P\{Z^*(t)=0|Z^*(0)=m\}<1.$$

Since for k, $n\geqq N$, $P\{Z^*(t+1)=k|Z^*(t)=n\}=P\{Z(t+1)=k|Z(t)=n\}$, then for every $j\geqq N$,

$$P\{\min_{t>0}Z(t)\geqq N|Z(0)=j\}=P\{\min_{t>0}Z^*(t)\geqq N|Z^*(0)=j\}=r_j.$$

Les us assume that for every $j\geqq N$,

$$r_j=P\{\min_{t>0}Z^*(t)\geqq N|Z^*(0)=j\}=0.$$

Then from (7) we obtain $P\{\lim\limits_{t\to\infty}Z^*(t)=\infty|Z^*(0)=j\}=0$. This is a contradiction to the non-extinction and transientness of the process.

Hence there exists $j\geqq N$ such that $r_j>0$. Since for every $t\geqq 0$, $P\{\min Z(t)>0|Z(0)=j\}\geqq r_j$, then it follows that $q_j<1$, i. e. $\{Z(t)\}$ does not extinct.

4. Comments

Remark 1. Theorem 1 can be formulated more briefly:

Theorem 1'. *Let $\varphi_{it}(n)=\alpha_{it}n(1+o(1))$ a. s. $n\to\infty$, for every $i\in I$. Hence under the Ass. A and if $\rho<0$, then $\{Z(t)\}$ extincts; under the Ass. B and if $\rho>0$, then $\{Z(t)\}$ does not extinct.*

Proof. Indeed, if $\rho<0$, then for any δ_1, $1<\delta_1\leqq e^{-\rho}$, there exists $N_1=N_1(\delta_1)$, such that $\varphi_{it}(n)\leqq\delta_1\alpha_{it}n$ a. s. when $n\geqq N_1$. Thus from Theorem 4 it follows that the process extincts since $\rho_1=E\log\delta_1\sum_{i\in I}\alpha_{i0}F'_{i0}\leqq 0$. If $\rho>0$, then for arbitrary δ_2, $e^{-\rho}<\delta_2<1$ there exists $N_2=N_2(\delta_2)$, such that $\varphi_{it}(n)\geqq\delta_2\alpha_{it}n$ a. s. $n\geqq N_2$. Now since $\rho_2=E\log\delta_2\sum_{i\in I}\alpha_{i0}F'_{i0}>0$, then from Theorem 6 it follows that $\{Z(t)\}$ does not extinct.

One can classify the process $\{Z(t)\}$ as a subcritical one when $\rho<0$, supercritical — when $\rho>0$ and critical — for $\rho=0$. In the subcritical case the process extincts, in the supercritical case it does not extinct and in the critical case it may either extinct or it may not extinct.

Remark 2. It is interesting to consider a special case of controlled branching processes in a random environment.

Assume that $\varphi_{it}(n)=[\beta_{it}n]$ a.s., where $\{\beta_{it}\}$ are positive and for every $i\in I$, i.i.d.r.v.

Here $[u]$ is the greatest integer less than or equal to u.

Define $$Y(t+1)=\sum_{i\in I}\sum_{j=1}^{[\beta_{it}Y(t)]} X_i(j,t),\quad t=0,1,2,\dots$$

Corollary. *i) Under Ass. A, if* $\rho=E\log\sum_{i\in I}\beta_{i0}F'_{i0}\leqq 0$, *then* $\{Y(t)\}$ *extincts;*

ii) Under Ass. B, if $\rho=E\log\sum_{i\in I}\beta_{i0}F'_{i0}>0$, *then* $\{Y(t)\}$ *does not extinct.*

Proof. i) Since $[\beta_{it}n]\leqq\beta_{it}n$, then the statement yields immediately from Theorem 3;

ii) For every $\varepsilon>0$ there exists $N<1$ such that for $n\geqq N$, $[\beta_{it}n]\geqq(\beta_{it}-\varepsilon)n$, $i\in I$, $t\geqq 0$. Since ε can be chosen small enough so that $E\log\sum_{i\in I}(\beta_{i0}-\varepsilon)F'_{i0}>0$, then the statement follows from Theorem 5.

Note that when $\beta_{it}<1$ a.s. the process $\{Y(t)\}$ allows a special case of emigration, while if $\beta_{it}>1$ a.s. $\{Y(t)\}$ admits a special case of immigration.

Remark 3. If the assumption for independence of $\{\alpha_{it}\}$ is omitted then following Bruss's construction [2] one can obtain the following criterion for almost sure extinction

Theorem 6. *Under the Ass. A and if*

i) α_{it}, $i\in I$, $t\geqq 0$ *are arbitrary non-negative r.v.;*

ii) *there exists N such that for every* $i\in I$ $\sup_{n\geqq N}\frac{\varphi_{it}(n)}{n}\leqq\alpha_{it}$ a.s., $t=0,1,2,\dots$;

iii) $E\log\sum_{i\in I}\alpha_{i0}EF'_{i0}\leqq\log\frac{m}{M}<0$, *where* $m=\inf_{i\in I}EF'_{i0}$ *and* $M=\sup_{i\in I}EF'_{i0}<\infty$, *then* $\{Z(t)\}$ *extincts.*

Since the proof of Theorem 6 is similar to [18, Th. 3], then it is omitted.

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Received 25.12.1989