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A Direct Theorem for the Best Algebraic Approximation in L_p $[-1,1]$, $(0<p<1)$

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The problem to characterize the best algebraic approximation in a finite interval is of importance for the approximation theory. For the spaces $L_p[-1,1]$, $1\leqq p\leqq\infty$ this problem is solved by K. Ivanov in [4, 5, 6], but for the case $0<p<1$ only the theory of best trigonometric approximation has reached a high level of completeness in [9, 10, 11]. The corresponding algebraic problem $(0<p<1)$ is unsolved till now, because of the so-called effect of the ends and the fact, that we work in the quasi-normed spaces. In this paper we prove a direct theorem for the best algebraic approximation in $L_p[-1,1]$, $0<p<1$.

1. Introduction

We shall consider functions, belonging to the sets $L_p[-1,1]$, $0<p<1$. As usual we denote:

$$\|f\|_{L_p[-1,1]}=\left\{\int_{-1}^{1}|f(x)|^p\mathrm{d}x\right\}^{\frac{1}{p}}.$$

Let $H_n(T_n)$ be the set of all algebraic (trigonometric) polynomials of a degree at most n. For the best trigonometric approximation in $L_p[0,2\pi]$, $0<p<1$ the following direct theorem is known:

Theorem A

$$E_n^T(f)_p\leqq c(k,p)\,.\,\omega_k\left(f,\frac{1}{n+1}\right)_p,\quad n\geqq k-1,$$

where

$$E_n^T(f)_p=\inf\{\|f-t_n\|_p,\ t_n\in T_n\}$$

$$\omega_k(f,\delta)_p=\sup_{|h|\leqq\delta}\|\Delta_h^k f(x)\|_p,\ f(x)\in L_p[0,2\pi],\ 0<p<1$$

$$\|f\|_p=\left\{\int_0^{2\pi}|f(x)|^p\mathrm{d}x\right\}^{\frac{1}{p}}.$$

This theorem is obtained for $k=1$ by V. Ivanov [11] and E. A. Storozenko, V. G. Krotov, P. Oswald [9]. The case k-arbitrary natural number is considered by E. A. Storozenko, P. Oswald [10].

V. Ivanov [11] has proved the following converse theorem for the best trigonometric approximation:

Theorem B

$$\omega_k\left(f, \frac{1}{n}\right)_p \leqq c(k,p).\frac{1}{n^k}\left\{\sum_{\nu=0}^{n}(\nu+1)^{k.p-1}.E_\nu^p(f)_p\right\}^{\frac{1}{p}}.$$

From theorems A. and B. we have:

Corollary C. *If* $f \in L_p[0, 2\pi]$, $0<\alpha<k$, $0<p<1$, *then*

$$E_n^T(f)_p = O(n^{-\alpha}) <=> \omega_k(f, \delta)_p = O(\delta^\alpha).$$

For the best algebraic approximation in $L_p[-1,1]$, $0<p<1$ only the following direct theorem, obtained by E. A. Storozenko [7] is known:

Theorem D. *Let* $k \in N$ *and* $n \geqq k-1$. *Then*

$$E_n(f)_p \leqq c(k,p).\omega_k\left(f, \frac{1}{n+1}\right)_p,$$

where

$$\omega_k(f,\delta)_p = \sup_{0\leqq h\leqq\delta}\left\{\int_{-1}^{1-kh}|\Delta_h^k f(x)|^p dx\right\}^{\frac{1}{p}}.$$

While theorems A and B give solution of the problem of characterizing the best trigonometric approximation in $L_p[0,2\pi]$, $0<p<1$, theorem D cannot be used for solving of the corresponding algebraic problem in $L_p[-1,1]$, $0<p<1$, because of the effect of the ends, which is not taken into account by the moduli $\omega_k(f,\delta)_p$. For that purpose we use another characteristic — the moduli $\tau_k(f, \Delta_n(x))_{p,p}$.

For $x \in [-1,1]$, we set $\Delta(\delta, x) = \delta.\sqrt{1-x^2}+\delta^2$; $\Delta_n(x) = \Delta(n^{-1}, x)$; $\delta = \text{const}$.

If $W \in C[-1,1]$, $W \geqq 0$, then we define the best algebraic approximation with weight W as follows:

$$E_n(W,f)_p := \inf\{\|W.(f-Q)\|_p : Q \in H_n\};\ E_n(1;f)_p := E_n(f)_p.$$

The following modulus is used in this paper:

(1.1) $$\tau_k(f, W; \delta)_{p,p} = \|W(.).\omega_k(f, .; \delta(\cdot))_p\|_p$$
$$\tau_k(f, 1; \delta)_{p,p} = \tau_k(f, \delta)_{p,p},$$

where the local L_p modulus of continuity is denoted by:

(1.2) $$\omega_k(f, x; \delta(x))_p = \left[(2.\delta(x))^{-1}.\int_{-\delta(x)}^{\delta(x)}|\Delta_v^k f(x)|^p dv\right]^{\frac{1}{p}}.$$

Here $0<p<1$ and $\Delta_v^k f(x)$ is defined as

$$\sum_{r=0}^{k}(-1)^{k-r}.\binom{k}{r}.f(x+r.v),$$

if x, $x+kv \in [-1,1]$ and as 0 otherwise. With the help of the moduli $\tau_k(f, \Delta_n)_{q,p}$; $1 \leqq q \leqq p \leqq \infty$ the following equivalence is established by K. Ivanov [4]:

(1.3) For $f \in L_p[-1,1]$, $1 \leqq p \leqq \infty$, $0<\alpha<k$ we have

$$E_n(f)_p = O(n^{-\alpha}) <=> \tau_k(f, 1; \Delta_n)_{1,p} = O(n^{-\alpha}).$$

In this paper we prove the following theorem:

Theorem 1. *Let* $f \in L_p[-1,1]$, $0<p<1$. *Then*

$$E_n(f)_p \leqq c(k,p) \,.\, \tau_k(f, A\,.\,\Delta_n)_{p,p}, \tag{1.4}$$

where $A=A(k,p)$ *is a constant, depending only on* k *and* p, $k \in N$.

2. Auxiliary results

We shall use the following auxiliary results:

Lemma 2.1 ([1]). *For every* $f \in L_p(\Delta)$, $0<p<1$, $k=1,2,\dots$ *there exists* $P_{k-1}(x)=P_{k-1}(x;\Delta,f,p) \in H_{k-1}$, *such that*

$$\|f-P_{k-1}\|_{L_p} \leqq c(k,p) . \left\{ |\Delta|^{-1} . \int_0^{\frac{|\Delta|}{k}} J_k(h,f)_{L_p} dh \right\}^{\frac{1}{p}}, \tag{2.1}$$

$$J_k(h,f)_{L_p} = \|\Delta_h^k f(x)\|^p_{L(a,b-kh)}, \quad 0 \leqq h \leqq \frac{|\Delta|}{k}, \quad \Delta=(a,b).$$

Lemma 2.2 ([2]). *If* $P_n(x) \in H_n$, $T_n(x)=\cos n \,.\, \arccos x$ *is the Chebyshev's polynomial,* $|P_n(x)| \leqq L$ *for* $x \in [a,b]$, *then for every* $x \overline{\in} [a,b]$ *the following inequality holds*:

$$|P_n(x)| \leqq L . \left| T_n\left(\frac{2x-a-b}{b-a}\right)\right|. \tag{2.2}$$

Lemma 2.3 ([3]). *Let*

$$\theta(x,y) = \begin{cases} 1, & x<y, \\ 0, & x \geqq y \end{cases} \qquad -1 \leqq x, y \leqq 1.$$

For $n=0,1,\dots;$ $l=1,2,\dots$ *and fixed* $y: -1 \leqq y \leqq 1$, *there exists a polynomial* $R(x,y) \in H_{2nl}$, *with respect to* x, *such that*:

$$|\theta(x,y)-R(x,y)| \leqq B_l \,.\, [n\,.\,|\arccos x - \arccos y| + 1]^{1-2.l}. \tag{2.3}$$

Lemma 2.4 [(4]). *For every* $x, y \in [-1,1]$, $|x-y| \leqq \lambda \,.\, \Delta_n(x)$, $\lambda>0$, $n \geqq 2\lambda$ *we have*

$$(4\lambda+2)^{-1} . \Delta_n(x) \leqq \Delta_n(y) \leqq (2\lambda+3/2) . \Delta_n(x). \tag{2.4}$$

Lemma 2.5 ([7]). (*An inequality of Nikolsky.*) *If* $P_n \in H_n$ *and* $0<p \leqq q \leqq \infty$, *then*

$$\|P_n\|_{L_q[a,b]} \leqq [2(p+1) . (b-a)^{-1}]^{\frac{1}{p}-\frac{1}{q}} n^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_n\|_{L_p[a,b]}. \tag{2.5}$$

We denote with $c(k,p)$ a constant, depending on k and p, which may differ at each occurrence. About some properties of the moduli $\tau_k(f,\Delta_n)_{p,p}$, $1 \leqq p \leqq \infty$, see [5].

Let

$$x_0 = -1,$$
$$x_1 = x_0 + \Delta_n(x_0) = -1 + \frac{1}{n^2},$$
$$x_2 = x_1 + \Delta_n(x_1),$$
$$\dots\dots\dots\dots\dots\dots$$
$$x_{i+1} = x_i + \Delta_n(x_i).$$

If $x_{\varphi(n)} < 1$ and $x_{\varphi(n)} + \Delta_n(x_{\varphi(n)}) > 1$, then we set $x_{\varphi(n)} = 1$. For every interval $\delta_i = (x_{i-1}, x_{i+1})$, $i = 1, 2, \dots, \varphi(n) - 1$, we take the polynomial $P_{i,k-1}(x)$ from Lemma 2.1. Let us denote by $S_{k-1,n}(x)$ the following piecewise function:

$$S_{k-1,n}(x) = P_{i,k-1}(x), \quad \text{for} \quad x \in [x_{i-1}, x_i), \quad i = 1, 2, \dots, \varphi(n).$$

One can write

$$S_{k-1,n}(x) = \sum_{i=1}^{\varphi(n)} P_i(x) . [\theta(x, x_i) - \theta(x, x_{i-1})] \tag{2.6}$$
$$= P_{\varphi(n)}(x) + \sum_{i=1}^{\varphi(n)-1} [P_i(x) - P_{i+1}(x)] . \theta(x, x_i).$$

We choose $l \in N$, such that

$$(2l-1) . p > 2(k-1) . p + 2, \quad \text{that is } (2l-k) . p > 1. \tag{2.7}$$

Let

$$Q_{n,k}(x) = P_{\varphi(n)}(x) + \sum_{i=1}^{\varphi(n)-1} [P_i(x) - P_{i+1}(x)] . R(x, x_i),$$

where $R(x, x_i) \in H_{2nl}$ is the polynomial from Lemma 2.3, that is $Q_{n,k}(x) \in H_{2nl+k-1}$. Let $h_i = (x_i, x_{i+1})$, $i = 0, 1, \dots, \varphi(n) - 1$ and $|h_i| = \Delta_n(x_i) = \Delta_i$.

Let us formulate and prove the following basic Lemma, which we shall use to prove Theorem 1.

Lemma 2.6. *Under above conditions and denotations we have*

$$I_i = \int_{[-1,1] \setminus h_i} \frac{\left| x - x_i - \frac{\Delta_i}{2} \right|^{(k-1).p}}{(n . |\arccos x - \arccos x_i| + 1)^{(2l-1).p}} dx \tag{2.8}$$
$$\leqq c(k, p) . \Delta_i^{(k-1).p+1}, \quad \textit{for} \quad i = 1, 2, \dots, \varphi(n) - 1.$$

Proof: We shall consider 3 cases, depending on i:

(A.) Let i: $-1/2 \leqq x_i \leqq 1/2$. Then $\Delta_i \asymp 1/n$.

$$I_i \leqq \int_{-1}^{1} \frac{|x - x_i - \Delta_i/2|^{(k-1).p}}{(n.|x - x_i| + 1)^{(2l-1).p}} dx \leqq 2 . \int_0^2 \frac{x^{(k-1).p}}{(nx+1)^{(2l-1).p}} dx + 2 . \int_0^2 \frac{\Delta_i^{(k-1).p}}{(nx+1)^{(2l-1).p}} dx$$

$= A_1 + A_2$. From (2.7) we obtain the estimations

$$A_1 \leqq 2.\frac{1}{n}.\int_1^{2n+1} \frac{\left(\frac{z-1}{n}\right)^{(k-1).p}}{z^{(2l-1).p}}\,dz \leqq 2.\left(\frac{1}{n}\right)^{(k-1).p+1}\int_1^{2n+1} z^{(k-1).p-(2l-1).p}dz$$

$$\leqq c(k,p).\Delta_i^{(k-1).p+1}.$$

$$A_2 \leqq 2.\Delta_i^{(k-1).p+1}.\int_1^{2n+1} z^{-(2l-1).p}dz \leqq c(k,p).\Delta_i^{(k-1).p+1}.$$

B.) Let $i: -1 \leqq x_i \leqq -\frac{1}{2}$. We shall consider 2 cases, depending on the positions of x.

a) $.-1 \leqq x \leqq x_i$. Then $|\arc\cos x - \arc\cos x_i| = (x_i - x).\frac{1}{\sqrt{1-\xi^2}}$, where $-1 \leqq x \leqq \xi \leqq x_i \leqq -\frac{1}{2}$ and $\frac{1}{\sqrt{1-\xi^2}} \geqq \frac{1}{\sqrt{1-x_i^2}}$

$$I_i' = \int_{-1}^{x_i} \frac{(x_i - x + \Delta_i/2)^{(k-1).p}}{(n.|\arc\cos x - \arc\cos x_i| + 1)^{(2l-1)p}}\,dx$$

$$\leqq c(k,p).\int_{-1}^{x_i} \frac{(x_i - x + \Delta_{i/2})^{(k-1).p}}{\left(\frac{n.(x_i - x)}{\sqrt{1-x_i^2}} + 1\right)^{(2l-1).p}}\,dx.$$

We set $x_i - x = y$

$$I_i' \leqq \int_0^{1+x_i} \frac{(y+\Delta_i/2)^{(k-1).p}}{\left(\frac{ny}{\sqrt{1-x_i^2}} + 1\right)^{(2l-1).p}}\,dy = \int_0^{\Delta_i/2} + \int_{\Delta_i/2}^{1+x_i} = I_{i,1}' + I_{i,2}'.$$

It is obvious, that

$$I_{i,1}' \leqq \int_0^{\Delta_i/2} \Delta_i^{(k-1).p}dy = \frac{1}{2}.\Delta_i^{(k-1).p+1},$$

$$I_{i,2}' \leqq \left(\frac{\sqrt{1-x_i^2}}{n}\right)^{(2l-1).p}.\int_{\Delta_i/2}^{1+x_i} y^{(k-1).p-(2l-1).p}dy$$

$$\leqq c(k,p).\Delta_i^{(2l-1).p}.\Delta_i^{(k-1).p-(2l-1).p+1} = c(k,p).\Delta_i^{(k-1).p+1}.$$

We obtain, that $I_i' \leqq c(k,p).\Delta_i^{(k-1).p+1}$.

b.) $x_{i+1} \leqq x \leqq 1$.

$$I_i'' = \int_{x_{i+1}}^{1} \frac{|x - x_i - \Delta_i/2|^{(k-1).p}}{(n.|\arc\cos x - \arc\cos x_i| + 1)^{(2l-1).p}}\,dx = \int_{x_{i+1}}^{0} + \int_0^1 = I_{i,1}'' + I_{i,2}''.$$

For $x \in [0,1]$ and $-1 \leqq x_i \leqq -\frac{1}{2}$ it is fulfilled

$$|\arc\cos x - \arc\cos x_i| \asymp \pi.$$

$$I''_{i,2} \leqq c(k,p).\left(\frac{1}{n}\right)^{(2l-1).p} \leqq c(k,p).\left(\frac{1}{n^2}\right)^{(k-1).p+1} \leqq c(k,p).\Delta_i^{(k-1).p+1},$$

where we have used the choice of l from (2.7).

Let's consider $I''_{i,1}$. We set $z_i = -x_i$; $y = x + z_i$.

$$I''_{i,2} = \int_{\Delta_i}^{z_i} \frac{(y-\Delta_i/2)^{(k-1).p}}{[n.(\arccos(-z_i) - \arccos(y-z_i)) + 1]^{(2l-1).p}} dy.$$

$y/2 \leqq y - \Delta i/2 \leqq y$. We have, that

$$\arccos(-z_i) - \arccos(y - z_i) = \arccos(z_i - y) - \arccos(z_i) = \frac{y}{\sqrt{1-\xi^2}},$$

where

$$0 \leqq z_i - y \leqq \xi \leqq z_i, \ \frac{1}{2} \leqq z_i \leqq 1, \ \frac{1}{\sqrt{1-\xi^2}} \geqq \frac{1}{\sqrt{1-(z_i-y)^2}}.$$

We go on with the estimation for $I''_{i,1}$.

$$I''_{i,1} \leqq c(k,p).\int_{\Delta_i}^{z_i} y^{(k-1).p}.\left(\frac{\sqrt{1-z_i^2+2.z_i.y-y^2}}{n.y}\right)^{(2l-1)p} dy$$

$$\left(\frac{\sqrt{1-z_i^2+2z_i.y-y^2}}{n.y}\right)^{(2l-1).p} \leqq c(k,p).\left[\left(\frac{\sqrt{1-z_i^2}}{n.y}\right)^{(2l-1).p} + \left(\frac{1}{n.\sqrt{y}}\right)^{(2l-1).p}\right]$$

$$I''_{i,1} \leqq c(k,p).\int_{\Delta_i}^{z_i} y^{(k-1).p-(2l-1)p} dy.\Delta_i^{(2l-1).p} + c(k,p).\int_{\Delta_i}^{z_i} y^{(k-1).p-\frac{(2l-1)}{2}.p} dy.\left(\frac{1}{n}\right)^{(2.l-1)p}$$

$$\leqq c(k,p).\left(\Delta_i^{(k-1).p+1} + \Delta_i^{(k-1).p+1-\frac{(2l-1).p}{2}}.\left(\frac{1}{n}\right)^{(2l-1).p}\right) \leqq c'(k,p).\Delta_i^{(k-1).p+1},$$

because from $\frac{1}{n^2} \leqq \Delta_i$, it follows:

$$\Delta_i^{-\frac{(2l-1).p}{2}}.\left(\frac{1}{n}\right)^{(2l-1).p} \leqq 1.$$

With this the case B.) is completed.

C.) Let $i: \frac{1}{2} \leqq x_i \leqq 1$.

We make in (2.8) the change $x_i = -y_i$, $x = -y$. We get

$$I_i = \int_{-1}^{y_i - \Delta_i} \frac{(y_i - y - \Delta_i/2)^{(k-1).p}}{(n.|\arccos y - \arccos y_i| + 1)^{(2l-1)p}} dy$$

$$+ \int_{y_i}^{1} \frac{(y - y_i + \Delta_i/2)^{(k-1)p}}{(n.|\arccos y - \arccos y_i| + 1)^{(2l-1)}} dy = \mathscr{I}'_i + \mathscr{I}''_i.$$

We have, that

$$\mathcal{I}'_i \leqq I'_i$$

$$\mathcal{I}''_i = I''_i + \int_{y_i}^{y_{i+1}} \frac{(y - y_i + \Delta_i/2)^{(k-1).p}}{(n.|\arccos y - \arccos y_i| + 1)^{(2l-1).p}} dy \leqq c(k,p)\,\Delta_i^{(k-1).p+1}.$$

From the estimations, established in case B.), we obtain the proof of (2.8). With this Lemma 2.6 is proved.

3. Proof of Theorem 1

From Lemma 2.1 we obtain

$$\|f(x) - P_{i,k-1}(x)\|^p_{L_p(\delta_i)} \leqq c(k,p).\frac{1}{(x_{i+1} - x_{i-1})}.\int_0^{\frac{x_{i+1}-x_{i-1}}{k}} \int_{x_{i-1}}^{x_{i+1}-k.h} |\Delta_h^k f(x)|^p dx\, dh.$$

From Lemma 2.4 it follows $\Delta_i \leqq (7/2).\Delta_{i-1}$ and

$$0 \leqq h \leqq \frac{x_{i+1} - x_{i-1}}{k} = \frac{\Delta_{i-1} + \Delta_i}{k} \leqq \frac{1 + 7/2}{k}.\Delta_{i-1} \leqq 5.\Delta_{i-1}.$$

(3.1) $$\|f(x) - P_{i,k-1}(x)\|^p_{L_p(\delta_i)} \leqq c(k,p).\int_{x_{i-1}}^{x_{i+1}} \frac{1}{10.\Delta_{i-1}}.\int_{-5.\Delta_{i-1}}^{5.\Delta_{i-1}} |\Delta_h^k f(x)|^p\, dh\, dx.$$

Using Lemma 2.4, we get for $x \in [x_{i-1}, x_i] : \Delta_{i-1}/6 \leqq \Delta_n(x) \leqq (7/2).\Delta_{i-1}$ and for $x \in [x_i, x_{i+1}] : \Delta_{i-1} \leqq 6.\Delta_i \leqq 36.\Delta_n(x)$

$$\Delta_{i-1} \geqq (2/7).\Delta_i \geqq (2/7)^2.\Delta_n(x).$$

So for $x \in \delta_i$ we have $(2/7)^2.\Delta_n(x) \leqq \Delta_{i-1} \leqq 36.\Delta_n(x)$.
From (3.1) we get

(3.2) $$\|f(x) - P_{i,k-1}(x)\|^p_{L_p(\delta_i)}$$
$$\leqq c(k,p).\int_{x_{i-1}}^{x_{i+1}} \frac{1}{10.36.\Delta_n(x)} \int_{-5.36\Delta_n(x)}^{5.36\Delta_n(x)} |\Delta_h^k f(x)|^p dh\, dx.$$

From the definition of $S_{k-1,n}(x)$, we obtain

$$\|f(x) - S_{k-1,n}(x)\|^p_{L_p[-1,1]} = \sum_{i=1}^{\varphi(n)} \|f(x) - P_{i,k-1}(x)\|^p_{L_p(h_{i-1})}$$
$$\leqq \sum_{i=1}^{\varphi(n)} \|f(x) - P_{i,k-1}(x)\|^p_{L_p(\delta_i)} \leqq c(k,p).\sum_{i=1}^{\varphi(n)} \int_{x_{i-1}}^{x_{i+1}} \frac{1}{360.\Delta_n(x)}.\int_{-180\Delta_n(x)}^{180\Delta_n(x)} |\Delta_h^k f(x)|^p dh\, dx$$
$$\leqq c(k,p).\int_{-1}^{1} \frac{1}{360.\Delta_n(x)} \int_{-180.\Delta_n(x)}^{180.\Delta_n(x)} |\Delta_h^k f(x)|^p\, dh\, dx = c(k,p).\tau_k^p(f, 180.\Delta_n(x))_{p,p}$$

and hence:

(3.3) $$\|f - S_{k-1,n}\|_{L_p[-1,1]} \leqq c(k,p).\tau_k(f,\ 180.\Delta_n)_{p,p}.$$

We go on with the following inequality:

(3.4) $$\|S_{k-1,n}(x)-Q_{n,k}(x)\|^p_{L_p[-1,1]}$$
$$\leqq \sum_{i=1}^{\varphi(n)-1}\left[\int_{h_i}+\int_{[-1,1]\setminus h_i}\right]|P_i(x)-P_{i+1}(x)|^p\,.\,|\theta(x,x_i)-R(x,x_i)|^p dx=M_1+M_2.$$

The proof of Theorem 1. will be completed, if we proof

(3.5) $$\|S_{k-1,n}(x)-Q_{n,k}(x)\|^p_{L_p[-1,1]}\leqq \gamma\,(k,p)\,.\sum_{i=1}^{\varphi(n)-1}\|P_i-P_{i+1}\|^p_{L_p(h_i)},$$

because from (3.2) and from

$$\|P_i-P_{i+1}\|^p_{L_p(h_i)}\leqq \|P_i-f\|^p_{L_p(\delta_i)}+\|P_{i+1}-f\|^p_{L_p(\delta_{i+1})}.$$

we shall get

(3.6) $$\|S_{k-1,n}-Q_{n,k}\|^p_{L_p[-1,1]}\leqq \gamma'(k,p)\,.\,\tau_k^p(f,180\,.\,\Delta_n)_{p,p}.$$

From (3.3) and (3.6) it follows, that:

(3.7) $$E_{2nl+k-1}(f)_p\leqq c(k,p)\,.\,\tau_k(f,180\Delta_n)_{p,p}.$$

Let's choose

$$m:2nl+k-1\leqq m\leqq 2(n+1)\,.\,l+k-1.$$
$$E_m(f)_p\leqq E_{2nl+k-1}(f)_p\leqq c(k,p)\,.\,\tau_k(f,180\,.\,\Delta_n)_{p,p}$$
$$m\leqq 2(n+1)\,.\,l+k-1\leqq 3nl,\ \ n\geqq 3$$
$$\Delta_n(x)\leqq \Delta_{\frac{m}{3l}}(x)$$
$$E_m(f)_p\leqq c(k,p)\,.\,\tau_k(f,\ 180\,.\,9l^2\,.\,\Delta_m(x))_{p,p},$$

that is (1.4) is proved with $A=180\,.\,9l^2$, $m\geqq 2nl+k-1$, $n\geqq 3$, where l is determined by (2.7).

So it is enough to prove (3.5). We shall evaluate first M_1 from (3.4). It is clear, that

(3.8) $$|\theta(x,x_i)-R(x,x_i)|\leqq B_l$$

and

(3.9) $$M_1\leqq B_l^p\,.\sum_{i=1}^{\varphi(n)-1}\|P_i-P_{i+1}\|^p_{L_p(h_i)}.$$

It remains to prove for M_2 an estimation, similar to (3.9), and thus the proofs of (3.5) and Theorem 1. will be completed.

From Lemma 2.5 for $q=\infty$ we have

(3.10) $$|P_i(x)-P_{i+1}(x)|^p\leqq 2(p+1)\,.\,\Delta_i^{-1}\,.\,(k-1)^2\,.\,\|P_i-P_{i+1}\|^p_{L_p(h_i)}$$
$$=\alpha(k,p)\,.\,\Delta_i^{-1}\,.\,\|P_i-P_{i+1}\|^p_{L_p(h_i)},\quad x\in h_i.$$

From Lemma 2.2, applied for the polynomial (P_i-P_{i+1}) and the interval h_i, and from (3.10), it follows, that for $x\in[-1,1]\setminus h_i$

(3.11)

$$|P_i(x)-P_{i+1}(x)|^p \leqq \alpha(k,p).\Delta_i^{-1}.\|P_i-P_{i+1}\|^p_{L_p(h_i)}.$$

$$T^p_{k-1}\left(\frac{2x-2x_i-\Delta_i}{\Delta_i}\right) \leqq \delta(k,p).\|P_i-P_{i+1}\|^p_{L_p(h_i)}.\Delta_i^{-(k-1).p-1}.\left|x-x_i-\frac{\Delta_i}{2}\right|^{(k-1).p}.$$

In the last inequality we have used, that

$$|T_k(t)|=\frac{1}{2}.((t+\sqrt{t^2-1})^k+(t-\sqrt{t^2-1})^k|\leqq 2^k.|t|^k, \quad |t|>1, \quad k\in N.$$

From (3.11) we get

$$(3.12) \qquad M_2 \leqq c(k,p).\sum_{i=1}^{\varphi(n)-1} \|P_i-P_{i+1}\|^p_{L_p(h_i)}.\Delta_i^{-(k-1).p-1}$$

$$\times \int_{[-1,1]\setminus h_i} \frac{|x-x_i-\Delta_{i/2}|^{(k-1).p}dx}{(n.|\arccos x-\arccos x_i|+1)^{(2l-1)p}}.$$

Using Lemma 2.6, (3.12) and (3.9) we obtain the proofs of (3.5) and Theorem 1.

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